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ORIGINAL RESEARCH

On the regularity for the simplified Ericksen-Leslie system in critical spaces

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Abstract In this paper, we present some new regularity criteria for suitable weak solutions to the 3D simplified Ericksen-Leslie system in critical spaces. Namely, it is shown that suitable weak solutions to this system are regular if the velocity field belongs to $L^{\infty}(-1,0;BMO^{-1}(\mathbb{R}^3))$ or the velocity field and direct field are in $L^{\infty}(-1,0;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$. Our results improve previous corresponding criteria due to Liu, Min and Zhang [J. Differential Equations **267**, 2643–2670 (2019)] and Men, Wang and Wu [Math. Methods Appl. Sci. **41**, 3672–3683 (2018)].

Keywords Simplified Ericksen-Leslie system · Local regularity · Suitable weak solutions · Critical spaces

1 Introduction

The flow of nematic liquid crystals can be treated as slow moving particles where the fluid velocity and the alignment of the particles influence each other. The hydrodynamic theory of liquid crystals was established by Ericksen [4,5] and Leslie [10,11] in the 1960's. As the general Ericksen-Leslie system is very complicated (see e.g., [23]), motivated by working on the harmonic heat flow, Lin and Liu [13] simplify the original equations. In this paper, we investigate the following simplified version for nematic liquid crystal flow in three-dimensions:

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = -\lambda \mathrm{div}(\nabla\mathbf{d}\odot\nabla\mathbf{d}), \\ \mathrm{div}\mathbf{u} = 0, \\ \partial_t \mathbf{d} + (\mathbf{u}\cdot\nabla\mathbf{d}) = \mu(\Delta\mathbf{d} - f(\mathbf{d})), \\ (\mathbf{u}(x,0),\mathbf{d}(x,0)) = (\mathbf{u}_0(x),\mathbf{d}_0(x)) \ \text{ with } \ \mathrm{div}\mathbf{u}_0 = 0. \end{cases} \tag{1.1}$$

Here, $\mathbf{u}:\mathbb{R}^3\times(-1,0)\to\mathbb{R}^3$ represents the field velocity, $P:\mathbb{R}^3\times(-1,0)\to\mathbb{R}$ is the field pressure, and $\mathbf{d}:\mathbb{R}^3\times(-1.0)\to\mathbb{R}^3$ is the director field denotes the alignment of the molecules. $f(\mathbf{d})=\nabla F(\mathbf{d})$ is the Ginzburg-Landau approximation function, where $F(\mathbf{d})$ is a scale function of $\mathbf{d}$ and is given by $F(\mathbf{d})=\frac{1}{\varepsilon^2}(|\mathbf{d}|^2-1)^2$, $\varepsilon>0$. As ε goes to zero, we see that $\mathbf{d}$ becomes an unit vector field. ν,λ,μ are positive constants that stand for the viscosity, the competition between kinetic energy and potential energy, and microscope elastic relaxation time for the molecular orientation field. Throughout the paper, due to the concrete values of the viscosity constants do not play a special role in our discussion, we assume that they are all equal to one for simplicity. The symbol

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$\nabla \mathbf{d} \odot \nabla \mathbf{d}$, which exhibits the property of the anisotropy of the material, represent the 3×3 matric and

$$(\nabla \mathbf{d} \odot \nabla \mathbf{d})_{ij} = \sum_{k=1}^{3} \partial_{x_i} \mathbf{d}_k \partial_{x_j} \mathbf{d}_k \ \text{ for } \ i, j = 1, 2, 3.$$

System (1.1) is a simplified version of the original Ericksen-Leslie model of variable length for the flow of liquid crystals. Lin and Liu [13] proved the existence of global weak solution to the system (1.1), moreover, they discuss some stability properties and time asymptotic of the system. Later, in [15], the authors studied the partial regularity theory for the system, and proved the following partial regularity result: there exists some ε_0 and $r < 1$ such that if $(\mathbf{u}, \mathbf{d}, P)$ is a suitable weak solution and satisfies

$$\frac{1}{r^2} \int_{Q_r(z_0)} (|\mathbf{u}|^3 + |\nabla \mathbf{d}|^3 + |P|^{\frac{3}{2}}) \mathrm{d}x \mathrm{d}t < \varepsilon_0, \tag{1.2}$$

then $z_0 = (x_0, t_0)$ is a regular point, where $Q_r(z_0) = (t_0 - r^2, t_0) \times B_r(x_0)$ and $B_r(x_0)$ is a ball of radius r centered at x_0. In particular, using this result implies that the one-dimensional parabolic Hausdorff measure of potential singular set of the suitable weak solutions is zero, which generalized the well-known result in [1] for Navier-Stokes system.

It is noted that the simplified Ericksen-Leslie system (1.1) reduces to the well-known Navier-Stokes equations when the director field $\mathbf{d} = (0, 0, 1)$. The mathematical theory of the Navier-Stokes equations extends at least as far back as Leray in 1930s. Pioneering works of Leray [12] and Hopf [8] proved the global existence of weak solution for Navier-Stokes equations in two or three dimensional spaces and the weak solution of two-dimensional Navier-Stokes equations is regular. In three dimensions, however, the question of regularity and uniqueness of weak solutions is an outstanding open problem in mathematical fluid mechanics. The sufficient regularity conditions for the three-dimensional Navier-Stokes equations originated in Serrin's [21], and it is established that if the weak solution $\mathbf{u}$ satisfies

$$\mathbf{u} \in L^p(0, T; L^q(\mathbb{R}^3)) \ \text{ with } \ \frac{2}{p} + \frac{3}{q} < 1, \ q > 3, \tag{1.3}$$

then $\mathbf{u}$ is a regular solution. The limit case $(p, q) = (\infty, 3)$ has been investigated later by Escauriaza, Seregin and V. Šverák [6]. i.e., if a weak solution $\mathbf{u} \in L^\infty(0, T; L^3(\mathbb{R}^3))$ additionally, then $\mathbf{u}$ is a regular solution. This result was improved by Tao [22]. The authors proved that as the solution $\mathbf{u}$ approaches a finite blowup time T, the critical norm $\|\mathbf{u}(t)\|_{L^3(\mathbb{R}^3)}$ must blow up at a rate $(\mathrm{loglog}\log \frac{1}{T-t})^c$ with some absolute constant $c > 0$. It is worth pointing out that the space $L^3(\mathbb{R}^3)$ is invariant under the nature scaling of the Navier-Stokes equations. The inclusion relations of scaling invariant spaces as follows:

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow L^{3,q}(\mathbb{R}^3) \hookrightarrow \dot{B}_{p,q}^{-1+\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow BMO^{-1}(\mathbb{R}^3) \hookrightarrow \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3) \ (3 < p, q < \infty). \tag{1.4}$$

There are many studies of well-posedness and the regularity of weak solutions in scaling invariant spaces to the Navier-Stokes equations (see [2,3,7,20,24,25] for example and references therein).

We observe that the simplified Ericksen-Leslie system (1.1) has the following scaling property. If $\mathbf{u}(x, t)$, $\mathbf{d}(x, t)$ and $P(x, t)$ solve (1.1), then for each $\lambda > 0$,

$$\mathbf{u}_\lambda = \lambda \mathbf{u}(\lambda x, \lambda^2 t), \mathbf{d}_\lambda = \mathbf{d}(\lambda x, \lambda^2 t), P_\lambda = \lambda^2 P(\lambda x, \lambda^2 t) \tag{1.5}$$

also solve (1.1) with $f(\mathbf{d})$ being replaced by $f_\lambda(\mathbf{d}_\lambda) = \lambda^2 f(\mathbf{d}_\lambda)$. For system (1.1), non-endpoint regularity criteria see [9,14,16,17,26]. [18,19] give the endpoint regularity criteria. In [19], the authors proved that if a weak solution $(\mathbf{u}, \mathbf{d})$ for system (1.1) satisfy the following assumption:

$$\mathbf{u} \in L^\infty((-1, 0), L^3(\mathbb{R}^3)),$$

then the solution $(\mathbf{u}, \mathbf{d})$ is smooth on $\mathbb{R}^3 \times (-1, 0)$.

The main goal of this paper is to derive the regularity criteria of suitable weak solutions to the simplified Ericksen-Leslie system in scaling invariant space BMO^{-1} and $\dot{B}_{\infty,\infty}^{-1}$. For convenience, we now introduce the definition of the suitable weak solution of the system (1.1) as follows.



Definition 1.1 *(Weak solution)* A pair $(\mathbf{u}, \mathbf{d})$ is a weak solution of (1.1), if $\mathbf{u} \in L^\infty(-1,0;L^2(\mathbb{R}^3)) \cap L^2(-1,0;H^1(\mathbb{R}^3))$ and $\mathbf{d} \in L^\infty(-1,0;H^1(\mathbb{R}^3)) \cap L^2(-1,0;H^2(\mathbb{R}^3))$, and for any test functions $\phi \in C_0^\infty(\Omega\times(0,T],\mathbb{R}^3)$ and $\psi \in C_0^\infty(\Omega\times(0,T],\mathbb{R}^3)$, with $\mathrm{div}\psi = 0$ in $\Omega_T := \mathbb{R}^3\times(-1,0]$, it holds that

$$\int_{\Omega_T} -\mathbf{d}\cdot\partial_t\phi - \Delta\mathbf{d}\cdot\phi - \mathbf{d}\cdot\mathbf{u}\otimes\nabla\phi \mathrm{d}x\mathrm{d}t = \int_{\Omega_T} -f(\mathbf{d})\cdot\phi \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^3}\mathbf{d}_0(x)\cdot\phi(x,0)\mathrm{d}x, \tag{1.6}$$

and

$$\int_{\Omega_T} -\mathbf{u}\cdot\partial_t\psi + \nabla\mathbf{u}\cdot\nabla\psi - \mathbf{u}\otimes\mathbf{u} : \nabla\psi \mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\nabla\mathbf{d}\odot\nabla\mathbf{d} : \nabla\phi \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^3}\mathbf{u}_0(x)\cdot\psi(x,0)\mathrm{d}x. \tag{1.7}$$

Definition 1.2 *(Suitable weak solution)* A weak solution of system (1.1) is a suitable weak solution, if, in addition, for any $0 \le \eta \in C_0^\infty(\mathbb{R}^3\times[-1,0))$, weak solution $(\mathbf{u},\mathbf{d},P)$ satisfies the following local energy inequality:

$$\begin{aligned}
&\int_{\mathbb{R}^3}(|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)\eta\mathrm{d}x + 2\int_{\Omega_T}(|\nabla\mathbf{u}|^2+|\nabla^2\mathbf{d}|^2)\eta\mathrm{d}x\mathrm{d}t\\
&\le \int_{\Omega_T}(|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)(\partial_t\eta+\Delta\eta)\mathrm{d}x\mathrm{d}t + \int_{\Omega_T}\big(|\mathbf{u}|^2+|\nabla\mathbf{d}|^2+2P\big)\mathbf{u}\cdot\nabla\phi\mathrm{d}x\mathrm{d}t\\
&\quad + \int_{\Omega_T}((\mathbf{u}\cdot\nabla)\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\eta\mathrm{d}x\mathrm{d}t - \int_{\Omega_T}\nabla f(\mathbf{d})\nabla\mathbf{d}\eta\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{1.8}$$

According to the scaling property of system (1.1), we introduce the following dimensionless quantities:

Definition 1.3 *(Scaled functionals).* For $1 \le p \le \infty$, let $z_0 = (x_0,t_0) \in \Omega_T$ and f equal to $\mathbf{u}$ or $\nabla\mathbf{d}$, define

$$A(f;r) = \frac{1}{r}\sup_{t_0-r^2\le t\le t_0}\int_{B_r(x_0)}|f|^2\mathrm{d}x,\ B(f;r) = \frac{1}{r}\int_{Q_r(z_0)}|\nabla f|^2\mathrm{d}x\mathrm{d}t,$$

$$C_p(f;r) = \frac{1}{r^{5-p}}\int_{Q_r(z_0)}|f|^p\mathrm{d}x\mathrm{d}t,\ D(P;r) = \frac{1}{r^2}\int_{Q_r(z_0)}|P|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t,$$

$$D_*(\nabla P;r) = \frac{1}{r}\int_{t_0-r^2}^{t_0}\left(\int_{B_r(x_0)}|\nabla P|^{\frac{3}{2}}\mathrm{d}x\right)^{\frac{2}{3}}\mathrm{d}t.$$

Let $A(\mathbf{u},\nabla\mathbf{d};r) = A(\mathbf{u};r)+A(\nabla\mathbf{d};r)$, $B(\mathbf{u},\nabla\mathbf{d};r)$ and $C_p(\mathbf{u},\nabla\mathbf{d};r)$ denote similar notations.

Our main results are as follow.

Theorem 1.1 *Let $(\mathbf{u},\mathbf{d},P)$ be a suitable weak solution of (1.1) in $Q_r(z_0)$. There exists $\varepsilon > 0$ such that if $\mathbf{u}\in L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))$ and*

$$\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))} < \varepsilon, \tag{1.9}$$

then $(\mathbf{u},\nabla\mathbf{d})$ is regular in $Q_{\frac{r}{2}}(z_0)$.

Theorem 1.2 *Let $(\mathbf{u},\mathbf{d},P)$ be a suitable weak solution of (1.1) in $Q_r(z_0)$. For any $M > 0$, there exists $\varepsilon_0(M)$ such that if $\nabla\mathbf{d}\in L^\infty(-1,0;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$, $\mathbf{u}\in L^\infty(-1,0;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$ and satisfies*

$$\|\mathbf{u}\|_{L^\infty(-1,0;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))} < \varepsilon_0(M),\ \|\nabla\mathbf{d}\|_{L^\infty(-1,0;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))} < M, \tag{1.10}$$

then $(\mathbf{u},\nabla\mathbf{d})$ is regular in $Q_{\frac{r}{2}}(z_0)$.

The paper is organized as follows. In section 2, we first introduce some notations and facts of the Lorentz spaces. In addition, we give some lemmas, which play an important role in the proof of Theorem 1.1 and Theorem 1.2. Section 3 is devoted to the proof of Theorem 1.1, and Section 4 is devoted to the proof of Theorem 1.2.

2 Preliminaries

Firstly, we recall that a local integrable function $f \in BMO(\mathbb{R}^3)$ if it satisfies

$$\sup_{x_0\in\mathbb{R}^3, R>0} \frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} |f(x) - f_{B_R(x_0)}|\mathrm{d}x < \infty.$$

We say that a local integrable function $V \in BMO^{-1}$, provided that there exists $(f_1, \cdots, f_n) \in BMO$ such that $V = \sum_{i=1}^n \frac{\partial f}{\partial x_i}$. Next, we give the definition of Besov spaces. Letting $e^{t\Delta}$ denotes the heat semigroup on $\mathbb{R}^3$. For $\alpha < 0$, a tempered distribution f on $\mathbb{R}^3$ is in the Besov space $\dot{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^3)$ if and only if

$$\|f\|_{\dot{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^3)} = \sup_{t>0} t^{-\frac{\alpha}{2}}\|e^{t\Delta}f\|_{L^\infty(\mathbb{R}^3)} < \infty.$$

The following trilinear estimate lemmas play an important role in the proof of Theorem 1.1.

Lemma 2.1 ([3]) *For* $1 < p_1, p_2 < \infty$, $f \in BMO^{-1}(\mathbb{R}^3)$, $g \in L^{p_1}(\mathbb{R}^3)$, $\nabla g \in L^{p_2}(\mathbb{R}^3)$, $\nabla h \in L^{\frac{q_1}{q_1-1}}(\mathbb{R}^3)$, $h \in L^{\frac{p_2}{p_2-1}}(\mathbb{R}^3)$, *there is an absolutely constant* $C > 0$ *such that*

$$\int_{\mathbb{R}^3} fgh\mathrm{d}x \le C\|f\|_{BMO^{-1}(\mathbb{R}^3)}\left(\|g\|_{L^{p_1}(\mathbb{R}^3)}\|\nabla h\|_{L^{\frac{p_1}{p_1-1}}(\mathbb{R}^3)} + \|\nabla g\|_{L^{p_2}(\mathbb{R}^3)}\|h\|_{L^{\frac{p_2}{p_2-1}}(\mathbb{R}^3)}\right). \tag{2.1}$$

Lemma 2.2 *For any* $0 < r \le \frac{\rho}{2}$, *there exists an absolutely constant* $C > 0$ *such that*

$$D_*(\nabla P; r) \le C(\frac{\rho}{r})B(\mathbf{u}, \nabla\mathbf{d}; \rho) + C(\frac{r}{\rho})^2 D_*(\nabla P; \rho), \tag{2.2}$$

$$D(P; r) \le C(\frac{\rho}{r})^2 A^{\frac{3}{4}}(\mathbf{u}, \nabla\mathbf{d}; \rho)B^{\frac{3}{4}}(\mathbf{u}, \nabla\mathbf{d}; \rho) + C(\frac{r}{\rho})^2 D(P; \rho), \tag{2.3}$$

$$D(P; r) \le C(\frac{\rho}{r})^2 C_3(\mathbf{u}, \nabla\mathbf{d}; \rho) + C(\frac{r}{\rho})^2 D(P; \rho), \tag{2.4}$$

Proof Taking the smooth cut-off function $\phi \in C_0^\infty(B_\rho(x_0))$ with $\phi \equiv 1$ on $B_{\frac{3}{4}}(x_0)$ and satisfy properties $|\nabla\phi| \le C\rho^{-1}$,$|\nabla^2\phi| \le C\rho^{-2}$. Using divergence-free condition, we can write

$$\partial_i\partial_i(\partial_k P\phi) = -\phi\partial_i\partial_j[\partial_k(\mathbb{U}_{i,j} + \mathbb{D}_{i,j})] + 2\partial_i\partial_i\partial_k P + \partial_k P\partial_i\partial_i\phi,$$

where $\mathbb{U}_{i,j} = (\mathbf{u}_i - \mathbf{u}_{i_{B_\rho(x_0)}})(\mathbf{u}_j - \mathbf{u}_{j_{B_\rho(x_0)}})$ and $\mathbb{D}_{i,j} = (\partial_i\mathbf{d} - (\partial_i\mathbf{d})_{B_\rho(x_0)})(\partial_j\mathbf{d} - (\partial_j\mathbf{d})_{B_\rho(x_0)})$. This yields that for each $x \in B_{\frac{3}{4}\rho}(x_0)$

$$\begin{aligned}\partial_k P(x) &= \mathscr{K} * \{-\phi\partial_i\partial_j[\partial_k(\mathbb{U}_{i,j} + \mathbb{D}_{i,j})] + 2\partial_i\partial_i\partial_k P + \partial_k P\partial_i\partial_i\phi\}\\ &= -\partial_i\partial_j\mathscr{K} * (\phi[\partial_k(\mathbb{U}_{i,j} + \mathbb{D}_{i,j})]) + 2\partial_i\mathscr{K} * (\partial_j\phi[\partial_k(\mathbb{U}_{i,j} + \mathbb{D}_{i,j})])\\ &\quad + \mathscr{K} * (\partial_i\partial_j\phi[\partial_k(\mathbb{U}_{i,j} + \mathbb{D}_{i,j})]) + 2\partial_i\mathscr{K} * (\partial_i\phi\partial_k P) - \mathscr{K} * (\partial_i\partial_i\phi\partial_k P)\\ &=: P_1(x) + P_2(x),\end{aligned} \tag{2.5}$$

where $\mathscr{K}$ stands for the normalized fundamental solution of Laplace's equation and

$$P_1(x) = -\partial_i\partial_j\mathscr{K} * (\phi[\partial_k(\mathbb{U}_{i,j} + \mathbb{D}_{i,j})]),$$

$$\begin{aligned}P_2(x) &= 2\partial_i\mathscr{K} * (\partial_j\phi[\partial_k(\mathbb{U}_{i,j} + \mathbb{D}_{i,j})]) + \mathscr{K} * (\partial_i\partial_j\phi[\partial_k(\mathbb{U}_{i,j} + \mathbb{D}_{i,j})])\\ &\quad + 2\partial_i\mathscr{K} * (\partial_i\phi\partial_k P) - \mathscr{K} * (\partial_i\partial_i\phi\partial_k P).\end{aligned} \tag{2.6}$$

Since $\phi(x) = 1$ when $x \in B_{\frac{3}{4}\rho}(x_0)$, we obtain that

$$\Delta P_2(x) = 0.$$

Hence, the property of the harmonic function yields

$$\begin{aligned}\int_{B_r(x_0)}|P_2(x)|^{\frac{3}{2}}\mathrm{d}x &\le C\left(\frac{r}{\rho}\right)^3\int_{B_{\frac{\rho}{2}}(x_0)}|P_2(x)|^{\frac{3}{2}}\mathrm{d}x\\ &\le C\left(\frac{r}{\rho}\right)^3\int_{B_{\frac{\rho}{2}}(x_0)}|\partial_k P(x)|^{\frac{3}{2}}\mathrm{d}x + C\int_{B_{\frac{\rho}{2}}(x_0)}|P_1(x)|^{\frac{3}{2}}\mathrm{d}x.\end{aligned} \tag{2.7}$$

On the other hand, applying the Calderon-Zygmund theorem, we have that

$$\begin{aligned}\int_{B_{\frac{\rho}{4}}(x_0)}|P_1(x)|^{\frac{3}{2}}\mathrm{d}x &\le C\int_{B_\rho(x_0)}|\partial_k\mathbf{u}(\mathbf{u}-\mathbf{u}_{B_\rho(x_0)})|^{\frac{3}{2}}+|\partial_k\nabla\mathbf{d}(\nabla\mathbf{d}-(\nabla\mathbf{d})_{B_\rho(x_0)})|^{\frac{3}{2}}\mathrm{d}x\\ &\le C\left(\int_{B_\rho(x_0)}|\nabla\mathbf{u}|^2\mathrm{d}x\right)^{\frac{3}{4}}\left(\int_{B_\rho(x_0)}|\mathbf{u}-\mathbf{u}_{B_\rho(x_0)}|^6\mathrm{d}x\right)^{\frac{1}{4}}\\ &\quad + C\left(\int_{B_\rho(x_0)}|\nabla^2\mathbf{d}|^2\mathrm{d}x\right)^{\frac{3}{4}}\left(\int_{B_\rho(x_0)}|\nabla\mathbf{d}-(\nabla\mathbf{d})_{B_\rho(x_0)}|^6\mathrm{d}x\right)^{\frac{1}{4}}\\ &\le C\left(\int_{B_\rho(x_0)}|\nabla\mathbf{u}|^2\mathrm{d}x\right)^{\frac{3}{2}}+C\left(\int_{B_\rho(x_0)}|\nabla^2\mathbf{d}|^2\mathrm{d}x\right)^{\frac{3}{2}}.\end{aligned} \tag{2.8}$$

Collecting (2.6)-(2.8) we get that

$$\begin{aligned}\int_{B_r(x_0)}|\nabla P|^{\frac{3}{2}}\mathrm{d}x &\le C\int_{B_r(x_0)}|P_1|^{\frac{3}{2}}+|P_2|^{\frac{3}{2}}\mathrm{d}x\\ &\le C\left(\int_{B_\rho(x_0)}|\nabla\mathbf{u}|^2+|\nabla^2\mathbf{d}|^2\mathrm{d}x\right)^{\frac{3}{2}}+C\left(\frac{r}{\rho}\right)^3\int_{B_\rho(x_0)}|\nabla P|^{\frac{3}{2}}\mathrm{d}x.\end{aligned} \tag{2.9}$$

Integrating above inequality in time on (t_0-r^2,t_0) we obtain that

$$\begin{aligned}&\int_{t_0-r^2}^{t_0}\left(\int_{B_r(x_0)}|\nabla P|^{\frac{3}{2}}\mathrm{d}x\right)^{\frac{2}{3}}\mathrm{d}t\\ &\le C\int_{Q_\rho(z_0)}|\nabla\mathbf{u}|^2+|\nabla^2\mathbf{d}|^2\mathrm{d}x\mathrm{d}t + C\left(\frac{r}{\rho}\right)^3\int_{t_0-\rho^2}^{t_0}\left(\int_{B_\rho(x_0)}|\nabla P|^{\frac{3}{2}}\mathrm{d}x\right)^{\frac{2}{3}}\mathrm{d}t,\end{aligned} \tag{2.10}$$

which yields the desired estimate (2.2). Noting that

$$\partial_i\partial_i(P\phi) = -\phi\partial_i\partial_j[\mathbb{U}_{i,j}+\mathbb{D}_{i,j}] + 2\partial_i\phi\partial_i P + P\partial_i\partial_i\phi \tag{2.11}$$

$$\int_{B_\rho(x_0)}|\mathbf{f}-\mathbf{f}_{B_\rho(x_0)}|^3\mathrm{d}x \le \int_{B_\rho(x_0)}|\mathbf{f}|^3\mathrm{d}x, \tag{2.12}$$

and

$$\int_{B_\rho(x_0)}|\mathbf{f}-\mathbf{f}_{B_\rho(x_0)}|^3\mathrm{d}x \le C\left(\int_{B_\rho(x_0)}|\mathbf{f}-\mathbf{f}_{B_\rho(x_0)}|^2\mathrm{d}x\right)^{\frac{3}{4}}\left(\int_{B_\rho(x_0)}|\nabla\mathbf{f}|^2\mathrm{d}x\right)^{\frac{3}{4}}, \tag{2.13}$$

where we take $\mathbf{f}=\mathbf{u}$ or $\nabla\mathbf{d}$. Using (2.11) and (2.13), similar with the process of the derivation of (2.2), we get (2.3). Repeating the deduction process of (2.2) and applying inequality (2.12) and (2.13), we can arrive at (2.4). Hence, we complete the proof of this lemma. □

Lemma 2.3 *([20]) For any $0<r<\infty$, there exists a positive constant C such that*

$$C_3(\mathbf{u};r) \le C\|\mathbf{u}\|^{\frac{3}{2}}_{L^\infty(-1,0;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))}(A^{\frac{3}{4}}(\mathbf{u};2r)+B^{\frac{3}{4}}(\mathbf{u};2r)). \tag{2.14}$$

We conclude this section by recalling the following partial regularity criterion.

Proposition 2.1 ([15]) *Let* $(\mathbf{u}, \mathbf{d}, P)$ *be a suitable weak solution of (1.1) in* $Q_r(z_0)$. *There exists* $\varepsilon_1 > 0$ *such that for some* $r > 0$,

$$\frac{1}{r^2}\int_{Q_r(z_0)} |\mathbf{u}|^3 + |\nabla \mathbf{d}|^3 + |P|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t \le \varepsilon_1, \tag{2.15}$$

then $z_0 = (x_0, t_0)$ *is a regular point.*

3 The proof of theorem 1.1

Taking the cutoff function ϕ, with smooth, supported in $Q_\rho(z_0)$, $\phi \equiv 1$ in $Q_{\frac{\rho}{2}}(z_0)$ and satisfying

$$|\partial_t \phi| + |\nabla^2 \phi| + |\nabla \phi|^2 \lesssim \frac{C}{\rho^2} \quad \text{in} \quad Q_\rho(z_0).$$

We let Γ denote backward heat function, which given by $\Gamma(x,t) = \frac{1}{4\pi(r^2-t)^{\frac{3}{2}}} e^{-\frac{|x|^2}{4(r^2-t)}}$, satisfies $\Gamma_t + \Delta\Gamma = 0$. By a direct computation, it is clear that for any $(x,t) \in Q_\rho(z_0)$,

$$|\Gamma(x,t)| \ge Cr^{-3},\ |\Gamma(x,t)\phi(x,t)| \le Cr^{-3},\ |\nabla\phi(x,t)\Gamma(x,t)| \le Cr^{-4},$$
$$|\phi(x,t)\nabla\Gamma(x,t)| \le Cr^{-4},\ |\phi(x,t)\nabla^2\Gamma(x,t)| \le Cr^{-5}.$$

And for any $(x,t) \in Q_\rho(z_0)\backslash Q_{\frac{\rho}{2}}(z_0)$, we can deduce that

$$\Gamma(x,t) \le C\rho^{-3},\ \nabla\Gamma(x,t) \le C\rho^{-4},$$

which leads to

$$|\Gamma(x,t)\partial_t\phi(x,t)| + |\Gamma(x,t)\Delta\phi(x,t)| + |\nabla\phi(x,t)\nabla\Gamma(x,t)| \le C\rho^{-5}.$$

Substituting $\psi(x,t) = \phi^2(x,t)\Gamma(x,t)$ into the local energy inequality (1.8) and using the maximum principle of $\mathbf{d}$ (see [15]) we obtain that

$$\begin{aligned}
&\sup_{t_0-\rho^2\le t\le t_0}\int_{B_\rho(x_0)} (|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2)\psi \mathrm{d}x + 2\int_{Q_\rho(z_0)} (|\nabla\mathbf{u}|^2 + |\nabla^2\mathbf{d}|^2)\psi \mathrm{d}x\mathrm{d}t \\
&\le \int_{Q_\rho(z_0)} (|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2)(\psi_t + \Delta\psi)\mathrm{d}x\mathrm{d}t + \int_{Q_\rho(z_0)} (|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2 + 2P)\mathbf{u}\cdot\nabla\psi \mathrm{d}x\mathrm{d}t \\
&\quad + \int_{Q_\rho(z_0)} ((\mathbf{u}\cdot\nabla)\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\psi \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.1}$$

We turn our attention back to controlling the nonlinear terms. A direct calculation gives that

$$\begin{aligned}
&\int_{Q_\rho(z_0)} ((\mathbf{u}\cdot\nabla)\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla(\phi^2\Gamma)\mathrm{d}x\mathrm{d}t \\
&= \int_{Q_\rho(z_0)} ((\mathbf{u}\cdot\nabla)\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla(\phi\Gamma)\phi \mathrm{d}x\mathrm{d}t + \int_{Q_\rho(z_0)} ((\mathbf{u}\cdot\nabla)\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\phi\phi\Gamma \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.2}$$

Applying Lemma 2.1, the properties of test function ϕ, Γ and the Hölder inequality we get that

$$\begin{aligned}
&\int_{B_\rho(x_0)} \big((\mathbf{u}\cdot\nabla)\mathbf{d}\odot\nabla\mathbf{d}\big)\cdot\nabla(\phi\Gamma)\phi \mathrm{d}x \\
&\quad\le C\|\mathbf{u}\|_{BMO^{-1}(\mathbb{R}^3)}[\|\nabla\mathbf{d}\cdot\nabla(\phi\Gamma)\|_{L^2(B_\rho(x_0))}\|\nabla(\nabla\mathbf{d}\phi)\|_{L^2(B_\rho(x_0))} \\
&\qquad + \|\nabla(\nabla\mathbf{d}\cdot\nabla(\phi\Gamma))\|_{L^2(B_\rho(x_0))}\|\nabla\mathbf{d}\phi\|_{L^2(B_\rho(x_0))}] \\
&\quad\le C\|\mathbf{u}\|_{BMO^{-1}(\mathbb{R}^3)}[r^{-4}\|\nabla\mathbf{d}\|_{L^2(B_\rho(x_0))}(\|\nabla^2\mathbf{d}\|_{L^2(B_\rho(x_0))} + r^{-1}\|\nabla\mathbf{d}\|_{L^2(B_\rho(x_0))}) \\
&\qquad + (r^{-4}\|\nabla^2\mathbf{d}\|_{L^2(B_\rho(x_0))} + r^{-5}\|\nabla\mathbf{d}\|_{L^2(B_\rho(x_0))})\|\nabla\mathbf{d}\|_{L^2(B_\rho(x_0))}] \\
&\quad\le C\|\mathbf{u}\|_{BMO^{-1}(\mathbb{R}^3)}[r^{-4}\|\nabla^2\mathbf{d}\|_{L^2(B_\rho(x_0))}\|\nabla\mathbf{d}\|_{L^2(B_\rho(x_0))} + r^{-5}\|\nabla\mathbf{d}\|^2_{L^2(B_\rho(x_0))}].
\end{aligned} \tag{3.3}$$

Integrating in time and using Young's inequality yields

$$\begin{aligned}&\int_{Q_\rho(z_0)}((\mathbf{u}\cdot\nabla)\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla(\phi\Gamma)\phi\mathrm{d}x\mathrm{d}t\\&\le C\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))}\left[r^{-4}\rho^2A^{\frac12}(\mathbf{u},\nabla\mathbf{d};\rho)B^{\frac12}(\mathbf{u},\nabla\mathbf{d};\rho)+r^{-5}\rho^3A(\mathbf{u},\nabla\mathbf{d};\rho)\right]\\&\le C\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))}r^{-5}\rho^3[A(\mathbf{u},\nabla\mathbf{d};\rho)+B(\mathbf{u},\nabla\mathbf{d};\rho)].\end{aligned}\tag{3.4}$$

Similarly,

$$\begin{aligned}&\int_{Q_\rho(z_0)}((\mathbf{u}\cdot\nabla)\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\phi\phi\Gamma\mathrm{d}x\mathrm{d}t\\&\le C\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))}r^{-5}\rho^3[A(\mathbf{u},\nabla\mathbf{d};\rho)+B(\mathbf{u},\nabla\mathbf{d};\rho)].\end{aligned}\tag{3.5}$$

Hence,

$$\begin{aligned}&\int_{Q_\rho(z_0)}((\mathbf{u}\cdot\nabla)\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla(\phi^2\Gamma)\mathrm{d}x\mathrm{d}t\\&\le C\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))}r^{-5}\rho^3[A(\mathbf{u},\nabla\mathbf{d};\rho)+B(\mathbf{u},\nabla\mathbf{d};\rho)].\end{aligned}\tag{3.6}$$

Proceeding analogously, we know that

$$\begin{aligned}&\int_{Q_\rho(z_0)}(|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)\mathbf{u}\cdot\nabla\psi\mathrm{d}x\mathrm{d}t\\&\le C\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))}r^{-5}\rho^3[A(\mathbf{u},\nabla\mathbf{d};\rho)+B(\mathbf{u},\nabla\mathbf{d};\rho)].\end{aligned}\tag{3.7}$$

It remains to control the term involving the pressure. In fact, it is straightforward to check that

$$\int_{B_\rho(x_0)}P\mathbf{u}\cdot\nabla(\phi^2\Gamma)\mathrm{d}x=\int_{B_\rho(x_0)}P\mathbf{u}\cdot\nabla(\phi\Gamma)\phi\mathrm{d}x+\int_{B_\rho(x_0)}P\mathbf{u}\cdot\nabla\phi\phi\Gamma\mathrm{d}x.\tag{3.8}$$

Applying Lemma 2.1 and the properties of test function ψ once again, we obtain that

$$\begin{aligned}&\int_{B_\rho(x_0)}P\mathbf{u}\cdot\nabla\phi\phi\Gamma\mathrm{d}x\\&\le C\|\mathbf{u}\|_{BMO^{-1}(\mathbb{R}^3)}\left[\|\nabla^2\phi\|_{L^3(B_\rho(x_0))}\|\phi\Gamma P\|_{L^{\frac32}(B_\rho(x_0))}+\|\nabla(\phi\Gamma P)\|_{L^{\frac32}(B_\rho(x_0))}\|\nabla\phi\|_{L^3(B_\rho(x_0))}\right]\\&\le C\|\mathbf{u}\|_{BMO^{-1}(\mathbb{R}^3)}\left[r^{-4}\|P\|_{L^{\frac32}(B_\rho(x_0))}+\left(r^{-4}\|P\|_{L^{\frac32}(B_\rho(x_0))}+r^{-3}\|\nabla P\|_{L^{\frac32}(B_\rho(x_0))}\right)\right]\\&\le C\|\mathbf{u}\|_{BMO^{-1}(\mathbb{R}^3)}\left[r^{-4}\|P\|_{L^{\frac32}(B_\rho(x_0))}+r^{-3}\|\nabla P\|_{L^{\frac32}(B_\rho(x_0))}\right].\end{aligned}\tag{3.9}$$

Integrating in time on $(t_0-\rho^2,t_0)$ and applying the Hölder inequality and Young's inequality we get

$$\begin{aligned}&\int_{Q_\rho(z_0)}P\mathbf{u}\cdot\nabla\phi\phi\Gamma\mathrm{d}x\mathrm{d}t\\&\le C\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))}\left[r^{-4}\|P\|_{L^1(t_0-\rho^2,t_0;L^{\frac32}(B_\rho(x_0)))}+r^{-3}\|\nabla P\|_{L^1(t_0-\rho^2,t_0;L^{\frac32}(B_\rho(x_0)))}\right]\\&\le C\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))}\left[r^{-4}\rho^{\frac23}\|P\|_{L^{\frac32}(Q_\rho(z_0))}+r^{-3}\|\nabla P\|_{L^1(t_0-\rho^2,t_0;L^{\frac32}(B_\rho(x_0)))}\right]\\&\le C\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))}\left[r^{-4}\rho^2D^{\frac23}(P;\rho)+r^{-3}\rho D_*(\nabla P;\rho)\right].\end{aligned}\tag{3.10}$$

Likewise, we have

$$\int_{Q_\rho(z_0)}P\mathbf{u}\cdot\nabla(\phi\Gamma)\phi\mathrm{d}x\mathrm{d}t\le C\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))}[r^{-5}\rho^3D^{\frac23}(P;\rho)+r^{-4}\rho^2D_*(\nabla P;\rho)]\tag{3.11}$$

Combining (3.8)-(3.10), we have proved that

$$\int_{Q_\rho(z_0)} P\mathbf{u}\cdot\nabla(\phi^2\Gamma)\mathrm{d}x\mathrm{d}t \le C\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))}[r^{-5}\rho^3 D^{\frac{2}{3}}(P;\rho)+r^{-4}\rho^2 D_*(\nabla P;\rho)]. \tag{3.12}$$

Substituting (3.6), (3.7), (3.12) into (3.1), we know that

$$\begin{aligned} &A(\mathbf{u},\nabla\mathbf{d};r)+B(\mathbf{u},\nabla\mathbf{d};r)\\ &\le C(\frac{r}{\rho})^2 A(\mathbf{u},\nabla\mathbf{d};\rho)+C(\frac{\rho}{r})^2\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))}[A(\mathbf{u},\nabla\mathbf{d};\rho)+B(\mathbf{u},\nabla\mathbf{d};\rho)]\\ &+C(\frac{\rho}{r})^3\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))}D^{\frac{2}{3}}(P;\rho)+C(\frac{\rho}{r})^2\|\mathbf{u}\|_{L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))}D_*(\nabla P;\rho) \end{aligned} \tag{3.13}$$

According to (2.2) and (2.3) in Lemma 2.2, we have

$$D_*(\nabla P;r)\le C(\frac{\rho}{r})(A(\mathbf{u},\nabla\mathbf{d};\rho)+B(\mathbf{u},\nabla\mathbf{d};\rho))+C(\frac{r}{\rho})^2 D_*(\nabla P;\rho), \tag{3.14}$$

$$D^{\frac{2}{3}}(P;r)\le C(\frac{\rho}{r})^{\frac{4}{3}}A^{\frac{1}{2}}(\mathbf{u},\nabla\mathbf{d};\rho)B^{\frac{1}{2}}(\mathbf{u},\nabla\mathbf{d};\rho)+C(\frac{r}{\rho})^{\frac{2}{3}}D^{\frac{2}{3}}(P;\rho). \tag{3.15}$$

Letting $F(r)=A(\mathbf{u},\nabla\mathbf{d};r)+B(\mathbf{u},\nabla\mathbf{d};r)+\eta D^{\frac{2}{3}}(P;r)+\eta D_*(\nabla P;\rho)$, according to the assumption in Theorem 1.1 we have

$$\begin{aligned} F(r)\le{}& C(\frac{r}{\rho})^2F(\rho)+C(\frac{\rho}{r})^2\varepsilon F(\rho)+C(\frac{\rho}{r})^3\varepsilon\eta^{-1}F(\rho)+C(\frac{\rho}{r})^2\varepsilon\eta^{-1}F(\rho)\\ &+C(\frac{\rho}{r})^{\frac{4}{3}}\eta F(\rho)+C(\frac{r}{\rho})^{\frac{2}{3}}F(\rho)+C(\frac{\rho}{r})\eta F(\rho)+C(\frac{r}{\rho})^2F(\rho). \end{aligned} \tag{3.16}$$

Taking $\eta=\varepsilon^{\frac{1}{2}}$, we have

$$\begin{aligned} F(r)&\le C(\frac{r}{\rho})^{\frac{2}{3}}F(\rho)+C(\frac{\rho}{r})^3\varepsilon^{\frac{1}{2}}F(\rho)\\ &=C\lambda^{-3}\varepsilon^{\frac{1}{2}}F(\rho)+C\lambda^{\frac{2}{3}}F(\rho), \end{aligned} \tag{3.17}$$

where $\lambda=\frac{r}{\rho}\le\frac{1}{8}$. We choose λ,ε such that

$$k=2C\lambda^{\frac{2}{3}}<1 \text{ and } \varepsilon<\left(\frac{k\lambda^3}{2C}\right)^2,$$

it follows from (3.17) that

$$F(\lambda\rho)\le kF(\rho).$$

Then a standard iterative argument ensures that there exist $r_1>0$ such that for $0<r<r_1$,

$$F(r)\le\varepsilon. \tag{3.18}$$

In the rest of this proof, we will show that there exists a $r_2>0$ such that

$$D(P;r)+C_3(\mathbf{u},\nabla\mathbf{d};r)\le\varepsilon \text{ for } 0<r<r_2.$$

Indeed, by using interpolation inequality and the Hölder inequality, we get

$$\begin{aligned}
&C_3(\mathbf{u},\nabla\mathbf{d};r)\\
&\le Cr^{-2}\int_{t_0-r^2}^{t_0}[(\int_{B_r(x_0)}|\nabla\mathbf{u}|^2\mathrm{d}x)^{\frac34}(\int_{B_r(x_0)}|\mathbf{u}|^2\mathrm{d}x)^{\frac34}+r^{-\frac32}(\int_{B_r(x_0)}|\mathbf{u}|^2\mathrm{d}x)^{\frac32}]\mathrm{d}t\\
&\quad+Cr^{-2}\int_{t_0-r^2}^{t_0}[(\int_{B_r(x_0)}|\nabla^2\mathbf{d}|^2\mathrm{d}x)^{\frac34}(\int_{B_r(x_0)}|\nabla\mathbf{d}|^2\mathrm{d}x)^{\frac34}+r^{-\frac32}(\int_{B_r(x_0)}|\nabla\mathbf{d}|^2\mathrm{d}x)^{\frac32}]\mathrm{d}t\\
&\le Cr^{-2}(\int_{Q_r(z_0)}|\nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}t)^{\frac34}(\int_{t_0-r^2}^{t_0}(\int_{B_r(x_0)}|\mathbf{u}|^2\mathrm{d}x)^3\mathrm{d}t)^{\frac14}\\
&\quad+Cr^{-\frac32}\sup_{t_0-r^2<t<t_0}(\int_{B_r(x_0)}|\mathbf{u}|^2\mathrm{d}x)^{\frac32}\\
&\quad+Cr^{-2}(\int_{Q_r(z_0)}|\nabla^2\mathbf{d}|^2\mathrm{d}x\mathrm{d}t)^{\frac34}(\int_{t_0-r^2}^{t_0}(\int_{B_r(x_0)}|\nabla\mathbf{d}|^2\mathrm{d}x)^3\mathrm{d}t)^{\frac14}\\
&\quad+Cr^{-\frac32}\sup_{t_0-r^2<t<t_0}(\int_{B_r(x_0)}|\nabla\mathbf{d}|^2\mathrm{d}x)^{\frac32}.
\end{aligned}\tag{3.19}$$

It follows from (3.18) that for $0<r<r_1$,

$$\begin{aligned}
C_3(\mathbf{u},\nabla\mathbf{d};r)&\le C(A^{\frac34}(\mathbf{u};r)B^{\frac34}(\mathbf{u};r)+A^{\frac32}(\mathbf{u};r)+A^{\frac34}(\nabla\mathbf{d};r)B^{\frac34}(\nabla\mathbf{d};r)+A^{\frac32}(\nabla\mathbf{d};r))\\
&\le C(A^{\frac32}(\mathbf{u},\nabla\mathbf{d};r)+B^{\frac32}(\mathbf{u},\nabla\mathbf{d};r))\\
&\le C\varepsilon^{\frac32}.
\end{aligned}\tag{3.20}$$

This together with (2.4) and a standard iterative process ensures that there exists $r_2>0$ such that

$$C_3(\mathbf{u},\nabla\mathbf{d};r)+D(P;r)\le\varepsilon\ \text{ for }\ 0<r<r_2.$$

Hence, we complete the proof of Theorem 1.1 due to Proposition 2.1. □

4 The proof of theorem 1.2

Under the assumption $\mathbf{u}\in L^\infty(-1,0;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))$ and $\nabla\mathbf{d}\in L^\infty(-1,0;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))$, we claim that the boundness of the following dimensionless quantities:

$$A(\mathbf{u},\nabla\mathbf{d};r)+B(\mathbf{u},\nabla\mathbf{d};r)+D(P;r)\le C(M,C_3(\mathbf{u},\nabla\mathbf{d};1),D(P;1)).\tag{4.1}$$

Using Lemma 2.3 we get

$$C_3(\mathbf{u};2r)\le C\|\mathbf{u}\|^{\frac32}_{L^\infty(-1,0;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}(A^{\frac34}(\mathbf{u};4r)+B^{\frac34}(\mathbf{u};4r)).\tag{4.2}$$

Similarly,

$$C_3(\nabla\mathbf{d};2r)\le C\|\nabla\mathbf{d}\|^{\frac32}_{L^\infty(-1,0;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}(A^{\frac34}(\nabla\mathbf{d};4r)+B^{\frac34}(\nabla\mathbf{d};4r)).\tag{4.3}$$

Applying the properties of cutoff function and local energy inequality (1.8), we know that

$$\begin{aligned}
&A(\mathbf{u},\nabla\mathbf{d};r)+B(\mathbf{u},\nabla\mathbf{d};r)\\
&\le CC_3^{\frac23}(\mathbf{u},\nabla\mathbf{d};2r)+CC_3(\mathbf{u},\nabla\mathbf{d};2r)+CD(P;2r)\\
&\le C\|(\mathbf{u},\nabla\mathbf{d})\|_{L^\infty(-1,0;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}(A^{\frac12}(\mathbf{u},\nabla\mathbf{d};4r)+B^{\frac12}(\mathbf{u},\nabla\mathbf{d};4r))\\
&\quad+C\|(\mathbf{u},\nabla\mathbf{d})\|^{\frac32}_{L^\infty(-1,0;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}(A^{\frac34}(\mathbf{u},\nabla\mathbf{d};4r)+B^{\frac34}(\mathbf{u},\nabla\mathbf{d};4r))+CD(P;4r).
\end{aligned}\tag{4.4}$$

It follows from (2.4) in Lemma 2.2 that

$$\begin{aligned}
&D(P;4r)\\
&\le C(\frac{\rho}{r})^2 C_3(\mathbf{u},\nabla\mathbf{d};\rho)+C(\frac{r}{\rho})D(P;\rho)\\
&\le C(\frac{\rho}{r})^2\|(\mathbf{u},\nabla\mathbf{d})\|^{\frac{3}{2}}_{L^\infty(-1,0;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))}(A^{\frac{3}{4}}(\mathbf{u},\nabla\mathbf{d};2\rho)+B^{\frac{3}{4}}(\mathbf{u},\nabla\mathbf{d};2\rho))+C(\frac{r}{\rho})D(P;2\rho)\\
&\le C(\frac{\rho}{r})^2\|(\mathbf{u},\nabla\mathbf{d})\|^{\frac{3}{2}}_{L^\infty(-1,0;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))}(A^{\frac{3}{4}}(\mathbf{u},\nabla\mathbf{d};\rho)+B^{\frac{3}{4}}(\mathbf{u},\nabla\mathbf{d};\rho))+C(\frac{r}{\rho})D(P;\rho).
\end{aligned}\tag{4.5}$$

We define $F_1(r)=A(\mathbf{u},\nabla\mathbf{d};r)+B(\mathbf{u},\nabla\mathbf{d};r)+D(P;r)$, then combining (4.4), (4.5) and applying the Young's inequality yields

$$\begin{aligned}
F_1(r)&\le C(\frac{\rho}{r})^{\frac{1}{2}}(A^{\frac{1}{2}}(\mathbf{u},\nabla\mathbf{d};\rho)+B^{\frac{1}{2}}(\mathbf{u},\nabla\mathbf{d};\rho))+C(\frac{\rho}{r})^{\frac{3}{4}}(A^{\frac{3}{4}}(\mathbf{u},\nabla\mathbf{d};4\rho)+B^{\frac{3}{4}}(\mathbf{u},\nabla\mathbf{d};4\rho))\\
&\quad+C(\frac{\rho}{r})^2(A^{\frac{3}{4}}(\mathbf{u},\nabla\mathbf{d};\rho)+B^{\frac{3}{4}}(\mathbf{u},\nabla\mathbf{d};\rho))+C(\frac{r}{\rho})D(P;\rho)\\
&\le C(\frac{\rho}{r})^2(F_1^{\frac{1}{2}}(\rho)+F_1^{\frac{3}{4}}(\rho))+C(\frac{r}{\rho})F_1(\rho)\\
&\le C(\frac{r}{\rho})F_1(\rho)+C(\frac{\rho}{r})^5+C(\frac{\rho}{r})^{11}\\
&\le C_1\lambda F_1(\rho)+C_2(\lambda),
\end{aligned}\tag{4.6}$$

where $\lambda=\frac{r}{\rho}\le\frac{1}{8}$. Choosing λ sufficient small such that $C_1\lambda^{\frac{1}{2}}<1$, then

$$F_1(\lambda\rho)\le\lambda^{\frac{1}{2}}F_1(\rho)+C_2(\lambda).$$

Iterating above inequality yields

$$F_1(\lambda^k\rho)\le\lambda^{\frac{k}{2}}F_1(\rho)+C_3(\lambda),$$

which means that for all $r\in(0,\frac{1}{2}]$,

$$F_1(r)\le C(M)(r^{\frac{1}{2}}F_1(1)+1),$$

this gives the desired estimate (4.1). Hence, Combining (3.1), (4.1) and (2.14) we have

$$\begin{aligned}
&A(\mathbf{u},\nabla\mathbf{d};r)+B(\mathbf{u},\nabla\mathbf{d};r)\\
&\le C(\frac{r}{\rho})^2C_2(\mathbf{u},\nabla\mathbf{d};r)+C(\frac{\rho}{r})^2C_3(\mathbf{u};\rho)\\
&\quad+C(\frac{\rho}{r})^2C_3^{\frac{1}{3}}(\mathbf{u};\rho)C_3^{\frac{2}{3}}(\nabla\mathbf{d};\rho)+C(\frac{\rho}{r})^2C_3^{\frac{1}{3}}(\mathbf{u};\rho)D^{\frac{2}{3}}(P;\rho)\\
&\le C(\frac{r}{\rho})^2A(\mathbf{u},\nabla\mathbf{d};2\rho)+C(\frac{\rho}{r})^2\|\mathbf{u}\|^{\frac{3}{2}}_{L^\infty(-1,0;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))}(A^{\frac{3}{4}}(\mathbf{u};2\rho)+B^{\frac{3}{4}}(\mathbf{u};2\rho))\\
&\quad+C(\frac{\rho}{r})^2\|\mathbf{u}\|^{\frac{1}{2}}_{L^\infty(-1,0;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))}(A^{\frac{3}{4}}(\mathbf{u};2\rho)+B^{\frac{3}{4}}(\mathbf{u};2\rho))^{\frac{1}{3}}.
\end{aligned}\tag{4.7}$$

An iteration argument similar with the proof in Theorem 1.1 yields the smallness of $C(\mathbf{u},\nabla\mathbf{d};r)+D(P;r)$. This together with Proposition 2.1 completes the proof of Theorem 1.2. □

Declarations

Conflict of interest The authors declared that they have no conflict of interest to this work.

References

1. L. Caffarelli, R. Kohn, L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Commun. Pure Appl. Math.* 35 (1982), 771–831.
2. A. Cheskidov, R. Shvydkoy, On the regularity of weak solutions of the 3D Navier-Stokes equations in $\dot{B}^{-1}_{\infty,\infty}$. *Arch. Ration. Mech. Anal.* 195 (2010), 159–169.
3. O. Chol-Jun, Regularity criterion for weak solutions to the 3D Navier-Stokes equations via two vorticity components in BMO^{-1}. *Nonlinear Anal.* 59 (2021), 103271.
4. J. L. Ericksen, Hydrostatic theory of liquid crystals. *Arch. Ration. Mech. Anal.* 9 (1962), 371–378.
5. J. L. Ericksen, Equilibrium theory of liquid crystals. *Adv. Liq. Cryst.* 2 (1976), 233–298.
6. L. Escauriaza, G. Seregin, V. Šverák, On $L^{\infty}L^{3}-$solutions to the Navier-Stokes equations and backward uniqueness. *Russian Math. Surveys* 58 (2003), 211–250.
7. I. Gallagher, G. S. Koch, F. Planchon, Blow-up critical Besov norms at a potential Navier-Stokes singularity. *Commun. Math. Phys.* 343 (2016), 39–82.
8. E. Hopf, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Math. Nachr.* 4 (1951), 213–231.
9. B. S. Lai, W. Y. Ma, On the interior regularity criteria for liquid crystal flows. *Nonlinear Anal. RWA* 40 (2018), 1–13.
10. F. M. Leslie, Some constitutive equations for anisotropic fluids. *Q. J. Mech. Appl. Math.* 19 (1966), 357–370.
11. F. M. Leslie, Some constitutive equations for liquid crystals. *Arch. Ration. Mech. Anal.* 28 (1968), 265–283.
12. J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.* 63 (1934), 193–248.
13. F. H. Lin, C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals. *Commun. Pure Appl. Math.* 48 (1995), 501–537.
14. Q. Li, B. Q. Yuan, Two regularity criteria of solutions to the liquid crystal flow. *Math. Methods Appl. Sci.* 46 (2023), 9167–9176.
15. F. H. Lin, C. Liu, Partial regularities of the nonlinear dissipative systems modeling the flow of liquid crystals. *Discrete Contin. Dyn. Syst.* 2 (1996), 1–22.
16. Q. Liu, Regularity of weak solutions and the number od singular points to the 3D simplified liquid crystal system. *J. Funct. Anal.* 277 (2019), 108294.
17. X. G. Liu, J. Z. Min, K. Wang, X. T. Zhang, Serrin's regularity results for the incompressible liquid crystals system. *Discrete Contin. Dyn. Syst.* 36 (2016), 5579–5594.
18. X. G. Liu, J. Z. Min, X. T. Zhang, $L^{3,\infty}$ solutions of the liquid crystal system. *J. Differential Equations* 267 (2019), 2643–2670.
19. Y. Y. Men, W. D. Wang, G. Wu, Endpoint regularity criterion for weak solutions of the 3D incompressible liquid crystals system. *Math. Methods Appl. Sci.* 41 (2018), 3672–3683
20. G. Seregin, D. G. Zhou, Regularity of solutions to the Navier-Stokes equations in $\dot{B}^{-1}_{\infty,\infty}$. *J. Math. Sci.* 244 (2020), 1003–1009.
21. J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations. *Arch. Ration. Mech. Anal.* 9 (1962), 187–195.
22. T. Tao, Quantitative bounds for critically bounded solutions to the Navier-Stokes equations. *arXiv:1908.04958v2.*
23. W. Wang, P. W. Zhang, Z. F. Zhang, Well-posedness of the Ericksen-Leslie system. *Arch. Ration. Mech. Anal.* 210 (2013), 837–855.
24. W. D. Wang, Z. F. Zhang, Blow-up of critical norms for the 3-D Navier-Stokes equations. *Sci. China Math.* 60 (2017), 637–650.
25. W. D. Wang, Z. F. Zhang, Regularity of weak solutions for the Navier-Stokes equations in the class $L_{\infty}(BMO^{-1})$. *Commun. Contemp. Math.* 14 (2012), 1250020–1250024.
26. H. Yu, Partial regularity criteria for suitable weak solutions of the three dimensional liquid crystal flow. *Math. Methods Appl. Sci.* 39 (2016), 4196–4207.

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