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ORIGINAL RESEARCH

Vallée-Poussin theorem for fractional functional differential equations with integral boundary condition

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Abstract This research paper focuses on the study of a Riemann-Liouville fractional functional differential equation and a linear continuous operator acting from the space of continuous functions to the space of essentially bounded functions with a boundary condition involving integral terms. We investigates the solvability and uniqueness of the equation under certain conditions on the coefficients. The paper utilizes techniques of Vallée-Poussin theorem, and Green's function sign constancy to establish the main results. Choosing a corresponding function within the context of the Vallée-Poussin theorem results in explicit criteria presented as algebraic inequalities. These inequalities, as we illustrate through examples, cannot be further improved.

Keywords Fractional differential equations · Riemann-Liouville derivative · Boundary value problems · Sign constancy of Green's function · Differential inequality

Mathematics Subject Classification 34K37 · 34K40 · 34K38 · 34K10

1 Introduction

In this paper, we consider the following fractional functional differential equation

$$(D_{a+}^{\alpha}x)(t) + (Tx)(t) = f(t), \quad t \in [a, b], 1 < \alpha \leq 2, \tag{1}$$

with boundary condition

$$x(a) = 0, \;\; x(b) - \lambda \int_a^b x(s)ds = 0, \;\; \lambda \geq 0. \tag{2}$$

In (1), D_{a+}^{α} is Riemann-Liouville fractional derivative of order $1 < \alpha \leq 2$, the operator $T : C \to L_{\infty}$ are linear continuous operators acting from the space of the continuous functions C to the space of essentially bounded

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functions L_∞ and $f \in L_\infty$. The operator T can be of the forms $(Tx)(t) = q(t)x(t-\tau(t)),\ x(\xi) = 0,$ if $\xi \neq [a,b]$, $(Tx)(t) = \int_a^b Q(t,s)x(\theta(s))ds$, or $(Tx)(t) = \int_a^b x(s)d_sQ(t,s)$. The conditions describing assumptions about all their coefficients are explained in [6], which allows acting of these operators $T : C \to L_\infty$.

In boundary condition (2), if $\lambda = 0$, then for $a = 0$ and $b = 1$, it reduces to

$$x(0) = 0, \quad x(1) = 0,$$

which is similar to one of the problems investigated in [17]. Thus, in some respect problem (1)–(2) in this work generalizes the problem studied in [17].

We consider equation (1)–(2) in the space D of functions $x : [a,b] \to \mathbb{R}$ such that $x^{(n-1)}$ is absolutely continuous on every interval $[\varepsilon, b]$, where $\varepsilon > 0$ and summable on $[0,1]$ and $x^{(n)}$ such that $tx^{(n)}$ is summable. The norm in the space D define as $\|x\|_D = \sum\limits_{i=0}^{n-2} \max\limits_{a \le t \le b} \left|x^{(i)}(t)\right| + \int\limits_a^b \left|x^{(n-1)}(t)\right| dt + \int\limits_a^b t \left|x^{(n)}(t)\right| dt$. Considering this space D looks naturally when fractional equations with the Riemann–Liouville derivatives (1) and the boundary conditions (2) are taken into account. We say that $x \in D$ is a solution of (1)–(2) if it satisfies this equation for almost every $t \in [a,b]$.

References to well-known monographs, [23,29], can be consulted for the utilization of fractional differential equations in different scientific and engineering domains. The application of problems featuring integral boundary conditions is elucidated in [35] concerning phase field models. Additionally, [33] provides insight into the significance of integral boundary conditions for heat equations with nonlinear gradient source terms that exhibit temporal variability. Furthermore, [28] offers examples of applications in biomedical computational fluid dynamics. Recently, in [34] authors, used fractional differential equation with an integral boundary condition to describe the world population growth.

In the last few decades, analysis of positive solutions and investigation of various inequalities for fractional differential equations has been an active area of research. Qualitative theory for fractional differential equations such as oscillation theory, zeros of solutions, disconjugacy and comparison theory for fractional differential equations can be studied and many results were obtained based on various inequalities, for example, Lyapunov-type inequalities, De La Vallée-Poussin inequalities and Hartman-Wintner-type inequalities (see, for example [3,5,18,22,25,27,31,36]). There are various methods such as topological methods, coincidence degree theory, upper and lower solution method and different numerical methods which are used to study fractional differential equations with different fractional derivatives.

Let us note some of the recent work that motivated us to study problem (1)–(2). In [19], Ferreira obtained Lyapunov inequality for fractional boundary value problem

$$\begin{cases} (D^\alpha_{a+}x)(t) + q(t)x(t) = 0, a < t < b, \\ x(a) = 0, \quad x(b) = \lambda \int_a^b x(s)ds, \quad \lambda \ge 0. \end{cases}$$

In [15], Cabada and Hamdi, studied the problem

$$\begin{cases} (D^\alpha x)(t) + \mu g(t) f(x(t)) = 0, \quad 0 \le t \le 1, \\ x(0) = x'(0) = 0, \quad x(1) = \lambda \int_0^1 x(s)ds, \quad 0 < \lambda < \alpha \end{cases}$$

where μ is a real positive parameter. Recently, Caballero et al. [16], studied existence and uniqueness of positive solutions for problem

$$\begin{cases} (D^\alpha_{0+}x)(t) + f(t, x(t), (Hx)(t)) = 0, \quad 0 < t < 1, \\ x(0) = x'(0) = \ldots = x^{n-2}(0) = 0, \quad x(1) = \lambda \int_0^1 x(s)ds, \end{cases}$$

where $n \ge 3$, $n-1 < \alpha \le n$, $\lambda \in (0,\alpha)$, H is an operator applying $C[0,1]$ into itself and $f : (0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$. One can find other work for fractional boundary value problems with various integral boundary conditions in [1,2,13,14,20,21,32].

Another motivation presents a corresponding new step. In the studying system: using representation of solution for one of the components of the solution vectors of the system to a scalar functional differential equation for another component of the solution-vector. For functional differential equation with the standard derivatives this idea was formulated in [4]. It should be stressed that even in the case of ordinary differential system, an equation for a corresponding component is an integro-differential (i.e., can be considered as the functional differential) equations. This motivates us to study problem (1)–(2). In this article, the Vallée-Poussin theorem on differential inequality is applied to investigate problem (1)–(2). The study yields explicit criteria for determining the negativity of Green's function in the form of algebraic inequalities. Regarding functional differential equations of order n, similar to the Vallée-Poussin theorem, the literature [7–10] provides results on the analog of this theorem and the sign-constancy of Green's functions based on it. Recent research on fractional functional differential equations, involving an operator T of a general form, employs the Vallée-Poussin theorem, and this can be found in [11,12,17,30].

2 Preliminaries

Monographs [23,29] provide a comprehensive overview of fractional calculus. For basic definitions and lemmas which are used in this work, they can be found for example in the recent publication [17]. Here, we discuss Green's function and its sign constancy to understand our main results better.

Lemma 1 (see [19]) *Given $h \in C([a, b], \mathbb{R})$, $1 < \alpha \leq 2$ and $\lambda \in \mathbb{R}$ such that $\alpha - \lambda(b - a) \neq 0$, the unique solution of the fractional differential equation*

$$\begin{cases} (D^{\alpha}_{a+}x)(t) = h(t), \quad a < t < b, \\ x(a) = 0, \quad x(b) = \lambda \int_a^b x(s)ds, \end{cases}$$

is

$$x(t) = \int_a^b G_0(t, s)h(s)ds$$

where

$$G_0(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} -\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\left(1 + \frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right) + (t - s)^{\alpha-1}, \quad a \leq s \leq t \leq b, \\ -\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\left(1 + \frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right), \qquad a \leq t < s \leq b. \end{cases} \tag{3}$$

Lemma 2 *Let $a, b, \alpha, \lambda \in \mathbb{R}$ with $a < b$, $1 < \alpha \leq 2$ and $\lambda \geq 0$ such that $\alpha - \lambda(b - a) > 0$. Then $G_0(t, s) < 0$ for all $t, s \in (a, b)$.*

3 Main results

Let us define the operator $K : L_\infty \to L_\infty$ by the equality

$$(Kz)(t) = -T\left[\int_a^b G_0(t, s)z(s)ds\right](t). \tag{4}$$

We use here and below the notation $T[\gamma(t)]$ meaning that the operator T acts on the continuous function γ, i.e., $T[\gamma(t)] = (T\gamma)(t)$. We assume in this paper the positivity of operators in the standard sense, i.e., the operator K is positive if $(Kz)(t) \geq 0$ for $t \in [a, b]$ for every nonnegative $z \in L_\infty$.

The following assertion can be considered as an analog of the Vallée-Poussin theorem on differential inequality [9].

Theorem 1 *Let $T : C \to L_\infty$, be positive operator, $1 < \alpha \leq 2$. Then the following assertions are equivalent:*

1) *there exist a positive number ε and a function $v \in D$ such that $v(t) - \lambda \int_a^b v(s)ds > 0$ for $t \in (a, b)$, $v(a) = 0$ and*

$$(D^{\alpha}_{a+}v)(t) + Tv(t) \equiv \psi(t) \leq -\varepsilon < 0 \ \ for \ \ t \in (a, b); \tag{5}$$

2) *the spectral radius $\rho(K)$ of the operator $K : L_\infty \to L_\infty$ is less than 1;*
3) *problem (1)–(2), is uniquely solvable for any $f \in L_\infty$ and its Green's function $G(t, s)$ is negative for $(t, s) \in (a, b) \times (a, b)$.*

Proof Let us start with the proof of the implication 1) $\Rightarrow$ 2). Consider the auxiliary problem

$$\begin{cases}(D^{\alpha}_{a+}x)(t) = z(t),\\ x(a) = v(a) = 0,\ x(b) - \lambda\int_a^b x(s)ds = v(b) - \lambda\int_a^b v(s)ds = 0,\end{cases} \tag{6}$$

where z is a function from L_∞. It is clear that

$$x(t) = \int_a^b G_0(t,s)z(s)ds + u(t), \tag{7}$$

where $G_0(t,s)$ is defined by (3) and u is a solution of the homogeneous equation $(D^{\alpha}_{a+}u)(t) = 0$ satisfying the conditions $u(a) = v(a)$, $u(b) - \lambda\int_a^b u(s)ds = v(b) - \lambda\int_a^b v(s)ds$. Substituting this representation instead of v into (5) yields

$$z(t) + T\left[\int_a^b G_0(t,s)z(s)ds\right] + (Tu)(t) = \psi(t). \tag{8}$$

$$z(t) - (Kz)(t) = \psi(t) - Tu(t),\quad t\in(a,b), \tag{9}$$

Let us prove that $u(t) > 0$ for $t\in(a,b)$. From assertion 1) of Theorem 1 we have $v(a) = 0$. The function u satisfies the boundary value problem

$$\begin{cases}(D^{\alpha}_{a+}u)(t) = 0,\quad t\in[a,b],\\ u(a) = 0,\quad u(b) - \lambda\int_a^b u(s)ds = \beta,\end{cases} \tag{10}$$

where β is a corresponding nonnegative number. Assume that $u(t)$ changes its sign at the point a_1 such that the solution changes its sign at the point a_1, i.e., for example, $u(t)\le 0$ for $t\in(0,a_1)$ and there exists a set S with mes$S > 0$, where $u(t) < 0$. In this case, we obtain that u satisfies the boundary value problem

$$\begin{cases}(D^{\alpha}_{a+}u)(t) = 0,\quad t\in[a,a_1],\\ u(a) = 0,\quad u(a_1) - \lambda\int_a^{a_1} u(s)ds = 0.\end{cases} \tag{11}$$

It is clear that $u(t)\equiv 0$ for $t\in[a,a_1]$ according to Lemma 1. We come to the contradiction with the assumption $u(t) < 0$ for $t\in S$. Now it is clear that $u(t)\ge 0$ for $t\in[a,b]$. Thus, $\Psi(t)\le -\varepsilon < 0$. Thus, there exists $\delta > 0$ such that $z(t)\le -\delta$ for $t\in(a,b)$. Put $w(t) = -z(t)\ge\delta > 0$. Then $w(t) > (Kw)(t)$. Then, according to [24, Theorem 5.8 on p. 84], we get $\rho(K) < 1$. Analogously other cases where $u(t)$ can change its sign can be proven as impossible. This completes the proof of 1)$\Rightarrow$2).

Let us prove now the implication 2) $\Rightarrow$ 3). Consider the boundary value problem (1)–(2). Let us substitute

$$x(t) = \int_a^b G_0(t,s)z(s)ds, \tag{12}$$

where $G_0(t,s)$ is Green's function of the problem consisting of the equation

$$(D^{\alpha}_{a+}x)(t) = z(t) \tag{13}$$

with the boundary conditions $x(a) = x(b) - \lambda\int_a^b x(s)ds = 0$. Substituting representation (12) into (1), we get (9), where $\psi(t) = f(t)$ and $u(t)\equiv 0$ for $t\in[a,b]$. If $\rho(K) < 1$, then (9) is uniquely solvable and its solution is

$$z(t) = (I-K)^{-1}\psi(t) = ((I + K + K^2 + K^3 + \ldots)\psi)(t). \tag{14}$$

We obtain that the solution x defined by (12) exists and is unique, and this proves that the problem (1)–(2) is uniquely solvable. We see also that $(I-K)^{-1}$ is a positive operator if K is positive. The assumption about positivity of the operator $T : C\to L_\infty$ and Lemma 1 about negativity of $G_0(t,s)$ imply that $K : L_\infty\to L_\infty$ is positive. Then from $f(t)\le 0$, it follows that $z(t)\le 0$ for $t\in[a,b]$. Thus, if $f(t)\le 0$, then $z(t)\le 0$ for

$t \in [a, b]$. If $z(t) \leq 0$, then from the fact of nonpositivity of Green's function $G_0(t, s)$ in the formula (12), we get $x(t) \geq 0$ for $t \in [a, b]$. From

$$\begin{aligned} x(t) &= \int_a^b G_0(t, s)z(s)ds \\ &= \int_a^b G_0(t, s)(I - K)^{-1} f(s)ds \\ &= \int_a^b G_0(t, s)[I + K + K^2 + K^3 + \ldots] f(s)ds \end{aligned}$$

positivity of operator K and the fact that $G_0(t, s) < 0$ according to Lemma 1, it follows that $G(t, s) \leq G_0(t, s) < 0$ for $t, s \in (a, b)$. This completes the proof of the implication 2) ⇒ 3).

In order to prove the implication 3) ⇒ 1), we set

$$v(t) = -\int_a^b G(t, s)ds. \tag{15}$$

Since $G(t, s) < 0$ for $(t, s) \in (a, b)$, we get $v(t) > 0$ for $t \in (a, b)$. It is clear that $v(a) = v(b) - \lambda \int_a^b v(s)ds = 0$, $\psi(t) = -1$. This completes the proof of the implication 3)⇒1).

The proof of Theorem 1 is now complete. □

Corollary 2 *Assume that* $1 < \alpha \leq 2$ *and*

$$T\left[(t-a)^{\alpha-1}\left(\left(\frac{(b-a)^{\alpha} - \lambda\left(\frac{(b-a)^{\alpha+1}}{\alpha+1}\right)}{(b-a)^{\alpha-1} - \lambda\left(\frac{(b-a)^{\alpha}}{\alpha}\right)}\right) - (t-a)\right)\right] < \Gamma(\alpha+1), \tag{16}$$

then problem (1)–(2) *is uniquely solvable for any* $f \in L_\infty$ *and its Green's function* $G(t, s)$ *is negative for* $t, s \in (a, b)$.

Proof Consider the auxiliary equation

$$\begin{cases} (D^{\alpha}_{a+}v)(t) = -\delta, \\ v(a) = 0, \quad v(b) - \lambda \int_a^b v(s)ds = 0, \quad \lambda \geq 0, \end{cases} \tag{17}$$

with $\lambda \geq 0$ and $\delta > 0$. Using Lemma 1 of [17], the solution v of the first equation in (17) can be given as

$$v(t) = c_1(t-a)^{\alpha-1} - \frac{\delta}{\Gamma(\alpha+1)}(t-a)^{\alpha}. \tag{18}$$

In order to ensure the inequality, it is necessary to establish a connection between c_1 and δ.

$$v(t) > 0. \tag{19}$$

To proceed with the proof, let's provide a more elaborate explanation of the concept referred to as the "connection." It is evident that the condition (19) is satisfied when c_1 is sufficiently large. However, to attain the inequality (5) from (18), we require c_1 to be sufficiently small. Hence, we have to select the smallest value of c_1 that ensures the fulfillment of inequality (19). Using the boundary condition $v(b) - \lambda \int_a^b v(s)ds = 0$, we obtain

$$c_1\left((b-a)^{\alpha-1} - \lambda\frac{(b-a)^{\alpha}}{\alpha}\right) + \frac{\lambda}{\Gamma(\alpha+1)}\left[(b-a)^{\alpha} - \lambda\frac{(b-a)^{\alpha+1}}{\alpha+1}\right] = 0$$

which gives

$$c_1 = \frac{\delta}{\Gamma(\alpha+1)}\left(\frac{(b-a)^{\alpha} - \lambda\left(\frac{(b-a)^{\alpha+1}}{\alpha+1}\right)}{(b-a)^{\alpha-1} - \lambda\left(\frac{(b-a)^{\alpha}}{\alpha}\right)}\right)$$

and putting this c_1 back in $v(t)$, we get

$$v(t)=\frac{\delta}{\Gamma(\alpha+1)}\left[\left(\frac{(b-a)^{\alpha}-\lambda\left(\frac{(b-a)^{\alpha+1}}{\alpha+1}\right)}{(b-a)^{\alpha-1}-\lambda\left(\frac{(b-a)^{\alpha}}{\alpha}\right)}\right)(t-a)^{\alpha-1}-(t-a)^{\alpha}\right].$$

Hence, taking into account $(D^{\alpha}_{a+}v)(t)=-\delta$ and (16), we find that assertion 1) of Theorem 1 holds. As this is equivalent to assertion 3) of Theorem 1, the proof is complete. □

Let us consider the equation where the operator T in form of deviations

$$(D^{\alpha}_{a+}x)(t)+\sum_{j=1}^{m}q_j(t)x(h_j(t))=f(t),\quad t\in(a,b), \tag{20}$$

where

$$x(\xi)=0,\quad \xi\notin(a,b), \tag{21}$$

$q_j, f\in L_\infty$, h is a measurable function. We obtain the following assertion.

Corollary 3 *Let $q(t)\geq 0$, $1<\alpha\leq 2$ and assume*

$$\sum_{j=1}^{m}\chi(a<h_j(t)<b)q_j(t)(h_j(t)-a)^{\alpha-1}\times\left(\left(\frac{(b-a)^{\alpha}-\lambda\left(\frac{(b-a)^{\alpha+1}}{\alpha+1}\right)}{(b-a)^{\alpha-1}-\lambda\left(\frac{(b-a)^{\alpha}}{\alpha}\right)}\right)-(h_j(t)-a)\right)<\Gamma(\alpha+1),$$
$$t\in(a,b),$$

where

$$\chi(a<h_j(t)<b)=\begin{cases}1, & h_j(t)\in(a,b),\\ 0, & h_j(t)\notin(a,b).\end{cases}$$

Then the problem consisting of equation (20) and the boundary conditions (2) is uniquely solvable for any $f\in L_\infty$ and its Green's function $G(t,s)$ is negative for $t,s\in(a,b)$.

4 Remarks and examples

Consider the particular case of (20) given as

$$\begin{cases}(D^{\alpha}_{a+}x)(t)+q(t)x(h(t))=f(t), & t\in[a,b],\\ x(a)=0,\ \ x(b)-\lambda\int_a^b x(s)ds=0,\end{cases} \tag{22}$$

where $1<\alpha\leq 2$, $q(t)\geq 0$ for $t\in[a,b]$.

Corollary 4 *If $1<\alpha\leq 2$ and*

$$q(t)\left[(h(t)-a)^{\alpha-1}\left(\left(\frac{(b-a)^{\alpha}-\lambda\left(\frac{(b-a)^{\alpha+1}}{\alpha+1}\right)}{(b-a)^{\alpha-1}-\lambda\left(\frac{(b-a)^{\alpha}}{\alpha}\right)}\right)-(h(t)-a)\right)\right]<\Gamma(\alpha+1), \tag{23}$$

is fulfilled, then problem (22) is uniquely solvable for any $f\in L_\infty$ and its Green?s function is negative for $(t,s)\in(a,b)\times(a,b)$.

Table 1 Calculated RHS of Inequality (28) with Varied ε Values

ε	$\frac{\Gamma(\alpha+1)}{(H-\varepsilon)\varepsilon^{\alpha-1}}$
0.09	1.76539
0.009	5.40813
0.005	7.24459
0.001	16.17446
0.0005	22.86974
0.0001	51.13043
0.00001	161.68301

For Simplicity, we now denote $H = \left(\frac{(b-a)^{\alpha} - \lambda\left(\frac{(b-a)^{\alpha+1}}{\alpha+1}\right)}{(b-a)^{\alpha-1} - \lambda\left(\frac{(b-a)^{\alpha}}{\alpha}\right)} \right)$.

We observe that the denominator of H is always nonzero. When we rewrite the expression as $(b-a)^{\alpha-1} - \lambda\frac{(b-a)^{\alpha}}{\alpha} = \frac{(b-a)^{\alpha-1}}{\alpha}\left[\alpha - \lambda(b-a)\right]$, it is clear that $(b-a) > 0$ and $\alpha > 1$. Consequently, from Lemma 2, $\alpha - \lambda(b-a) > 0$, which clarifies why the denominator is never zero.

Remark 1 Taking into account that $\max\limits_{a \le t \le b} (t-a)^{\alpha-1}(H-(t-a)) = \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}} H^{\alpha}$, which is achieved when $t = a + \frac{(\alpha-1)}{\alpha}H$, we get

$$q(t) < \frac{\alpha^{\alpha}}{(\alpha-1)^{\alpha-1}H^{\alpha}}\Gamma(\alpha+1). \tag{24}$$

Remark 2 Inequality (24) generalizes corresponding results obtained in [17] which follows in the case of $a = 0,\ b = 1$, and $\lambda = 0$.

Remark 3 Let's provide an illustration demonstrating inequality (24) cannot be improved. We consider the case $a = 0,\ b = 1$, resulting in $H_{01} = \frac{\alpha(\alpha-1-\lambda)}{(\alpha+1)(\alpha-\lambda)}$. Assume that $h(t) = a + \frac{\alpha-1}{\alpha}H_{01}$ in (22), we examine the equation

$$(D^{\alpha}_{0+}x)(t) + \frac{\alpha^{\alpha}}{(\alpha-1)^{\alpha-1}(H_{01})^{\alpha}}\Gamma(\alpha+1)x\left(a + \frac{\alpha-1}{\alpha}H_{01}\right) = 0, \quad t \in [a,b], \quad 1 < \alpha \le 2. \tag{25}$$

The function

$$x(t) = (t^{\alpha-1}H_{01} - t^{\alpha}) \tag{26}$$

satisfies this equation (25) and boundary conditions

$$x(0) = 0, \quad x(1) - \lambda\int_0^1 x(s)ds = 0, \tag{27}$$

and problem (25), (27) has infinite number of solutions of the form $(t^{\alpha-1}H_{01} - t^{\alpha})$ for every real number c.

Corollary 5 *If $1 < \alpha \le 2$, $h(t) - a < \varepsilon$ for $a \le t \le b$, then the inequality*

$$q(t) < \frac{\Gamma(\alpha+1)}{(H-\varepsilon)(\varepsilon)^{\alpha-1}} \tag{28}$$

implies the unique solvability for any $f \in L_{\infty}$ of the problem (22) and negativity of its Green's function.

Example 1 If we take $\varepsilon = 0.09,\ 0.009,\ 0.005,\ 0.001,\ 0.0005,\ 0.0001,\ 0.00001, \lambda = 1.2,\ \alpha = 1.5$, in inequality (28) of Corollary 5, then the right-hand sides (RHS) of inequality (28) calculated in Table 1 and represented in figure 1.

We see that $q < 1.76539$ for $\varepsilon = 0.09$ and achieves $q < 161.68301$ for $\varepsilon = 0.00001$.

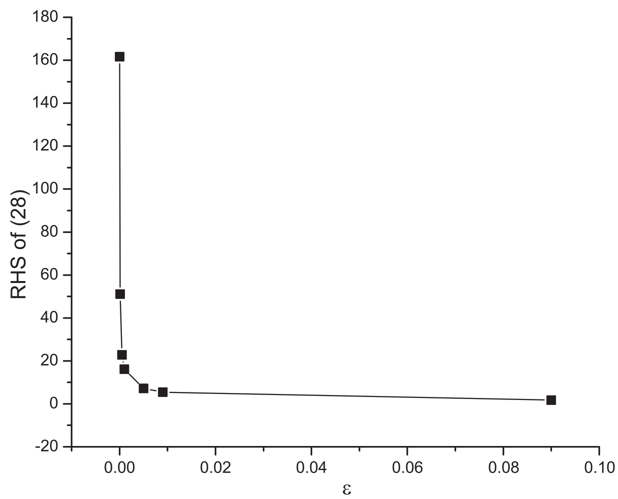

Fig. 1 Graphical Analysis of RHS in Inequality (28) with Varying ε

5 Conclusion

This paper considers a fractional functional differential equation with an integral boundary condition. The equation involves a Riemann-Liouville fractional derivative and a linear continuous operator T. We provide the necessary and sufficient conditions for negativity of Green's function in the form of Vallée-Poussin theorem on differential inequalities. We derive explicit sufficient tests for the negativity of the Green's function and discuss the application of their technique to various cases of functional differential equations. The results obtained in this paper extend previous inequalities available in the literature for fractional boundary value problems [17].

Next steps can be in numerical methods and computational techniques for solving fractional functional differential equations with integral boundary conditions and investigating the convergence, accuracy, and stability of these methods are important areas for their practical implementation. It can be stressed that assertion on positivity/negativity of Green's functions which are used as a basis of so-called Chaplygin's integration method were discovered in the well-known paper by N. Luzin [26]. This direction of future research can contribute to the advancement of the theory and applications of fractional functional differential equations with integral boundary conditions, which will further help to understand important mathematical models.

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References

1. Abbas, M. I., Fečkan, M.: Investigation of an implicit Hadamard fractional differential equation with Riemann-Stieltjes integral boundary condition. Mathematica Slovaca, **72**(4), 925-934 (2022)
2. Agarwal, R. P., Baleanu, D., Hedayati, V., Rezapour, S.: Two fractional derivative inclusion problems via integral boundary condition. Applied Mathematics and Computation, **257**, 205-212 (2015)
3. Agarwal, R. P., Bohner, M., Özbekler, A.: Lyapunov inequalities and applications. Cham, Switzerland: Springer (2021)
4. Agarwal, R. P., Domoshnitsky, A.: On positivity of several components of solution vector for systems of linear functional differential equations. Glasgow Mathematical Journal, **52**(1), 115-136 (2010)
5. Akgö, S. D., Özbekle, A.: De la Vallée Poussin inequality for impulsive differential equations. Mathematica Slovaca, **71**(4), 881-888 (2021)
6. Azbelev, N. V., Rakhmatullina, L.: Introduction to the theory of functional differential equations: methods and applications (Vol. 3). Hindawi Publishing Corporation. (2007)
7. Azbelev, N. V., Domoshnitskii, A. I.: A question concerning linear-differential inequalities. 1. Differential Equations, **27**(3), 257-263 (1991)
8. Azbelev, N. V., Domoshnitskii, A. I.: A question concerning linear-differential inequalities. 2. Differential Equations, **27**(6), 641-647 (1991)
9. Azbelev, N. V., Domoshnitskii, A. I.: On de la Vallée Poussin's differential inequality. Differentsial'nye Uravneniya, **22**(12), 2041-2045 (1986)

10. Berezansky, L., Domoshnitsky, A., Koplatadze, R.: Oscillation, nonoscillation, stability and asymptotic properties for second and higher order functional differential equations. CRC Press, (2020)
11. Bohner, M., Domoshnitsky, A., Litsyn, E. Padhi, S., Srivastava, S. N.: Vallée-Poussin theorem for Hadamard fractional functional differential equations. Applied Mathematics in Science and Engineering **31**(1), 2259057 (2023)
12. Bohner, M., Domoshnitsky, A., Padhi, S., Srivastava, S. N.: Vallée-Poussin theorem for equations with Caputo fractional derivative. Mathematica Slovaca, **73**(3), 713-728 (2023)
13. Bourguiba, R., Toumi, F., Wanassi, O. K.: Existence and nonexistence results for a system of integral boundary value problems with parametric dependence. Filomat, **34**(13), 4453-4472 (2020)
14. Cabada, A., Hamdi, Z.: Nonlinear fractional differential equations with integral boundary value conditions. Applied Mathematics and Computation, **228**, 251-257 (2014)
15. Cabada, A., Hamdi, Z.: Multiplicity results for integral boundary value problems of fractional order with parametric dependence. Fractional Calculus and Applied Analysis, **18**, 223-237 (2015)
16. Caballero, J., Harjani, J., Sadarangani, K.: Existence and Uniqueness of Positive Solutions to a Class of Singular Integral Boundary Value Problems of Fractional Order. Mediterranean Journal of Mathematics, **20**(2), 86 (2023).
17. Domoshnitsky, A., Padhi, S., Srivastava, S. N.: Vallée-Poussin theorem for fractional functional differential equations. Fractional Calculus and Applied Analysis, **25**(4), 1630-1650 (2022)
18. Ferreira, R.: A Lyapunov-type inequality for a fractional boundary value problem. Fractional calculus and applied analysis, **16**(4), 978-984 (2013)
19. Ferreira, R. A.: Lyapunov inequalities for some differential equations with integral-type boundary conditions. Advances in Mathematical Inequalities and Applications, 59-70 (2018)
20. Graef, J. R., Kong, L., Kong, Q., Wang, M.: Uniqueness of positive solutions of fractional boundary value problems with non-homogeneous integral boundary conditions. Fractional Calculus and Applied Analysis, **15**, 509-528 (2012)
21. Graef, J. R., Kong, L., Kong, Q., Wang, M.: Fractional boundary value problems with integral boundary conditions. Applicable Analysis, **92**(10), 2008-2020 (2013)
22. Jleli, M., O'Regan, D., Samet, B.: Lyapunov-type inequalities for coupled systems of nonlinear fractional differential equations via a fixed point approach. Journal of Fixed Point Theory and Applications, **21**, 1-15 (2019)
23. Kilbas, A. A., Srivastava, H. M., Trujillo, J. J.: Theory and applications of fractional differential equations (Vol. 204). Elsevier (2006)
24. Krasnosel'skii, M.A., Vainikko, G.M., Zabreyko, R.P., Ruticki, Y.B., Stet?senko, V.V.: Approximate Solution of Operator Equations. Springer, Dordrecht (2012)
25. Łupińska, B., Odzijewicz, T.: A Lyapunov?type inequality with the Katugampola fractional derivative. Mathematical Methods in the Applied Sciences, **41**(18), 8985-8996 (2018)
26. Luzin, N. N.: On the method of approximate integration of academician SA Chaplygin. Uspekhi matematicheskikh nauk, **6**(6), 3-27 (1951)
27. Mitrinovic, D. S., Pecaric, J., Fink, A. M.: Inequalities involving functions and their integrals and derivatives (Vol. 53). Springer Science & Business Media, (1991)
28. Nicoud, F., Schönfeld, T.: Integral boundary conditions for unsteady biomedical CFD applications. International Journal for Numerical Methods in Fluids, **40**(3-4), 457-465 (2002).
29. Podlubny, I.: Fractional Differential Equations. Academic, San Diego (1999)
30. Srivastava, S. N., Domoshnitsky, A., Padhi, S., Raichik, V.: Unique solvability of fractional functional differential equation on the basis of Vallée-Poussin theorem. Archivum Mathematicum, **59**(1), 117-123 (2023).
31. Srivastava, S. N., Pati, S., Padhi, S., Domoshnitsky, A.: Lyapunov inequality for a Caputo fractional differential equation with Riemann-Stieltjes integral boundary conditions. Mathematical Methods in the Applied Sciences, **46**, 13110-13123, (2023).
32. Sun, Y., Sun, Y.: Positive solutions and monotone iterative sequences for a fractional differential equation with integral boundary conditions. Advances in Difference Equations, **2014**(1), 1-10 (2014)
33. Szymanska-Debowska, K.: On the existence of solutions for nonlocal boundary value problems. Georgian Mathematical Journal, **22**(2), 273-279 (2015)
34. Wanassi, O. K., Torres, D. F.: An integral boundary fractional model to the world population growth. Chaos, Solitons & Fractals, **168**, 113151 (2023)
35. Xu, X., Zhang, L., Shi, Y., Chen, L. Q., Xu, J.: Integral boundary conditions in phase field models. Computers & Mathematics with Applications, **135**, 1-5 (2023)
36. Zohra, B. F., Benaouda, H., Mokhtar, K.: Lyapunov-and Hartman-Wintner-type inequalities for a nonlinear fractional BVP with generalized Ψ-Hilfer derivative. Mathematical Methods in the Applied Sciences, **44**(3), 2637-2649 (2021)