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ORIGINAL RESEARCH

Higher order numerical methods for fractional delay differential equations

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Abstract In this paper, we present a new family of higher-order numerical methods for solving non-linear fractional delay differential equations (FDDEs) along with the error analysis. Further, we solve various non-trivial systems of FDDEs to illustrate their applicability and utility. By using the proposed numerical methods, computational time is reduced drastically. These methods take only 5 to 10 percent of the time required for other methods such as the fractional Adams method (FAM). Furthermore, these methods converge for very small values of fractional derivative while FAM and the new predictor-corrector method (NPCM) introduced by Daftardar-Gejji *et al.* [1] do not converge. The order of convergence of the proposed methods is $r + \alpha$, where r is the order of fractional backward difference formulae and α denotes the order of the fractional derivative. Thus these methods have a higher order of accuracy than FAM or NPCM.

Keywords Caputo derivative · Fractional delay differential equations · Error analysis · Numerical solutions

1 Introduction

Fractional differential equations (FDEs) find applications in various fields for modeling complex phenomena that exhibit non-local and non-integer order behaviors. The use of fractional derivatives in modeling offers more flexibility to capture memory effects, long-range dependencies, and anomalous behaviors that are not adequately addressed by traditional integer-order differential equations [2,3]. Moreover, time delay is one of the phenomena that can be seen in various systems such as economic systems, control systems, biological systems, chemical systems, and so on [4]. Hence, combining fractional derivatives with time delays in mathematical models allows for a more comprehensive representation of systems with both fractional order dynamics and delayed responses. This combination is motivated by the need to capture the complexities observed in real-world systems where both fractional order effects and time delays play a role [5]. During recent decades, fractional delay differential equations (FDDEs) have found applications in several fields of physics, biology, finance, chemistry, economics

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and engineering such as wave propagation, population dynamics, dynamic systems, control systems, financial systems, neural networks and neuroscience, chemical reactions and reaction-diffusion systems and so on [6–10]. Recently, various models such as complex-valued single neuron model [11], model for the chemotactic interactions between one species and two competing chemicals through Keller-Segel system [12,13], genetic regulatory network model [14], consensus of multi-agent systems [15], eco-epidemiological model [16], epidemic model [17,18], economic growth model [19], model for ebola infection [20], prey-predator model with inter-specific competition [16], model of small-world networks [21], food chain model [22], model for tumor-immune system [23] and so on have been developed using FDDEs. Several authors have discussed the existence and uniqueness theorems for FDDEs [24–28]. Delay differential equations (DDEs) are infinite-dimensional in nature as they require the history in a certain time interval. So these equations are very difficult to analyze analytically [29]. Due to the non-local nature of fractional derivatives, the computations involved in solving FDDEs take enormous time. Hence developing accurate and time-efficient numerical methods is a challenging task.

In the literature, various traditional methods used for solving ordinary differential equations have been modified to solve fractional differential equations (FDEs) and further extended to solve fractional delay differential equations (FDDEs). Diethelm *et al.* have modified the Adams method for solving FDEs, termed as fractional Adams methods (FAM) [30]. Researchers have used FAM to solve various systems of nonlinear FDEs. Further, Bhalekar and Daftardar-Gejji [31] extended FAM to solve FDDEs. Moghaddam and Mostaghim have generalized the finite difference method to solve FDDEs [32]. Wang *et al.* [33] have given an algorithm based on Grünwald-Letnikov derivative for solving FDEs with time delay. Pandey *et al.* [34] have introduced the approximation method to solve FDDEs which is based on the application of Bernstein's operational matrix of fractional differentiation. Further, Diethelm *et al.* [35] have discussed various numerical algorithms for FDEs. Li *et al.* [36] have designed higher-order algorithms in this context. A numerical algorithm based on the fractional Euler method is presented in [37].

Daftardar-Gejji *et al.* [1,38] modified FAM and proposed a new predictor-corrector method (NPCM) to solve FDEs, which is extended for FDDEs. It is observed that NPCM is more time efficient as compared to FAM. Further, Jhinga and Daftardar-Gejji [39,40] proposed a new finite difference predictor-corrector method to solve FDEs and later extended it for FDDEs. In [41], Kumar applied the L1-scheme to solve these equations. Furthermore, Kumar and Daftardar-Gejji [42] have proposed a new family of predictor-corrector methods based on fractional backward difference formulae denoted by (FBD-PCMr), where $r = 1, 2, \ldots, 6$; for solving FDEs. FBD-PCMr are more accurate, time efficient, and computationally economical as compared to FAM or NPCM. In this paper, we extend FBD-PCMr to solve FDDEs and present its errOr analysis. These methods are based on fractional backward difference formulae (FBDF). The proposed family of numerical methods provides very accurate solutions in much less time taken by the existing methods such as FAM and NPCM. Moreover, they converge for very small values of α, while FAM and NPCM fail. Apart from this, the proposed methods have a higher order of accuracy. The order of the new methods is $r + \alpha$, $(1 \le r \le 6)$ where r is the order of fractional backward difference formulae and α is the order of the fractional derivative. Numerical simulations were executed using Python 2.1 on a computer featuring a core-i5 processor.

The present paper is organized as follows: Basic definitions and notations are given in section 2. In section 3, new numerical methods are introduced. Error analysis of these methods has been presented in section 4. In section 5, the new numerical methods have been applied to solve various non-linear systems of FDDEs. Finally, in section 6 the conclusions are drawn.

2 Fractional derivatives

In this section, we give some definitions and set up notations [43,44] which are used in the later sections.

Definition 2.1 Riemann-Liouville (RL) fractional derivative of a real-valued function $y(t)$ of order α is defined as

$$ {}^{RL}D_t^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} y(\xi) d\xi, \quad 0 < \alpha < 1. \tag{1} $$

Definition 2.2 Caputo derivative of a real valued function $y(t)$ of order α is defined as:

$$ D_t^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} \frac{dy(\xi)}{d\xi} d\xi, \quad 0 < \alpha < 1. \tag{2} $$

The relation between RL and Caputo derivatives (for $0 < \alpha < 1$) is given by:

$$\begin{aligned} D_t^\alpha y(t) &= {}^{RL}D_t^\alpha \Big(y(t) - y(0)\Big) \\ &= {}^{RL}D_t^\alpha y(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(0). \end{aligned} \tag{3}$$

Definition 2.3 Mittag-Leffler function with one parameter α is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad Re(\alpha) > 0, z \in \mathbb{C}.$$

3 Genesis of the methods

In this section, we develop a new class of higher-order numerical methods for solving the fractional delay differential equation (FDDEs) having the following form:

$$\left.\begin{aligned} &D_t^\alpha y(t) = f\Big(t, y(t), y_\tau(t)\Big),\ t \in [0, T],\ \tau > 0,\ T > 0,\ 0 < \alpha < 1, \\ &y(t) = \phi(t),\ \ t \in [-\tau, 0], \end{aligned}\right\} \tag{4}$$

where D_t^α is the fractional Caputo derivative of order α, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a sufficiently smooth function and $y_\tau(t) = y(t - \tau)$ a delay term.

We propose higher order numerical methods based on fractional backward difference formulae (FBDFr) of order r, $r = 1, 2, \ldots, 6$ as described below. Consider a partition of $[0, T]$ with uniform grid points $t_j = jh : j = -M, -M + 1, \ldots, -1, 0, 1, \ldots, N$; where M and N are positive integers such that $\tau = Mh$ and $T = Nh$. Further, let $y_\tau(t_j) = y(t_j - \tau) = y(jh - Mh) = y((j - M)h) = y(t_{j-M}), \quad j = 0, 1, \ldots, N$ and $y(t_j) = \phi(t_j), \quad j = -M, -M + 1, \ldots, 0$.

The Riemman-Liouville fractional derivative of $y(t)$ $i.e$ ${}^{RL}D_t^\alpha y(t)$ at $t = t_n$ can be approximated as [43]

$${}^{RL}D_t^\alpha y(t_n) \approx \widetilde{{}^{RL}D_t^\alpha} y(t_n) = h^{-\alpha} \sum_{j=0}^{n} \omega_{n-j}^r y(t_j), \tag{5}$$

where w_j^r, $(j = 0, 1, 2, \ldots)$ denote convolution weights of order r $(r = 1, 2, \ldots 6)$. The convolution weights w_j^r are calculated from the corresponding generating functions $W_\alpha^r(z) = \sum_{j=0}^{\infty} \omega_j^r z^j$ as described in [45]. The generating functions $W_\alpha^r(z)$ presented in [43,45] are given below:

$$\begin{aligned} W_\alpha^1(z) &= (1 - z)^\alpha, \\ W_\alpha^2(z) &= \Big(\frac{3}{2} - 2z + \frac{1}{2}z^2\Big)^\alpha, \\ W_\alpha^3(z) &= \Big(\frac{11}{6} - 3z + \frac{3}{2}z^2 - \frac{1}{3}z^3\Big)^\alpha, \\ W_\alpha^4(z) &= \Big(\frac{25}{12} - 4z + 3z^2 - \frac{4}{3}z^3 + \frac{1}{4}z^4\Big)^\alpha, \\ W_\alpha^5(z) &= \Big(\frac{137}{60} - 5z + 5z^2 - \frac{10}{3}z^3 + \frac{5}{4}z^4 - \frac{1}{5}z^5\Big)^\alpha, \\ W_\alpha^6(z) &= \Big(\frac{147}{60} - 6z + \frac{15}{2}z^2 - \frac{20}{3}z^3 + \frac{15}{4}z^4 - \frac{6}{5}z^5 + \frac{1}{6}z^6\Big)^\alpha. \end{aligned}$$

We observe that $|(\omega_0^r)^{-1}| \leq 1$. Suppose that y_j denotes the approximate solution of $y(t)$ at $t = t_j$. In view of (3) and (5) at $t = t_n$;we get

$$D_t^\alpha y(t) \approx h^{-\alpha} \sum_{j=0}^{n} \omega_{n-j}^r y(t_j) - \frac{t_n^{-\alpha}}{\Gamma(1-\alpha)} y(0) \tag{6}$$

Approximating the Caputo derivative on the left side of Eqn. (4) by Eqn. (6), we obtain the following FBDFr.

$$h^{-\alpha}\sum_{j=0}^{n}\omega_{n-j}^{r}y_j-\frac{t_n^{-\alpha}}{\Gamma(1-\alpha)}\phi(0)=f\Big(t_n,y_n,y_\tau(t_n)\Big). \tag{7}$$

We write Eqn. (7) in the following way

$$y_n=\frac{h^{\alpha}t_n^{-\alpha}}{\omega_0^r\Gamma(1-\alpha)}\phi(0)-\frac{1}{\omega_0^r}\sum_{j=0}^{n-1}\omega_{n-j}^{r}y_j+\frac{h^{\alpha}}{\omega_0^r}f\Big(t_n,y_n,y_\tau(t_n)\Big). \tag{8}$$

Put $t_n=nh$ and $\frac{1}{\omega_0^r}=A_r$ in Eqn. (8) to obtain

$$y_n=A_r\frac{n^{-\alpha}}{\Gamma(1-\alpha)}\phi(0)-A_r\sum_{j=0}^{n-1}\omega_{n-j}^{r}y_j+A_rh^{\alpha}f\Big(t_n,y_n,y_\tau(t_n)\Big). \tag{9}$$

Equation (9) can be written in the following form

$$y_n=g+N\Big(t_n,y_n,y_\tau(t_n)\Big), \tag{10}$$

where

$$g=A_r\frac{n^{-\alpha}}{\Gamma(1-\alpha)}\phi(0)-A_r\sum_{j=0}^{n-1}\omega_{n-j}^{r}y_j,$$
$$N\Big(t_n,y_n,y_\tau(t_n)\Big)=A_rh^{\alpha}f\Big(t_n,y_n,y_\tau(t_n)\Big).$$

Observe that Eqn. (10) is implicit in nature. We employ the iterative method introduced by Daftardar-Gejji and Jafari [46] to solve Eqn. (10) and get

$$\begin{aligned}
y_{n0}&=A_r\frac{n^{-\alpha}}{\Gamma(1-\alpha)}\phi(0)-A_r\sum_{j=0}^{n-1}\omega_{n-j}^{r}y_j,\\
y_{n1}&=N(y_{n0})=A_rh^{\alpha}f\Big(t_n,y_{n0},y_\tau(t_n)\Big),\\
y_{n2}&=N(y_{n0}+y_{n1})-N(y_{n0}).
\end{aligned} \tag{11}$$

We consider three term approximate solution of Eqn. (10) *i.e.* $y_n\approx y_{n0}+y_{n1}+y_{n2}$, which yields six new predictor-corrector methods corresponding to $r=1,\ldots,6$. Hence a new family of predictor-corrector methods termed as FBD-PCMr is developed as follows:

$$\left.\begin{aligned}
y_n^p&=A_r\frac{n^{-\alpha}}{\Gamma(1-\alpha)}\phi(0)-A_r\sum_{j=0}^{n-1}\omega_{n-j}^{r}y_j,\\
z_n^p&=N(y_n^p)=A_rh^{\alpha}f\Big(t_n,y_n^p,y_\tau(t_n)\Big),\\
y_n^c&=y_{n0}+y_{n1}+y_{n2}=y_n^p+A_rh^{\alpha}f\Big(t_n,y_n^p+z_n^p,y_\tau(t_n)\Big),
\end{aligned}\right\} \tag{12}$$

where $r=1,2,\ldots,6$. Here the convolution weights w_j^r are calculated from the corresponding generating functions $W_\alpha^r(z)$ as described earlier. These weights can also be calculated using the recursive relation given in [45].

Remark 3.1 For solving equation (9) in the paper, methods such as the Newton–Raphson method, Adomian decomposition method (ADM), new iterative method introduced by Daftardar-Gejji and Jafari (DGJM), homotopy perturbation method, etc. can be used. But DGJM is used because it avoids the computations of derivative terms of the function in the Newton Raphson method. Also, ADM includes tedious calculations of Adomian polynomials which again involves high-order differentials of the function whereas DGJM is simple, easy to implement, and computer-friendly. Thus DGJM is used as it reduces the computational part.

4 Error Analysis

First, we state some lemmas which are useful for proving the basic theorem regarding error estimates of the newly proposed methods.

Lemma 4.1 [47–49] *Let $y(t)$ be a real-valued function, then the following inequality holds*

$$\left| {}^{RL}D_t^{\alpha} y(t_n) - h^{-\alpha} \sum_{j=0}^{n} \omega_{n-j}^{r} y(t_j) \right| \leq C' h^{r}, \; n = 1, 2, \ldots k, \quad t \in [0, T], \tag{13}$$

where C' is a positive constant independent of h and n.

We set:

$$R_n = A_r h^{\alpha} \left[{}^{RL}D_t^{\alpha} y(t_n) - h^{-\alpha} \sum_{j=0}^{n} \omega_{n-j}^{r} y(t_j) \right]. \tag{14}$$

It is clear from Eqn. (13) and (14),

$$\left| R_n \right| \leq C' h^{r+\alpha}. \tag{15}$$

Lemma 4.2 [48] *Suppose $b_{j,n} = (n-j)^{\alpha-1}$, for $j = 1, 2, \ldots, n-1$ and $b_{j,n} = 0$ for $j \geq n$; $0 < \alpha < 1,\; h,\; K,\; T > 0,\; kh \leq T$, $k \in \mathbb{N}$ and $\sum_{j=m}^{n} b_{j,n} |e_j| = 0$ for $m > n \geq 1$. If*

$$\left| e_n \right| \leq K h^{\alpha} \sum_{j=1}^{n-1} b_{j,n} \left| e_j \right| + \left| \xi_0 \right|, \quad n = 1, 2, \ldots, k, \tag{16}$$

then

$$\left| e_k \right| \leq C \left| \xi_0 \right|, \quad k = 1, 2, \ldots, \tag{17}$$

where $C > 0$ and independent of h and k.

Lemma 4.3 [49] *Let ω_j^r, $(j = 0, 1, 2, \ldots)$ be the convolution weights of generating functions $W_{\alpha}^{r}(z)$ of fractional backward formulae of order r, $r = 1, 2, \ldots, 6$. Then*

$$\omega_j^r = O\left(j^{-\alpha-1} \right). \tag{18}$$

Next, we prove a theorem that gives the error estimates of the proposed methods.

Theorem 4.1 *Suppose $f(t, x, y)$ satisfies the Lipschitz condition with respect to its variables as follows:*

$$\left| f(t, x_1, y_1) - f(t, x_2, y_2) \right| \leq L_1 \left| x_1 - x_2 \right| + L_2 \left| y_1 - y_2 \right|, \tag{19}$$

where L_1 and L_2 are positive constants. Let $y(t)$ denote the exact solution and y_k^c $(k = 1, 2, \ldots N)$ the approximate solutions obtained by the proposed methods (12) *of the FDDE* (4)*. For the sufficiently small h the error estimate of the proposed methods is of $O(h^{r+\alpha})$ i.e.*

$$\max_{0 \leq k \leq N} \left| y(t_k) - y_k^c \right| \leq O(h^{r+\alpha}), \quad r = 1, 2, \ldots, 6. \tag{20}$$

Proof. We prove this result using mathematical induction. Suppose that the result is true for $k = 1, 2, \dots, n-1$. Then we prove that it is also true for $k = n$. In view of (4) and (15), the error equation of the proposed method (12) is:

$$y(t_k) - y_k^c = \left(y(t_k) - y_k^p\right) + A_r h^\alpha \Bigg(f\Big(t_k, y(t_k) + N\big(y(t_k)\big), y(t_{k-M})\Big) - f\Big(t_k, y_k^p + N\big(y_k^p\big), y_{k-M}\Big)\Bigg) + R_k. \tag{21}$$

Hence

$$\begin{aligned}
\Big|y(t_k) - y_k^c\Big| &\le \Big|y(t_k) - y_k^p\Big| + h^\alpha \Big| f\Big(t_k, y(t_k) + N(y(t_k)), y(t_{k-M})\Big) \\
&\quad - f\Big(t_k, y_k^p + N(y_k^p), y_{k-M}\Big)\Big| + \Big|R_k\Big| \\
&\le \Big|y(t_k) - y_k^p\Big| + L_1 h^\alpha \Big|y(t_k) + N(y(t_k)) - y_k^p - N(y_k^p)\Big| \\
&\quad + L_2 h^\alpha \Big|y(t_{k-M}) - y_{k-M}\Big| + \Big|R_k\Big| \\
&\le \Big|y(t_k) - y_k^p\Big| + L_1 h^\alpha \Big|y(t_k) - y_k^p\Big| + L_1 h^\alpha \Big|N(y(t_k)) - N(y_k^p)\Big| \\
&\quad + L_2 h^\alpha \Big|y(t_{k-M}) - y_{k-M}\Big| + \Big|R_k\Big| \\
&\le \Big|y(t_k) - y_k^p\Big| + L_1 h^\alpha \Big|y(t_k) - y_k^p\Big| + L_1^2 h^{2\alpha} \Big|y(t_k) - y_k^p\Big| \\
&\quad + L_2 h^\alpha \Big|y(t_{k-M}) - y_{k-M}\Big| + \Big|R_k\Big| \\
&\le \Big(1 + L_1 h^\alpha + L_1^2 h^{2\alpha}\Big)\Big|y(t_k) - y_k^p\Big| + L_2 h^\alpha \Big|y(t_{k-M}) - y_{k-M}\Big| + \Big|R_k\Big|.
\end{aligned} \tag{22}$$

Using the result (15) and induction hypothesis in Eqn. (22), we get

$$\Big|y(t_k) - y_k^c\Big| \le \lambda_1 \Big|y(t_k) - y_k^p\Big| + \lambda_2 h^{r+\alpha}, \tag{23}$$

where $\lambda_1 = \Big(1 + L_1 h^\alpha + L_1^2 h^{2\alpha}\Big)$ and $\lambda_2 = C'\Big(1 + L_2 h^\alpha\Big)$. Further we find a bound for $\Big|y(t_k) - y_k^p\Big|$,

$$\begin{aligned}
y(t_k) - y_k^p &= A_r \frac{k^{-\alpha}}{\Gamma(1-\alpha)}\Big(y(t_0) - \phi(0)\Big) - A_r \sum_{j=0}^{k-1} \omega_{k-j}^r \Big(y(t_j) - y_j\Big) \\
&= A_r \frac{k^{-\alpha}}{\Gamma(1-\alpha)}\Big(y(t_0) - \phi(0)\Big) - A_r \omega_k^r \Big(y(t_0) - \phi(0)\Big) \\
&\quad - A_r \sum_{j=1}^{k-1} \omega_{k-j}^r \Big(y(t_j) - y_j\Big).
\end{aligned} \tag{24}$$

In view of lemma (4.3), there exists positive constant K_r such that $|\omega_j^r| \le K_r j^{-\alpha-1} \le K_r j^{\alpha-1}$ for $j > 0$ and $r = 1, 2, \dots, 6$, we get

$$\begin{aligned}
\Big|y(t_k) - y_k^p\Big| &\le \frac{k^{-\alpha}}{\Gamma(1-\alpha)}\Big|y(t_0) - \phi(0)\Big| + K_r k^{\alpha-1}\Big|y(t_0) - \phi(0)\Big| \\
&\quad + K_r \sum_{j=1}^{k-1} (k-j)^{\alpha-1}\Big|y(t_j) - y_j\Big|, \\
&\le \Big(\frac{k^{-\alpha}}{\Gamma(1-\alpha)} + K_r k^{\alpha-1}\Big)\Big|y(t_0) - \phi(0)\Big| + K_r \sum_{j=1}^{k-1} (k-j)^{\alpha-1}\Big|y(t_j) - y_j\Big|.
\end{aligned} \tag{25}$$

Since $\left|y(t_0) - \phi(0)\right| = 0$, we have

$$\left|y(t_k) - y_k^p\right| \leq K_r \sum_{j=1}^{k-1} (k-j)^{\alpha-1} \left|y(t_j) - y_j\right|. \tag{26}$$

In view of Eqn. (23) and Eqn. (26), we get

$$\left|y(t_k) - y_k^c\right| \leq \lambda_1 K_r \sum_{j=1}^{k-1} (k-j)^{\alpha-1} \left|y(t_j) - y_j\right| + \lambda_2 h^{r+\alpha}. \tag{27}$$

In view of lemma (4.2), we get

$$\left|y(t_k) - y_k^c\right| \leq C h^{r+\alpha}, \tag{28}$$

where C is a constant free from h and k. Thus, the error estimate for FBD-PCMr is of $O(h^{r+\alpha})$, where $r = 1, 2, \ldots, 6$.

Remark 4.1 The error estimate for FBD-PCMr is of $O(h^{r+\alpha})$, $1 \leq r \leq 6$. It may be noted that the proposed methods give better bounds than the existing ones [1,31,50] for $r \geq 2$.

Remark 4.2 In fractional Adams method (FAM), the two sets of parameters $a_{j,n}$ and $b_{j,n}$ $(0 \leq j \leq n-1)$ (cf. eq. (12) and (14) in [30]) for nth iteration are involved in the predictor-corrector approach, whereas 3-term new predictor-corrector method (NPCM) involves only one set of parameter $a_{j,n}$ (cf. Section (4.1) in [38]) for the computations of solution. Therefore, the number of arithmetic operations in FAM is double that of NPCM. Hence NPCM takes almost 50% time of the time taken by FAM. It can be observed from the table (4), (5) and (6) for different numbers of grid points that execution time is almost half that of FAM. Also, in the case of the newly proposed family of methods, there is only one set of parameters (or weights) ω_j^r (where $1 \leq r \leq 6$ and $0 \leq j \leq n-1$ for nth iteration) (cf. [45]) as compared to FAM and follows a recursive relation. The terms involved in ω_j^r are simpler as compared to the terms involved in FAM/NPCM and the operations count in the newly proposed method is much less as compared to FAM and NPCM. Thus the execution time gets reduced. Also, our recent papers on new numerical methods [see Table (2) and (6) in section (6) in Chapter 1 [50]) have clearly shown the execution time comparison between newly developed methods and existing one. The intention of finding new time-efficient numerical methods is that they are very useful especially when one wants to do simulations pertaining to fractional order dynamical systems and study phenomena such as chaos.

5 Illustrative Examples

In this section, we solve various non-trivial systems of FDDEs using FBD-PCMr. We also compare FBD-PCMr solutions with exact, FAM and NPCM. Further, we compare the execution time taken by these methods.

Example 5.1 Consider the following fractional delay differential equation:

$$\left.\begin{aligned} D_t^\alpha y(t) + y^4(t) &= \frac{\Gamma(3)}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha} + y(t) \\ &\quad - y_\tau(t) - 2t\tau + \tau^2 + \tau + (t^2 - t)^4, \\ y(t) &= 0, \quad -\tau \leq t \leq 0. \end{aligned}\right\} \tag{29}$$

The exact solution of (29) is $(t^2 - t)$. We solve the system (29) for $\alpha = 0.50, h = 0.001, \tau = 0.01$ using FBD-PCMr ($r = 1, 2, 3, 4$). Exact solutions and approximate solutions obtained by FBD-PCMr, FAM, and NPCM are plotted in Fig. 1. It is clear that these solutions are in very good agreement. Further, it is also observed that for very small values of α, the FBD-PCMr converge while FAM and NPCM do not converge (see Table 1). Also, we notice the discontinuity of the derivative at $t = 0$ as $0 = D_t^\alpha y(0) \neq D_t^\alpha \phi(0) = \tau^2 + \tau$, where $\tau > 0$.

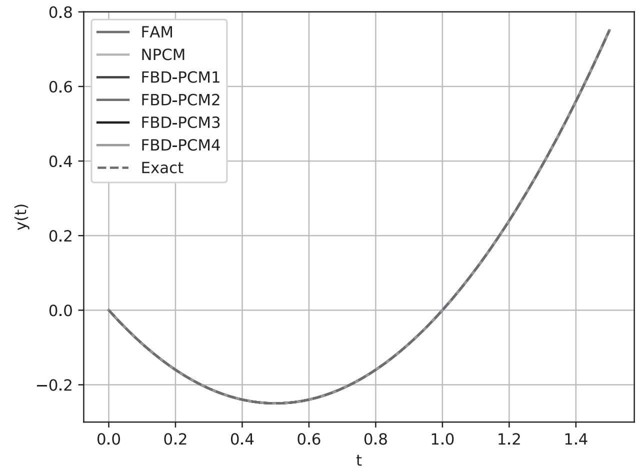


Fig. 1 Solutions of the IVP (29)

Table 1 Solutions of (29) when $\tau = 1.0,\ t = 1.0$

Step size	α	FAM	NPCM	FBD-PCM2	FBD-PCM3	Exact
10^{-2}	0.01	diverges	diverges	0.047695933	0.048269213	0
	0.001	diverges	diverges	0.002533904	0.002560065	0
10^{-3}	0.01	diverges	diverges	0.053043766	0.053458057	0
	0.001	diverges	diverges	0.002698425	0.002721550	0

Table 2 Absolute errors in numerical solutions of Eqn. (30)

t	FAM	NPCM	FBD-PCM2
0.2	2.1968×10^{-5}	2.1005×10^{-5}	1.9429×10^{-5}
0.4	1.5571×10^{-5}	1.4326×10^{-5}	1.0606×10^{-5}
0.6	1.3550×10^{-5}	1.2037×10^{-5}	5.4097×10^{-6}
0.8	1.3538×10^{-5}	1.1662×10^{-5}	9.7216×10^{-7}
1.0	1.5224×10^{-5}	1.2789×10^{-5}	3.9765×10^{-6}

Example 5.2 Consider the following fractional delay differential equation:

$$\left.\begin{aligned} D_t^{\alpha} y(t) &= \frac{\Gamma(2.5)}{\Gamma(2.5-\alpha)} t^{3/2-\alpha} + y_{\tau}^2(t) - y(t) - (t-\tau)^3 + t^{3/2}, \\ y(t) &= 0, \quad -\tau \le t \le 0. \end{aligned}\right\} \tag{30}$$

The exact solution of (30) is $y(t) = t^{3/2}$. We employ FAM, NPCM and FBD-PCM2 to solve the system (30) numerically when $\alpha = 0.75$, $h = 0.001$, and $\tau = 0.1$. Absolute and relative errors are calculated in Tables 2-3. It is noted that FBD-PCM2 gives more accurate solutions as compared to the existing methods such as FAM and NPCM. Further, we measure the execution time taken by FAM, NPCM, and FBD-PCM2 which results in 27.7942, 14.5396, and 1.6673 seconds respectively. Hence FBD-PCM2 takes very little execution time as compared to FAM and NPCM. Further, we observe the discontinuity of $D_t^{\alpha} y(t)$ at $t = 0$ (as $0 = D_t^{\alpha} y(0) \ne D_t^{\alpha} \phi(0) = \tau^3,\ \tau > 0$).

Example 5.3 Consider the following fractional-order delay differential equation

$$\left.\begin{aligned} D_t^{\alpha} y(t) &= k y(t-\tau), \quad 0 < \alpha < 1 \\ y(0) &= E_{\alpha}(-k(-\tau)^{\alpha}), \quad -\tau \le t \le 0. \end{aligned}\right\} \tag{31}$$

The exact solution of (31) is $y(t) = E_{\alpha}(-k(t-\tau)^{\alpha})$. For $k = 1$, $\alpha = 0.75$, $h = 0.001$ and $\tau = 0.1$, we apply FAM, NPCM, FBD-PCM2 and FBD-PCM3 to solve the system (31) numerically. Additionally, we calculate the absolute and relative errors in the solutions of (31) which are given in Tables 4-5. It is observed that FBD-PCM2 and FBD-PCM3 give more accurate results than FAM and NPCM.

Table 3 Relative errors in numerical solutions of Eqn. (30)

t	FAM	NPCM	FBD-PCM2
0.2	2.4560×10^{-4}	2.3484×10^{-4}	2.1722×10^{-4}
0.4	6.1549×10^{-5}	5.6631×10^{-5}	4.1925×10^{-5}
0.6	2.9155×10^{-5}	2.5899×10^{-5}	1.1640×10^{-6}
0.8	1.8919×10^{-5}	1.6298×10^{-5}	1.3586×10^{-6}
1.0	1.5224×10^{-5}	1.2789×10^{-5}	3.9765×10^{-6}

Table 4 Absolute errors in numerical solutions of Eqn. (31)

t	FAM	NPCM	FBD-PCM2	FBD-PCM3
0.2	0.118432042	0.118432042	0.047534848	0.009789798
0.4	0.099819692	0.099819692	0.057141448	0.034359415
0.6	0.069280011	0.069280011	0.039172428	0.023094789
0.8	0.04086963	0.04086963	0.018210838	0.006108765
1.0	0.017317701	0.017317701	0.001703553	0.011002541

Table 5 Relative errors in numerical solutions of Eqn. (31)

t	FAM	NPCM	FBD-PCM2	FBD-PCM3
0.2	0.131834346	0.131834346	0.052914106	0.010897656
0.4	0.136382808	0.136382808	0.078071882	0.046944981
0.6	0.114741833	0.114741833	0.064877533	0.038249683
0.8	0.081064233	0.081064233	0.036120896	0.012116635
1.0	0.04066076	0.04066076	0.003999825	0.025833202

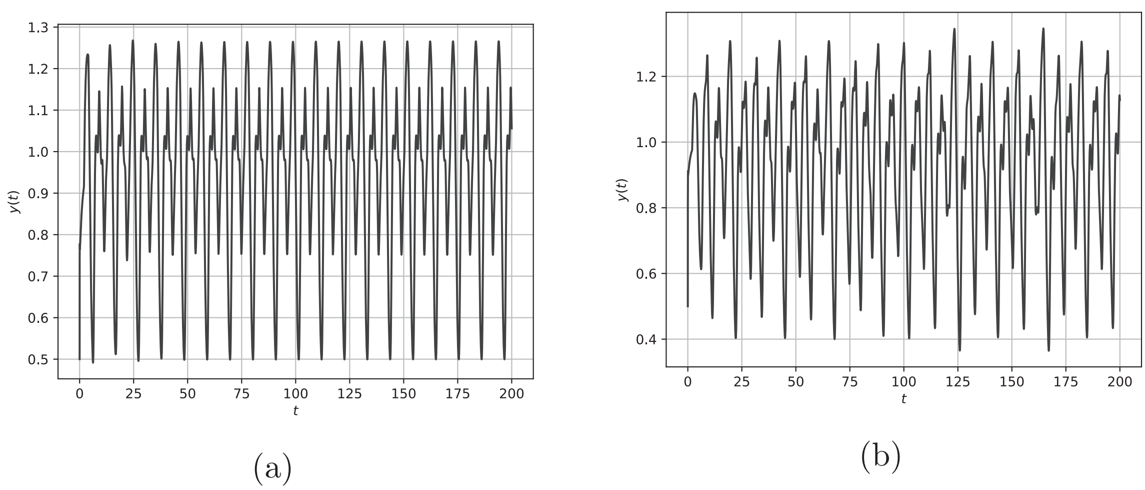


Fig. 2 FBD-PCM3 solutions of (32) for (a) $\alpha = 0.84$ (b) $\alpha = 0.98,\ \tau = 2$

Example 5.4 Consider the following fractional version of Mackey-Glass delay differential equation [51]:

$$\left.\begin{aligned} D_t^{\alpha} y(t) &= \frac{a y_\tau(t)}{1 + y_\tau^k(t)} - b y(t), \quad 0 < \alpha < 1, \\ y(t) &= 0.5, \quad t \leq 0. \end{aligned}\right\} \tag{32}$$

We perform the numerical simulations for the system (32) for $a = 2,\ b = 1,\ \tau = 2.0,\ k = 9.65$ using FBD-PCMr ($r = 1, 2, 3$). For $\alpha = 0.84,\ 0.98$ and $h = 0.01$, FBD-PCM3 solutions are plotted in Fig. 2 whereas the phase portraits are depicted in Fig. 3. It is observed that Fig. 3 (a) shows the stable behavior whereas the Fig. 3 (b) is chaotic. Moreover, it is noticed that the numerical solutions obtained by FBD-PCMr are in very

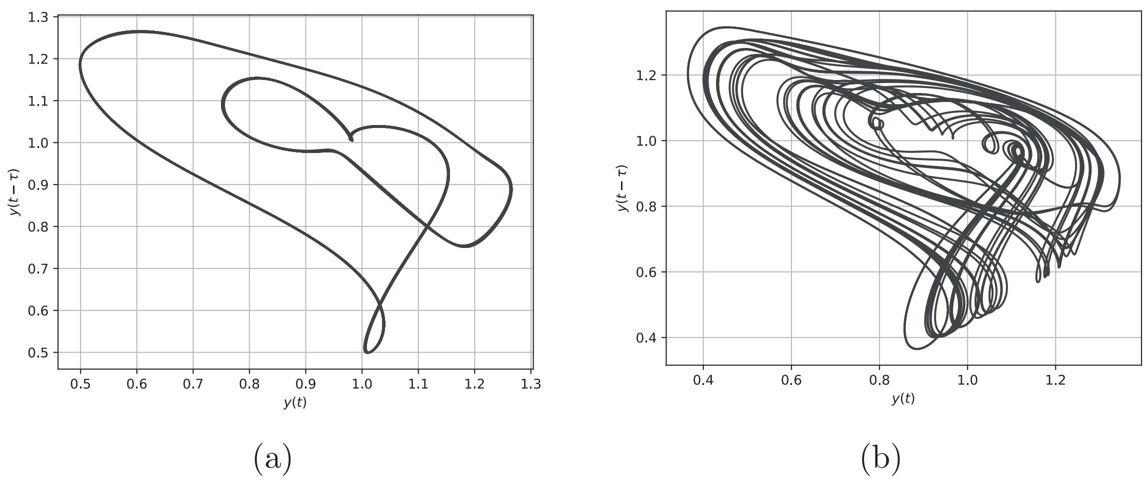


Fig. 3 Phase portraits of (32) for (a) $\alpha = 0.84$, stable orbit (b) $\alpha = 0.98$, chaotic behavior, $\tau = 2$

Table 6 Execution time (in seconds) of (32); Itr: Iterations

Itr	α	FAM	NPCM	FBD-PCM1	FBD-PCM2	FBD-PCM3
2500	0.84	22.2484	12.0660	2.1557	2.2437	2.4406
	0.98	22.7179	12.0416	2.1599	2.2379	2.6984
5000	0.84	82.9255	49.7488	9.0338	9.0208	10.7248
	0.98	88.4409	48.6216	9.2197	9.4875	9.7624
7500	0.84	213.8403	111.3022	19.3286	19.7739	20.5842
	0.98	203.7307	108.0494	21.6629	21.4026	19.9927
10000	0.84	384.4382	190.7846	37.1363	36.5477	37.4468
	0.98	362.8194	193.3436	35.5670	35.2556	37.2983

good agreement with solutions obtained by FAM and NPCM [1]. FBD-PCMr takes very less execution time as compared to NPCM and FAM (*cf.* Table 6).

Example 5.5 Consider the following FDDE [40,52]:

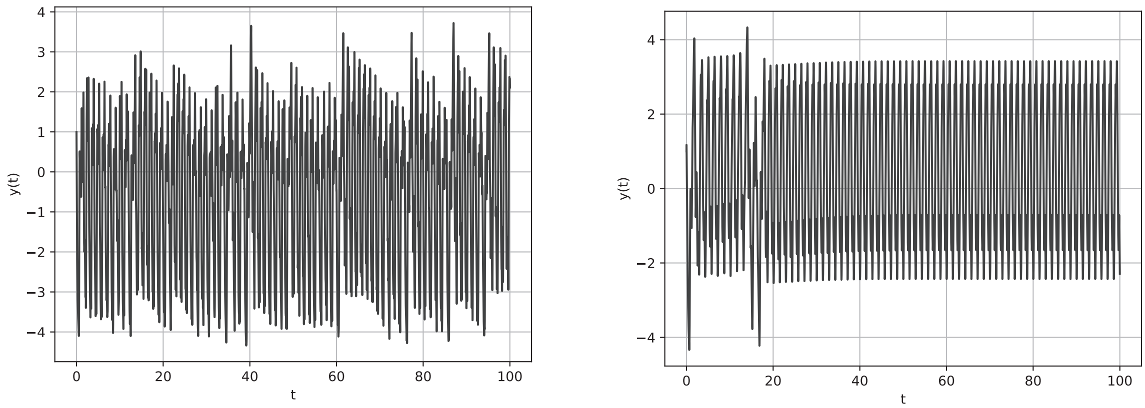
$$\left.\begin{aligned} D_t^{\alpha} y(t) &= 12y_{\tau}(t) + 2.4y(t) - 0.07\Big(y(t) - 2.34y_{\tau}(t)\Big)^3, \\ y(t) &= 1, \quad t \le 0, \quad 0 < \alpha < 1. \end{aligned}\right\} \tag{33}$$

We solve the system (33) using FBD-PCMr ($r = 1, 2, 3$) by taking $\tau = 0.5$ and $h = 0.01$. Solutions of (33) for $\tau = 0.5$, and $\alpha = 0.7,\ 0.9$ are plotted in Fig. 4. Further, its phase portraits for $\alpha = 0.7$ and 0.9 are drawn in Fig. 5. It is also observed that this equation shows chaotic behavior for $\alpha = 0.7$ and stable orbits for $\alpha = 0.9$. In Table 7, execution time (in seconds) is compared. Further, it is noticed that FBD-PCMr takes very less time as compared to FAM and NPCM.

Example 5.6 Consider the following fractional order Uçar system [53,54]:

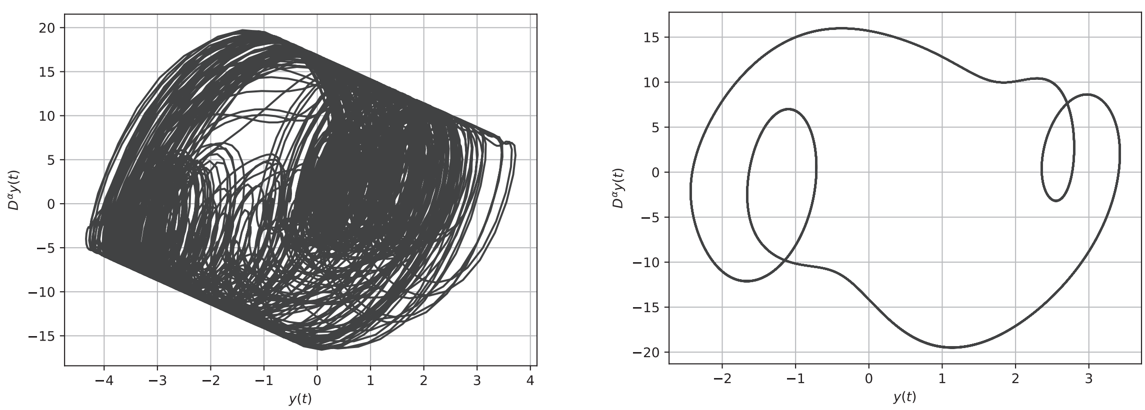
$$\left.\begin{aligned} D_t^{\alpha} y(t) &= y_{\tau}(t)\Big(a - by_{\tau}^2(t)\Big), \quad 0 < \alpha < 1, \\ y(t) &= \phi(t), \quad t \le 0. \end{aligned}\right\} \tag{34}$$

We take $a = 1,\ b = 1, h = 0.01, \phi(t) = 0.5$ and perform various numerical simulations using FBD-PCM2 for the system (34). Chaotic trajectory t versus $y(t)$, chaotic phase portraits $y(t)$ versus $y(t-\tau)$ and $y(t)$ versus $D^{\alpha} y(t)$ are depicted for $\alpha = 0.99$, $\tau = 1.7$ in Fig. 6, for $\alpha = 0.97$, $\tau = 1.7$ in Fig. 7 and for $\alpha = 0.90,\ \tau = 2$ in Fig. 8. Moreover, the phase-portraits for $\alpha = 0.50,\ \tau = 2$ and $\alpha = 0.40,\ \tau = 4.5$ and the limit cycle are represented in Fig. 9. We obtain the limit cycles for $\alpha = 0.97,\ \tau = 1.3$ and $\alpha = 0.99,\ \tau = 1.5$ and stable trajectory for $\alpha = 0.99,\ \tau = 0.7$ (cf. Fig. 10), whereas the stable trajectory for $\alpha = 0.90,\ \tau = 0.8$ (cf. Fig. 11). We noticed that the obtained results are in support as proposed in [54].



(a) (b)

Fig. 4 FBD-PCMr solutions of (33) for (a) $\alpha = 0.7$ (b) $\alpha = 0.9,\ \tau = 0.5$



(a) (b)

Fig. 5 Chaotic behavior and stable orbit of (33) for (a) $\alpha = 0.7$ (b) $\alpha = 0.9,\ \tau = 0.5$

Table 7 Execution time (in seconds) of (33); Itr: Iterations

Itr	α	FAM	NPCM	FBD-PCM1	FBD-PCM2	FBD-PCM3
2500	0.70	28.39	15.29	2.74	2.86	2.88
	0.90	25.11	14.24	2.18	2.30	2.31
5000	0.70	107.91	58.41	9.15	9.60	9.66
	0.90	104.45	55.41	10.84	10.89	11.01
7500	0.70	251.84	133.44	22.23	22.85	20.00
	0.90	229.39	131.82	23.21	23.72	23.88
10000	0.70	449.21	220.33	35.24	35.96	36.47
	0.90	404.90	216.23	35.44	36.14	36.66

Example 5.7 Consider the following fractional version of the system of FDDEs [55]:

$$\left.\begin{aligned} &D_t^{\alpha}x(t) = 35\Big(y(t) - x(t)\Big),\\ &D_t^{\alpha}y(t) = -7x(t) - x(t)\Big(0.2z(t) + z_{\tau}(t) - 5\sin\big(z_{\tau}(t)\big)\Big) + 28y(t),\\ &D_t^{\alpha}z(t) = x(t)y(t) - 3z(t),\\ &\quad x(t) = 1,\ y(t) = 1,\ z(t) = 30, \quad -\tau \le t \le 0. \end{aligned}\right\} \tag{35}$$

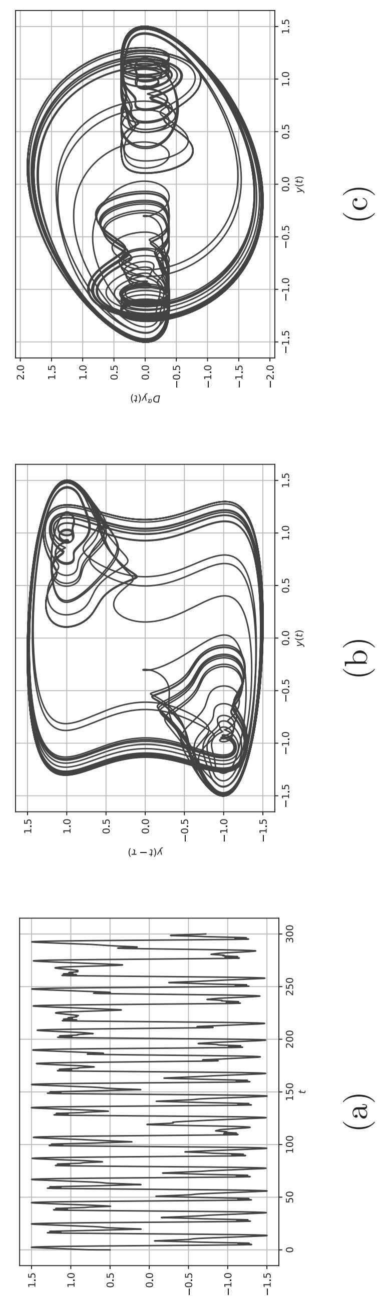
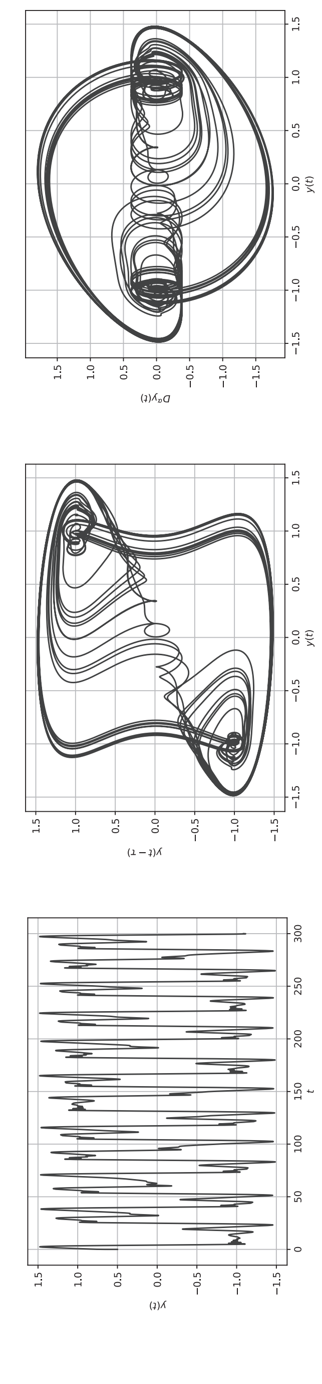


Fig. 6 For $\alpha = 0.99,\ \tau = 1.7$, chaotic (a) trajectory (t versus $y(t)$) (b) phase-portrait ($y(t)$ versus $y(t-\tau)$) (c) phase-portrait ($y(t)$ versus $D^{\alpha}y(t)$) of (34)



(a) (b) (c)

Fig. 7 For $\alpha = 0.97,\ \tau = 1.7$, chaotic (a) trajectory (t versus $y(t)$) (b) phase-portrait ($y(t)$ versus $y(t-\tau)$) (c) phase-portrait ($y(t)$ versus $D^{\alpha}\, y(t)$) of (34)

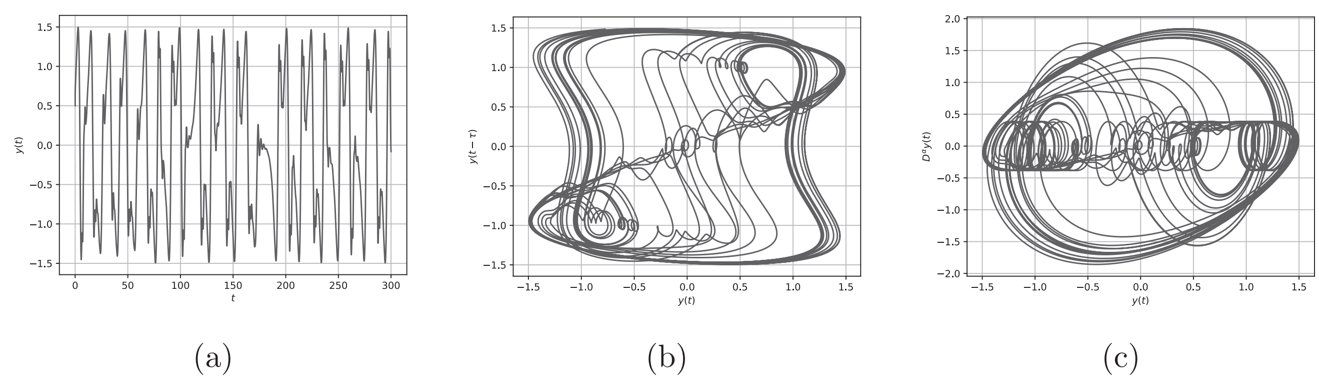


Fig. 8 For $\alpha = 0.90,\ \tau = 2$, chaotic (a) trajectory (t versus $y(t)$) (b) phase-portrait ($y(t)$ versus $y(t-\tau)$) (c) phase-portrait ($y(t)$ versus $D^{\alpha}y(t)$) of (34)

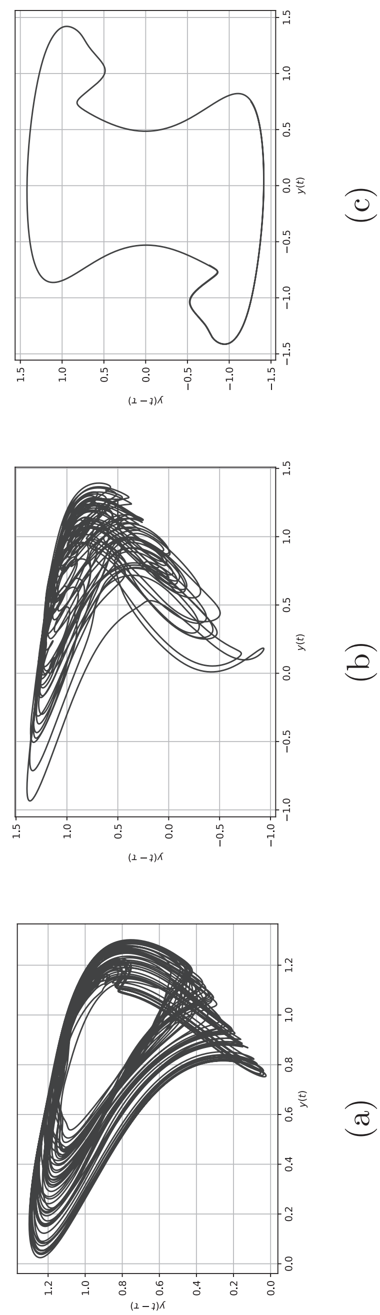


Fig. 9 Phase-portraits for (a) $\alpha = 0.50,\ \tau = 2$ (b) $\alpha = 0.40,\ \tau = 4.5$ (c) limit-cycle, $\alpha = 0.90,\ \tau = 1.9$ of (34)

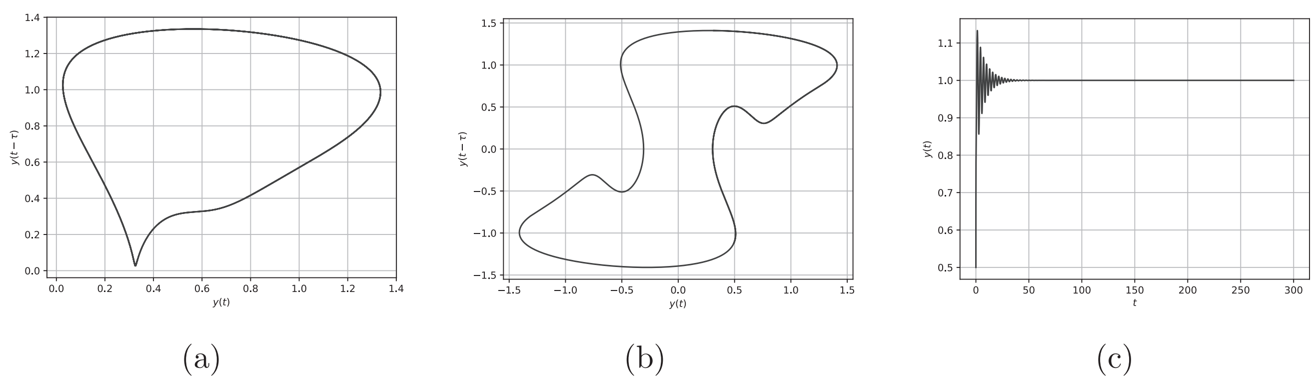


Fig. 10 limit cycles for (a) $\alpha = 0.97,\ \tau = 1.3$ (b) $\alpha = 0.99,\ \tau = 1.5$ (c) and stable trajectory, $\alpha = 0.99,\ \tau = 0.7$ of (34)

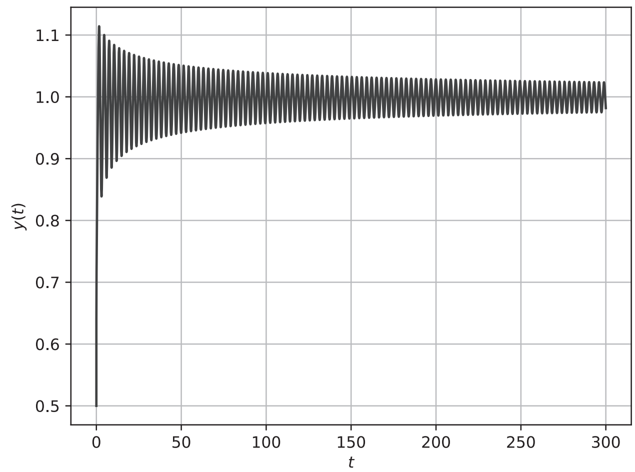


Fig. 11 Stable trajectory for $\alpha = 0.90,\ \tau = 0.8$ of (34)

We solve the fractional order system of DDEs (35) numerically using FBD-PCMr ($r = 2, 3$). The phase portraits of the system (35) in xyz space xz, yz and xy planes, for $\alpha = 0.98, 0.94,\ \tau = 0.1$ and $h = 0.005$ are plotted in Figs. 12, 13 and 14 (b). It is noted that the system (35) shows chaotic behavior for $\alpha = 0.98, 0.94$ and stable orbit when $\alpha = 0.70$ Fig. 14(c). Further, we compare execution time taken by FAM, 3-term NPCM, and FBD-PCMr in Table 8. We observe that FBD-PCMr is time time-efficient method as compared to FAM and NPCM.

6 Conclusions

In this paper, a new family of six new predictor-corrector methods (FBD-PCMr) is proposed for solving nonlinear FDDEs along with error analysis. The error estimates of FBD-PCMr are $O(h^{r+\alpha}),\ r = 1, 2, \ldots, 6$. The proposed methods have a higher order of accuracy. Various nontrivial examples have been solved to demonstrate the utility and efficiency of the methods. The proposed methods are accurate and time-efficient. These methods take only 5 to 10 percent of the time required for other methods such as the fractional Adams method (FAM). Thus, the execution time is reduced drastically. Further, FBD-PCMr converge even for very small values of α, while FAM and NPCM fail.

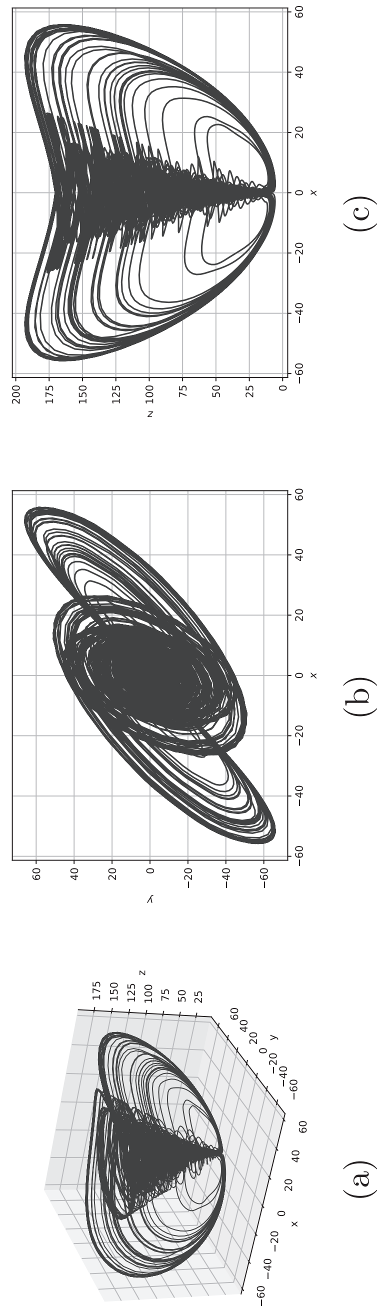


Fig. 12 For $\alpha = 0.98$, $\tau = 0.1$, phase-portraits of (35) in (a) xyz space (b) xy plane (c) xz plane

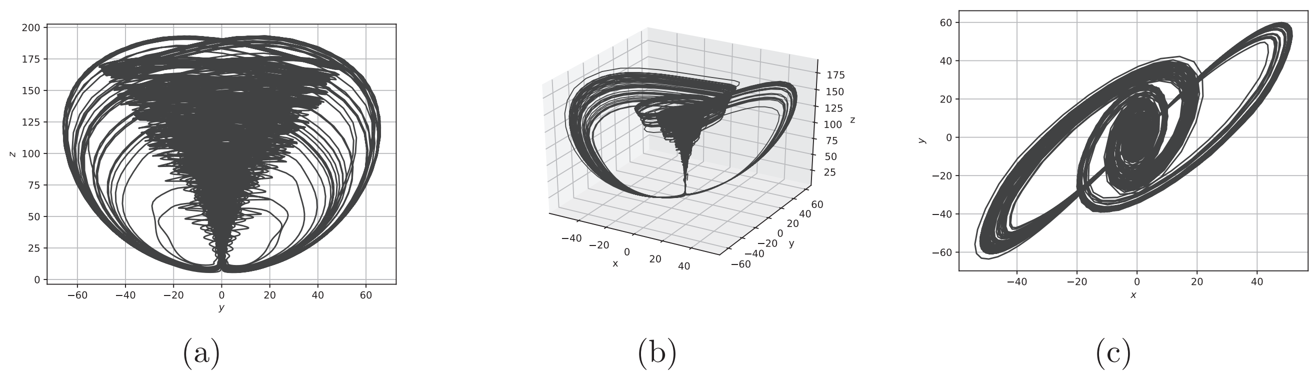


Fig. 13 Phase-portraits of (35), $\tau = 0.1$ (a) $\alpha = 0.98$, yz plane (b) $\alpha = 0.94$, xyz plane (c) $\alpha = 0.94$, xy plane

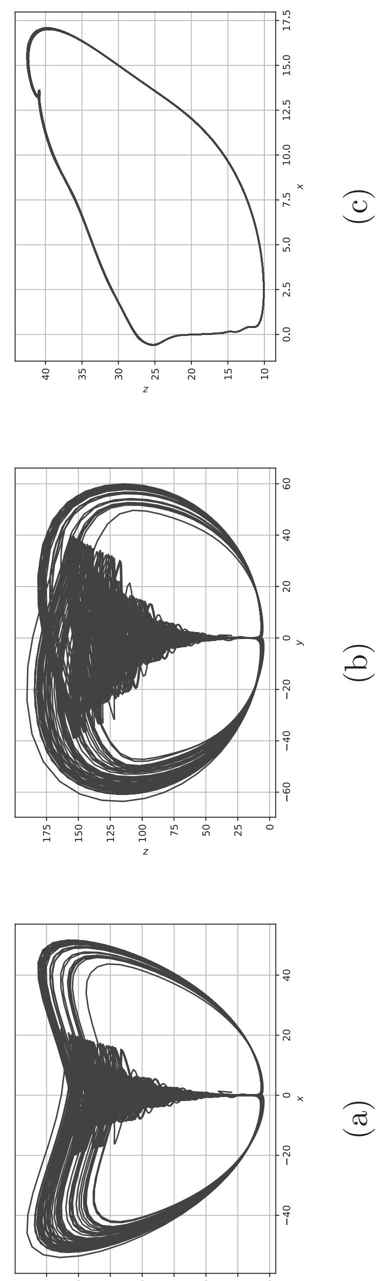


Fig. 14 Phase-portraits of (35), $\tau = 0.1$ (a) $\alpha = 0.94$, xz plane (b) $\alpha = 0.94$, yz plane (c) stable orbit $\alpha = 0.70$, xz plane

Table 8 For $\tau = 0.1$, execution time (in seconds) of (35)

Iterations	α	FAM	NPCM	FBD-PCM2	FBD-PCM3
5000	0.70	391.24	203.04	28.25	28.54
	0.98	393.35	212.51	27.77	28.07
7500	0.70	831.40	442.70	61.95	62.10
	0.98	969.21	476.44	61.69	65.89
10000	0.70	1629.23	871.07	114.47	114.53
	0.98	1714.46	873.01	108.93	110.57
15000	0.70	3494.75	1953.64	243.14	245.52
	0.98	3592.14	1960.49	245.99	249.95

Declarations

Conflict of Interest The authors declare no conflict of interest.

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