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ORIGINAL RESEARCH

Advancing convergence analysis: extending the scope of a sixth order method

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Abstract In this article, we aim to emphasize the critical role of extended convergence analysis in advancing research and understanding in the interdisciplinary fields of Applied and Computational Mathematics, Physics, Engineering, and Chemistry. By gaining a comprehensive understanding of the convergence behavior of numerical methods, one can make informed decisions regarding algorithm selection, optimization, and convergence domains, leading to more accurate and reliable scientific results in diverse applications. The conventional approach to assessing the convergence order of higher order methods for solving systems of non-linear equations relied on the Taylor series expansion, necessitating the computation of higher order derivatives that were typically absent in the method. This limitation not only constrained the method's applicability but also increased the computational cost of solving the problem. In contrast, our study introduces a unique and innovative approach, where we demonstrate the improvised convergence of the method using only first order derivatives. Our new method offers several advantages over the traditional approach, providing valuable information regarding the radii of the convergence region and precise estimates of error boundaries. Furthermore, we establish the notion of semi-local convergence, which proves to be particularly significant as it allows for the identification of the specific domain in which the iterates converge. We have validated the convergence requirements through carefully selected numerical examples.

Keywords High-order methods · Nonlinear equations · Convergence analysis · Banach space

Mathematics Subject Classification 37N30 · 47J25 · 49M15 · 65H10 · 65J15

1 Introduction

A great deal of problems in the Computational Sciences [1–8] can be simplified with the help of tools in mathematical modelling into the form

$$J(u) = 0 \tag{1}$$

where J is a continuously Fréchet differentiable non-linear operator defined on an open convex subset Λ of a Banach space W_1 with images in another Banach space W_2.

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It's quite unlikely to find a direct closed-form solution to (1) most of the time. This signified the development of iterative techniques for resolving equations with nonlinear coefficients of the kind (1). Most iterative techniques created to date are built on the Newton-Raphson approach [2–9]. If the starting point t_0 is close enough to the root, it quadratically converges to the solution. Researchers then developed higher order methods including Euler's, Chebyshev's, Halley's, and Super Halley's [2–4] for faster convergence, but these methods were not very useful in practise because they required computing the second derivative, which increased the cost of evaluation. Analysts developed the concept of higher order multi-point approaches to address this disadvantage. Additionally, many multi-step techniques of higher order in the style of Newton were developed at the expense of auxiliary functions, derivatives, and change at iteration points [1,6–16].

Local and semi-local convergence are the two main categories typically used to categorise studies about the characteristics of various iterative methods [6–8,10–16]. The semi-local convergence analysis is built on the information surrounding a starting point and provides some conditions that guarantee the iterative process' convergence. While the local convergence is built on the information around a solution, estimating the radii of the convergence domain.

The convergence domain, error estimates, and uniqueness of the solution are all examined in these kinds of studies. The findings from these studies are important because they broaden the iterative methods' application space.

In this research article, we look into the local and semi-local convergence aspects of a sixth order Newtonian approach for solving (1). Here, in a Banach space region, the following method suggested by Kalyanasundaram et al [1] is taken into consideration:

$$\begin{aligned} w_k &= t_k - \frac{2}{3}J'(t_k)^{-1}J(t_k),\\ s_k &= t_k - Z_1 J'(t_k)^{-1}J(t_k),\\ t_{k+1} &= s_k - Z_2 J'(t_k)^{-1}J(s_k),\\ \text{where,}\quad Z_1 &= 1 - \frac{3}{4}(\rho(t_k,w_k)-1) + \frac{9}{8}(\rho(t_k,w_k)-1)^2,\\ Z_2 &= 1 - \frac{3}{2}(\rho(t_k,w_k)-1) + \frac{1}{2}(\rho(t_k,w_k)-1)^2\\ \text{and}\quad \rho(t_k,w_k) &= J'(t_k)^{-1}J'(w_k). \end{aligned} \tag{2}$$

for $t_0 \in \Lambda$ and $\forall\, k = 0, 1, 2, \cdots$

The equivalent approach to (2) is demonstrated in [1] to have an order of convergence of six and have been proven using Taylor series expansions with hypotheses on the derivatives of J up to seventh order (not included in the method). This limits the scheme's scope of application. Furthermore, even in the absence of these higher order derivatives, iterates in the algorithm (2) may converge. By defining J on $\Lambda = [-0.5, 1.5]$, consider a model example with

$$J(u) = \begin{cases} u^3\ln(u) + 3u^5 - 3u^4, & \text{if } u \neq 0\\ 0, & \text{if } u = 0 \end{cases}$$

In Λ, we can see that the solution $\xi = 1$ and the third derivative can be calculated as

$$J'''(u) = 11 - 72u + 180u^2 + 6\ln(u).$$

The fact that J''' is unbounded on Λ is clearly visible. Therefore, adopting the findings in [1] may not necessarily guarantee convergence.

Higher order derivatives not included in the method are needed to demonstrate the convergence using Taylor's series. Furthermore, the initial point is chosen at random, and in studies of this kind, the conclusions about the convergence domain are not explored. Additionally, there was no mention of error estimates or the uniqueness of the solution. This is where the importance of our current study of convergence properties comes into play, as it uses simply the first order derivative to demonstrate convergence regardless of the approach.

The estimation of solutions to non-linear equations defined on the domain $\mathbb{R}$ is the focus of a plethora of iterative methods [2–6,8]. All of these techniques call for the beginning point t_0 to be very close to the solution ξ of (1) so that the method converges to ξ. But here, the following queries surface:

(i) What is the shortest distance that can exist between t_0 and ξ for the method to converge on ξ?

(ii) Can we specify the range of convergence where the iterates converges?
(iii) Can we guarantee that the sequence of iterates converges?

In this study, we introduce hypotheses based solely on the first derivatives of the function, with a focus on all the problems raised above. Also, Lipschitz conditions, which only require first order derivatives, are used to derive the convergence of the schemes instead of Taylor's series expansion. As a result, the numerical schemes' relevance is enhanced. We also give estimates for the domain of convergence's radii, upper bounds for error estimates, and findings demonstrating the root's uniqueness in the obtained domain of convergence. This strategy can be used with other iterative techniques.

This article's content is organised as follows: In Sect. 3, local convergence analysis is provided. In Sect. 4, majorizing sequences are presented and discussed for the Semi-Local Convergence Analysis of (2). Sect. 5 contains numerical applications that make use of the convergence findings developed in the earlier sections. This article is concluded by the remarks in Sect. 6.

2 Basic concepts

For the reader's understanding, we give the statement of Banach Lemma on Invertible Operators and also the definition of a majorizing sequence, which is continuously utilized in the subsequent theorems and proofs.

Theorem 2.1 (Banach Lemma on Invertible Operators) *[4] "If W is a bounded linear operator in $\mathcal{X}$, then W^{-1} exists if and only if there is another bounded linear operator B in $\mathcal{X}$ such that B^{-1} exists and*

$$\|I - BW\| < 1. \tag{3}$$

If W^{-1} exists, then we can write

$$W^{-1} = \sum_{k=0}^{\infty}(I - BW)^k B \quad (\textit{Neumann Series}) \tag{4}$$

and

$$\|W^{-1}\| \leq \frac{\|B\|}{1 - \|I - BW\|}." \tag{5}$$

Definition 2.2 [6] "Consider a sequence $\{x_k\}$ in a normed space $\mathcal{X}$. If there exists a nonnegative sequence $\{v_k\}$ such that for all $k \geq 0$, the inequality

$$\|x_{k+1} - x_k\| \leq \|v_{k+1} - v_k\|$$

holds, then $\{v_k\}$ is a **majorizing sequence** for $\{x_k\}$".

3 Convergence type 1: local

This section delves into the local convergence analysis of the sixth-order iterative method. It focuses on establishing the conditions under which the method converges locally, determining the convergence domain and radii, and proving the uniqueness of the solution within this domain. A detailed examination of the convergence behavior in the vicinity of the solution is provided.

Introduction of a number of functions and constants plays a part in the analysis of local convergence of the algorithm (2). Set $V = [0, +\infty)$.
Assume:

(P_1) There exist continuous and non-decreasing function $\eta_0 : V \to \mathbb{R}$ such that the equation

$$\eta_0(u) - 1 = 0$$

have a minimal solution in the interval $V - \{0\}$. Denote by μ_0 the smallest such zero. Set $V_0 = [0, \mu_0)$.

(P_2) There exist continuous and non-decreasing function $\eta : V_0 \to V$ such that the equations

$$\Delta_m(u) - 1 = 0, \quad m = 1, 2, 3$$

have minimal solutions denoted by $\kappa_m \in V_0 - \{0\}$, respectively, where

$$\Delta_1(u) = \frac{\int_0^1 \eta((1-\lambda)u)d\lambda + \frac{1}{3}(1 + \int_0^1 \eta_0(\lambda u))d\lambda}{1 - \eta_0(u)},$$

$$\Delta_2(u) = \left[\frac{\int_0^1 \eta((1-\lambda)u)d\lambda}{1-\eta_0(u)} + \left(\frac{3}{4}\left(\frac{\bar{\eta}(u)}{1-\eta_0(u)}\right) + \frac{9}{8}\left(\frac{\bar{\eta}(u)}{1-\eta_0(u)}\right)^2\right)\frac{1+\int_0^1 \eta_0(\lambda u)}{1-\eta_0(u)}\right]u$$

and

$$\Delta_3(u) = \left[\frac{\int_0^1 \eta((1-\lambda)\Delta_2(u)u)d\lambda}{1-\eta_0(\Delta_2(u)u)} + \frac{\bar{\bar{\eta}}(u)}{(1-\eta_0(u))(1-\eta_0(\Delta_2(u)u))}[1 + \int_0^1 \eta_0(\lambda\Delta_2(u)u)d\lambda] + \frac{1}{2}[3\frac{\bar{\eta}(u)}{(1-\eta_0(u))} + (\frac{\bar{\eta}(u)}{(1-\eta_0(u))})^2]\frac{1+\int_0^1 \eta_0(\lambda\Delta_2(u)u)}{1-\eta_0(u)}d\lambda\right]\Delta_2(u),$$

where

$$\bar{\eta}(u) = \begin{cases} \eta((1+\Delta_1(u))u) \\ \text{or} \\ \eta_0(u) + \eta_0(\Delta_1(u)u) \end{cases},$$

$$\bar{\bar{\eta}}(u) = \begin{cases} \eta((1+\Delta_2(u))u) \\ \text{or} \\ \eta_0(u) + \eta_0(\Delta_2(u)u) \end{cases}.$$

Set

$$\kappa = min\{\kappa_m\} \tag{6}$$

and $V_1 = [0, \kappa)$.
Then, the constant κ is a radius of convergence for the method (2) (see Theorem 3.1). Notice that these definitions imply that for each $u \in V_1$

$$0 \le \eta_0(u) < 1 \tag{7}$$

and

$$0 \le \Delta_m(u) < 1. \tag{8}$$

The following local convergence hypotheses are linked to the aforementioned functions:

(P_3) For each $u \in \Lambda$

$$\|J'(\xi)^{-1}(J'(u) - J'(\xi))\| \le \eta_0(\|u - \xi\|);$$

Set $\Lambda_0 = \Lambda \cap U(\xi, \mu_0)$;

(P_4)

$$\|J'(\xi)^{-1}(J'(u) - J'(v))\| \le \eta(\|u - v\|)$$

for each $u, v \in \Lambda_0$.
and

(P_5) $U[\xi, \kappa] \subset \Lambda$.

The local convergence analysis of the method (2) is then proposed using the terms defined previously and the conditions (P_1)-(P_5). Define $p_i = \|x_i - \xi\|$.

In the following theorem, we establish the conditions under which the sixth-order iterative method converges locally. We will determine the convergence domain and radii. The proposition that follows the theorem proves the uniqueness of the solution in the obtained convergence domain.

Theorem 3.1 *Assume that t_0 is in $U(\xi,\kappa)-\xi$ and that the conditions (P_1)-(P_5) are true. The following things then hold:*

$$\{t_i\} \subset U(\xi,\kappa), \tag{9}$$
$$\|w_i-\xi\| \leq \Delta_1(p_i)p_i \leq p_i \leq \kappa, \tag{10}$$
$$\|s_i-\xi\| \leq \Delta_2(p_i)p_i \leq p_i, \tag{11}$$
$$p_{i+1} \leq \Delta_3(p_i)p_i \leq p_i \tag{12}$$

and the sequence $\{t_i\}$ is convergent to the solution ξ of the equation $J(u)=0$.

Proof By induction, the items (9)-(12) are validated. Choose x arbitrarily inside $U(\xi,\kappa)-\{\xi\}$. By putting the conditions (P_1), (P_3), the definition (6), and the estimate (7) into practise, it follows that

$$\|J'(\xi)^{-1}(J'(x)-J'(\xi))\| \leq \eta_0(\|x-\xi\|) \leq \eta_0(\kappa) < 1.$$

The invertibility of the operator J is implied by the previous estimate as well as the standard lemma on linear operators and their invertibility attributed to Banach [2–5] and hence we have

$$\|J'(x)^{-1}J'(\xi)\| \leq \frac{1}{1-\eta_0(\|x-\xi\|)}. \tag{13}$$

In particular, if $i=0$, the estimate (13) holds for $x=t_0$ and the iterate w_0, s_0, t_1 is clearly characterised by the method's (2) sub-steps. Additionally, one may write in turn that using the first sub-step

$$\begin{aligned} w_0-\xi &= t_0-\xi-J'(t_0)^{-1}J(t_0)+\frac{1}{3}J'(t_0)^{-1}J(t_0) \\ &= -(J'(t_0)^{-1}J'(\xi))\int_0^1 J'(\xi)^{-1}[J'(\xi+\lambda(t_0-\xi))-J'(t_0)]d\lambda(t_0-\xi) \\ &\quad +\frac{1}{3}(J'(t_0)^{-1}J'(\xi))\int_0^1 J'(\xi)^{-1}J'(\xi+\lambda(t_0-\xi))d\lambda(t_0-\xi). \end{aligned} \tag{14}$$

Using (6), (8) (for $m=1$), (13) (for $x=t_0$), (P_4), and (14), we can derive

$$\begin{aligned} \|w_0-\xi\| &\leq \frac{\int_0^1 \eta((1-\lambda)p_0)d\lambda+|\frac{1}{3}|(1+\int_0^1 \eta_0(\lambda p_0)d\lambda)}{1-\eta_0(p_0)}p_0 \\ &\leq \Delta_1(p_0)p_0 \\ &\leq p_0 < \kappa. \end{aligned} \tag{15}$$

Therefore, (10) holds true for $i=0$ and $w_0 \in U(\xi,\kappa)$. We may now formulate the second sub-step of (2) as,

$$\begin{aligned} s_0-\xi &= t_0-\xi-J'(t_0)^{-1}J(t_0)-\left[-\frac{3}{4}(J'(t_0)^{-1}J'(w_0)-1)+\frac{9}{8}(J'(t_0)^{-1}J'(w_0)-1)^2\right]J'(t_0)^{-1}J(t_0) \\ &= t_0-\xi-J'(t_0)^{-1}J(t_0)-\left[-\frac{3}{4}(J'(t_0)^{-1}(J'(w_0)-J'(t_0)))+\frac{9}{8}(J'(t_0)^{-1}(J'(w_0)-J'(t_0)))^2\right] \\ &\quad J'(t_0)^{-1}J(t_0). \end{aligned} \tag{16}$$

Using (6), (8) (for $m=2$), (13) (for $x=t_0, s_0$), (15) and (16), we have the following

$$\begin{aligned} \|s_0-d\| &\leq \left[\frac{\int_0^1 \eta((1-\lambda)p_0)d\lambda}{1-\eta_0(p_0)}+\left(\frac{3}{4}\left(\frac{\bar{\eta}(p_0)}{1-\eta_0(p_0)}\right)+\frac{9}{8}\left(\frac{\bar{\eta}(p_0)}{1-\eta_0(p_0)}\right)^2\right)\frac{1+\int_0^1 \eta_0(\lambda p_0)d\lambda}{1-\eta_0(p_0)}\right]p_0 \\ &\leq \Delta_2(p_0)p_0 \leq p_0. \end{aligned} \tag{17}$$

As a result, (11) is valid for $i=0$ and $s_0 \in U(\xi,\kappa)$ and also $J'(s_0)^{-1}$ exists. We may now form the final sub-step of (2) as,

$$\begin{aligned} t_1 - \xi = s_0 - \xi - J'(s_0)^{-1}J(s_0) + J'(s_0)^{-1}(J'(t_0) - J'(s_0))J'(t_0)^{-1}J(s_0) \\ - \left[-\frac{3}{2}(J'(t_0)^{-1}J'(w_0) - 1) + \frac{1}{2}(J'(t_0)^{-1}J'(w_0) - 1)^2 \right] J'(t_0)^{-1}J(s_0) \end{aligned}$$

and as a result, we have

$$\begin{aligned} p_1 \leq & \left[\frac{\int_0^1 \eta(\|(1-\lambda)(s_0 - \xi)\|)d\lambda}{1 - \eta_0(\|s_0 - \xi\|)} + \frac{\bar{\bar{\eta}}(p_0)}{(1 - \eta_0(\|s_0 - \xi\|))(1 - \eta_0(p_0))} \left[1 + \int_0^1 \eta_0(\lambda\|s_0 - \xi\|)d\lambda \right] \right. \\ & \left. + \frac{1}{2}\left[3\frac{\bar{\eta}(p_0)}{(1 - \eta_0(p_0))} + \left(\frac{\bar{\eta}(p_0)}{(1 - \eta_0(p_0))} \right)^2 \right] \frac{1 + \int_0^1 \eta_0(\lambda\|s_0 - \xi\|)d\lambda}{1 - \eta_0(p_0)} \right] \|s_0 - \xi\| \\ & \leq \Delta_3(p_0)p_0 \leq p_0. \end{aligned}$$

Therefore, the iterate $t_1 \in U(\xi, \kappa)$, i.e., the item (9) and (12), both hold if $i = 0$. As a result, if $i = 0$, all items (9) to (12) hold.

The induction is then complete if one replaces the roles of t_0, w_0, s_0, t_1 in the earlier calculations with t_i, w_i, s_i, t_{i+1}. Lastly, based on the estimation

$$p_{i+1} \leq cp_i \leq \kappa, \quad \text{where} \quad c = \Delta_3(p_0) \in [0, 1)\,,$$

it follows that p_{i+1} is in $U(\xi, \kappa)$ and $\lim_{i \to +\infty} t_i = \xi$. □

Next, the distinctive nature of the solution domain is specified.

Proposition 3.2 *Presume:*

(i) *The equation $J(u) = 0$ admits a solution $\bar{x} \in U(\xi, k_1)$ for a certain value $k_1 > 0$.*
(ii) *In the ball $U(\xi, k_1)$, the condition (P_3) is true.*
(iii) *There happen to be cases of $k_2 > k_1$ wherein*

$$\int_0^1 \eta_0(\lambda k_2)d\lambda \leq 1.$$

Then, in the region Λ_1, where $\Lambda_1 = \Lambda \cap U[\xi, k_2]$, the only solution to equation (1) is ξ.

Proof Let's define the linear operator $\mathscr{Y} = \int_0^1 J'(\xi + \lambda(\bar{x} - \xi))d\lambda$. Using the criteria (ii) and (iii), we eventually accomplish that

$$\begin{aligned} \|J'(\xi)^{-1}(\mathscr{Y} - J'(\xi))\| &\leq \int_0^1 \eta_0(\lambda\|\bar{x} - \xi\|)d\lambda \\ &\leq \int_0^1 \eta_0(\lambda k_2)d\lambda \\ &< 1. \end{aligned}$$

Consequently, we conclude that $\bar{x} = \xi$ because the linear operator $\mathscr{Y} \in \mathscr{L}(W_2, W_1)$ and

$$\bar{x} - \xi = \mathscr{Y}^{-1}(J(\bar{x}) - J(\xi)) = \mathscr{Y}^{-1}(0) = 0.$$

□

Remark 1 In accordance with the criteria of Theorem 3.1, set $\bar{x} = \xi$ and $k_1 = \kappa$.

4 Convergence type 2: semi-local

In this section, we explore the semi-local convergence properties of the proposed iterative method. The analysis extends beyond the immediate neighborhood of the solution, highlighting the broader applicability and robustness of the method under less restrictive initial conditions. Additionally, we derive error estimates to quantify the accuracy and reliability of the method in these scenarios.

Likewise, with the local convergence, suppose:

(Q_1) There is a continuous, not diminishing function $\vartheta_0 : V \to V$ such that the expression

$$\vartheta_0(u) - 1 = 0$$

holds a minimum solution in $V - \{0\}$. The smallest such zero is denoted by μ_1. Represent $V_2 = [0, \mu_1)$. Additionally, let $\vartheta : V_2 \to V$ represent a continuous and not diminishing function.

For every $k = 0, 1, 2, \ldots$, we propose the following scalar sequences for $\sigma_0 = 0$ and $\tau_0 \in [0, \mu_1]$ as

$$\begin{aligned}
\nu_k &= \tau_k + \frac{3}{2}\left(\frac{5}{3} + \frac{3}{4}\left(\frac{\bar{\vartheta}_k}{1-\vartheta_0(\sigma_k)}\right) + \frac{9}{8}\left(\frac{\bar{\vartheta}_k}{1-\vartheta_0(\sigma_k)}\right)^2\right)(\tau_k - \sigma_k),\\
\gamma_k &= \left(1 + \int_0^1 \vartheta_0(\sigma_k + \lambda(\nu_k - \sigma_k))\right) d\lambda(\nu_k - \sigma_k) + \frac{3}{2}(1 + \vartheta_0(\sigma_k))(\tau_k - \sigma_k),\\
\epsilon_k &= 1 + \frac{3}{2}\left(\frac{\bar{\vartheta}_k}{1-\vartheta_0(\sigma_k)}\right) + \frac{1}{2}\left(\frac{\bar{\vartheta}_k}{1-\vartheta_0(\sigma_k)}\right)^2,\\
\sigma_{k+1} &= \nu_k + \frac{\epsilon_k \gamma_k}{1-\vartheta_0(\sigma_k)},\\
\beta_{k+1} &= \left(1 + \int_0^1 \vartheta_0(\sigma_k + \lambda(\sigma_{k+1} - \sigma_k))\right) d\lambda(\sigma_{k+1} - \sigma_k) + \frac{3}{2}(1 + \vartheta_0(\sigma_k))(\tau_k - \sigma_k),\\
\tau_{k+1} &= \sigma_{k+1} + \frac{\beta_{k+1}}{1-\vartheta_0(\sigma_{k+1})},\\
\text{where } \bar{\vartheta}_k &= \begin{cases} \vartheta(\sigma_k + \tau_k), \\ \text{or} \\ \vartheta_0(\sigma_k) + \vartheta_0(\tau_k) \end{cases}.
\end{aligned} \tag{18}$$

For these scalar sequences, a criteria for convergence is required. The following Lemma gives the convergence of the majorizing sequence $\{\sigma_k\}$, which plays a crucial role in the convergence analysis of the method (2).

Lemma 4.1 *Presume*

(Q_2) *there is* $\delta^* > 0$*, so that for each* $k = 0, 1, 2, \ldots$

$$\vartheta_0(\sigma_k) < 1 \quad \textit{and} \quad \sigma_k < \delta^*. \tag{19}$$

Then, the sequence $\{\sigma_k\}$ *induced by (18) is limited from above by* δ^**, non-declining and converges to some* $\kappa^* \in [0, \delta^*]$.

Proof The equations in (18) and (19) directly lead to the conclusions of the Lemma. □

Remark 2 (i) The supremum of the sequence $\{\sigma_k\}$ is the limit point κ^*.
(ii) μ_1 is a preferable alternative for the upper limit δ^*.
(iii) For applications, it is recommended that $\bar{\vartheta}_k$ have the lowest value possible.

In the Theorem 4.2, the sequence σ_k is demonstrated to be majorizing for the technique (2). Scalar functions and sequences correspond to the method's operators just like in the local case.

(Q_3) For t_0 and each $v \in \Lambda$, $J'(t_0)^{-1} \in \mathscr{L}(W_2, W_1)$ and $\|J'(t_0)^{-1}J(t_0)\| \le \tau_0$,

$$\|J'(t_0)^{-1}(J'(v) - J'(t_0))\| \le \vartheta_0(\|v - t_0\|).$$

Denote $\Lambda_2 = \Lambda \cap U(t_0, \mu_1)$.

(Q_4) $\|J'(t_0)^{-1}(J'(v_2) - J'(v_1))\| \leq \vartheta(\|v_2 - v_1\|)\ \forall v_1, v_2 \in \Lambda_2$.
(Q_5) $U[t_0, \kappa^*] \subset \Lambda$.

Following that, the method's major semi-local result makes use of the proposed terminology in conjunction with the conditions (Q_1)-(Q_5).

In the following theorem, we establish the conditions under which the sixth-order iterative method converges semi-locally. We will demonstrate the error bounds and provide the necessary criteria to ensure the convergence of the method.

Theorem 4.2 *Suppose that (Q_1)-(Q_5) are valid. The subsequent arguments are also valid*

$$\{t_i\} \subset U(t_0, \kappa^*), \tag{20}$$

$$\|w_i - t_i\| \leq \tau_i - \sigma_i, \tag{21}$$

$$\|s_i - w_i\| \leq \nu_i - \tau_i, \tag{22}$$

$$\|t_{i+1} - s_i\| \leq \sigma_{i+1} - \nu_i. \tag{23}$$

Moreover, ξ is present in $U[t_0, \kappa*]$ establishing $J(\xi) = 0$.

Proof The proof is comparable to the local case proof provided in Theorem 3.1. With t_0, the role of ξ is switched, though. According to the criterion (Q_3),

$$\begin{aligned}\|w_0 - t_0\| &\leq \|J'(t_0)^{-1}J(t_0)\| \\ &\leq \tau_0 \\ &= \tau_0 - \sigma_0 < \kappa^*.\end{aligned}$$

As a result, $w_0 \in U(t_0, \kappa^*)$ and hence (21) is valid if $i = 0$. Let x be arbitrary but within the range $U(t_0, \kappa^*)$. Consequently, it follows from the condition (Q_3) and the corresponding definition of κ^* that

$$\|J'(t_0)^{-1}(J'(x) - J'(t_0))\| \leq \vartheta_0(\|x - t_0\|) \leq \vartheta_0(\kappa^*) < 1.$$

This implies that $J'(x)$ is invertible and

$$\|J'(x)^{-1}J'(t_0)\| \leq \frac{1}{1 - \vartheta_0(\|x - t_0\|)}. \tag{24}$$

Consider that the inequality (21) is valid for all $k < i$.

Following that, using the second sub-step of (2), we may write

$$\begin{aligned}s_k - w_k &= \left[\frac{2}{3}I + I - \frac{3}{4}(J'(t_k)^{-1}J'(w_k) - 1) + \frac{9}{8}(J'(t_k)^{-1}J'(w_k) - 1)^2\right]J'(t_k)^{-1}J(t_k) \\ &= \left[\frac{5}{3}I + \frac{3}{4}[J'(t_k)^{-1}(J'(t_k) - J'(w_k))] + \frac{9}{8}[J'(t_k)^{-1}(J'(w_k) - J'(t_k))]^2\right] \\ &\quad \times (-\frac{3}{2}(w_k - t_k)),\end{aligned}$$

hence, we have

$$\begin{aligned}\|s_k - w_k\| &\leq \frac{3}{2}\left[\frac{5}{3} + \frac{3}{4}\left(\frac{\bar{\vartheta}_k}{1 - \vartheta_0(\|t_k - t_0\|)}\right) + \frac{9}{8}\left(\frac{\bar{\vartheta}_k}{1 - \vartheta_0(\|t_k - t_0\|)}\right)^2\right]\|w_k - t_k\| \\ &\leq \nu_k - \tau_k\end{aligned} \tag{25}$$

followed by

$$\begin{aligned}\|s_k - t_0\| &\leq \|s_k - w_k\| + \|w_k - t_0\| \\ &\leq \nu_k - \tau_k + \tau_k - \sigma_0 \\ &= \nu_k < \kappa^*.\end{aligned} \tag{26}$$

As a result, $s_k \in U(t_0, \kappa^*)$ and (22) holds true.

We require the following estimates

$$
\begin{aligned}
J(s_k) &= J(s_k) - J(t_k) + J(t_k) \\
&= J(s_k) - J(t_k) - \frac{3}{2}J'(t_k)(w_k - t_k) \\
&= \int_0^1 J'(t_k + \lambda(s_k - t_k))(s_k - t_k)d\lambda - \frac{3}{2}J'(t_k)(w_k - t_k), \\
\|J'(t_0)^{-1}J(s_k)\| &\le \left(1 + \int_0^1 \vartheta_0(\|s_k + \lambda(s_k - t_k) - t_0\|)d\lambda\right)\|s_k - t_k\| + \frac{3}{2}\left(1 + \vartheta_0(\|t_k - t_0\|)\right)\|w_k - t_k\| \\
&= \bar{\gamma}_k \le \gamma_k
\end{aligned}
$$

and

$$
\begin{aligned}
\|Z_2(t_k)\| &\le \left[1 + \frac{3}{2}[J'(t_k)^{-1}(J'(t_k) - J'(w_k))] + \frac{1}{2}[J'(t_k)^{-1}(J'(w_k) - J'(t_k))]^2\right] \\
&\le \left[1 + \frac{3}{2}\left(\frac{\bar{\vartheta}_k}{1 - \vartheta_0(\|t_k - t_0\|)}\right) + \frac{1}{2}\left(\frac{\bar{\vartheta}_k}{1 - \vartheta_0(\|t_k - t_0\|)}\right)^2\right] \\
&\le \epsilon_k.
\end{aligned}
$$

Using the final step of (2), we can now formulate that

$$
\begin{aligned}
\|t_{k+1} - s_k\| &\le \|Z_2\|\|J'(t_k)^{-1}J'(t_0)\|\|J'(t_0)^{-1}J(s_k)\| \\
&\le \frac{\epsilon_k \gamma_k}{1 - \zeta_0(\|t_k - t_0\|)} \\
&\le \sigma_{k+1} - \nu_k
\end{aligned}
\tag{27}
$$

followed by

$$
\begin{aligned}
\|t_{k+1} - t_0\| &\le \|t_{k+1} - s_k\| + \|s_k - t_0\| \\
&\le \sigma_{k+1} - \nu_k + \nu_k - \sigma_0 \\
&= \sigma_{k+1} < \kappa^*.
\end{aligned}
\tag{28}
$$

As a result, $t_{k+1} \in U(t_0, \kappa^*)$ and (23) holds true. We may now establish the first sub-step of (2) as

$$
\begin{aligned}
J(t_{k+1}) &= J(t_{k+1}) - J(t_k) - \frac{3}{2}J'(t_k)(w_k - t_k) \\
&= \int_0^1 J'(t_k + \lambda(t_{k+1} - t_k))(t_{k+1} - t_k)d\lambda - \frac{3}{2}(J'(t_k) - J'(t_0) + J'(t_0))(w_k - t_k),
\end{aligned}
\tag{29}
$$

accordingly, we can write

$$
\begin{aligned}
\|J'(t_0)^{-1}J(t_{k+1})\| &\le (1 + \int_0^1 \vartheta_0(\|t_k + \lambda(t_{k+1} - t_k) - t_0\|)d\lambda\|t_{k+1} - t_k\| + \frac{3}{2}(1 + \vartheta_0(\|t_k - t_0\|))\|w_k - t_k\| \\
&= \bar{\beta}_{k+1} \le \beta_{k+1}.
\end{aligned}
$$

As a result, we may derive from the first sub-step of (2) and (24) for $x = t_{k+1}$ that

$$
\begin{aligned}
\|w_{k+1} - t_{k+1}\| &\le \|J'(t_{k+1})^{-1}J'(t_0)\|\|J'(t_0)^{-1}J(t_{k+1})\| \\
&\le \frac{\beta_{k+1}}{1 - \zeta_0(\sigma_{k+1})} \\
&= \tau_{k+1} - \sigma_{k+1}
\end{aligned}
$$

and so forth we have

$$
\begin{aligned}
\|w_{k+1} - t_0\| &\le \|w_{k+1} - t_{k+1}\| + \|t_{k+1} - t_0\| \\
&\le \tau_{k+1} - \sigma_{k+1} + \sigma_{k+1} - \tau_0 \\
&= \tau_{k+1} < \kappa^*.
\end{aligned}
$$

Induction is therefore accomplished. As a result, it follows that the sequence $\{t_i\}$ is fundamental in the Banach space W_1 (because by the requirements (Q_1) and (Q_2) the sequence $\{\tau_i\}$ is fundamental and convergent as well). Consequently, there is $\xi \in U[t_0, \kappa^*]$ such that $lim_{i\to+\infty} t_i = \xi$.

Finally, use the continuity of the operator J to derive $J(\xi) = 0$ and demonstrate that ξ is in fact a solution of the equation $J(u) = 0$. This can be done by letting $k \to +\infty$ in the estimate (29). The following estimation is also helpful:

$$\|t_{i+k} - t_i\| \leq \tau_{i+k} - \tau_i \quad \text{for} \quad k \geq 0, \tag{30}$$

leading to

$$\|\xi - t_i\| \leq \kappa^* - \tau_i$$

provided that $k \to +\infty$ in (30). □

The uniqueness of the solution domain is subsequently established.

Proposition 4.3 *Assume:*

(i) *For some $r > 0$, the equation $J(u) = 0$ has a solution $\xi^* \in U(t_0, r)$.*
(ii) *On $U(t_0, r)$, (Q_3) holds true.*
(iii) *There exists $r^{'} > r$ such that*

$$\int_0^1 \vartheta_0((1-\lambda)r + \lambda r^{'})d\lambda < 1.$$

Set $\Lambda_3 = \Lambda \cap U[t_0, r^{'}]$.

Then one and only point in the domain Λ_3 that can satisfy the equation $J(u) = 0$ is ξ^*.

Proof Assume that there exists $\xi^{'} \in \Lambda_3$ satisfying $J(u) = 0$. Condition (ii) and (iii) allow us to derive

$$\begin{aligned}
\|J'(t_0)^{-1}(A - J'(t_0))\| &\leq \int_0^1 \vartheta_0((1-\lambda)\|\xi^* - t_0\| + \lambda\|\xi^{'} - t_0\|)d\lambda \\
&\leq \int_0^1 \vartheta_0((1-\lambda)r + \lambda r^{'})d\lambda \\
&< 1, \\
\text{where } A &= \int_0^1 J'(\xi^* + \lambda(\xi^{'} - \xi^*))d\lambda
\end{aligned}$$

and therefore it follows that $\xi^{'} = \xi^*$. □

Remark 3 Set $\xi^* = \xi$ and $r = \kappa^*$ in accordance with the requirements of Theorem 4.2.

5 Numerical illustrations

This section presents numerical examples to illustrate the practical effectiveness of the sixth-order iterative method. Through various test cases, we demonstrate the convergence characteristics and validate the theoretical findings from the previous sections. The convergence outcomes can be explained using the examples that follow:

Example 1 Along with the maximum norm, set $W_1 = W_2 = C[0, 1]$. Assume that $\Lambda = U[0, 1]$. The following definition applies to the operator J on Λ:

$$J(\Gamma)(u) = \Gamma(u) - 5\int_0^1 u\lambda\Gamma(\lambda)^3 d\lambda.$$

From this definition,

$$J'(\Gamma(\zeta))(u) = \zeta(u) - 15\int_0^1 u\lambda\Gamma(\lambda)^2\zeta(\lambda)d\lambda, \text{ for each } \zeta \in \Lambda.$$

$\xi = 0$ provides a solution. Then, $\eta_0(q) = 7.5q$, $\eta(q) = 15q$ satisfy criteria (P_1) through (P_5). As a result, we obtain the radii to be $\kappa_1 = 0.0410256$, $\kappa_2 = 0.0607758$ and $\kappa_3 = 0.0464258$. Consequently, by using (6), we have $\kappa = 0.0410256$.

Table 1 Estimates for Example (4)

k	0	1	2	3	4	5	6
$\vartheta_0(\sigma_k)$	0	0.0356996	0.0386414	0.0407413	0.0407446	0.0407607	0.0407607
σ_k	0	0.0228791	0.0247644	0.0261102	0.0261123	0.0261226	0.0261226

Here, $\kappa^* = 0.0261226$

Table 2 Estimates for Example (5)

$\|t_0 - \xi\|$	$\|t_1 - \xi\|$	$\|t_2 - \xi\|$
$1.14418 * 10^{-1}$	$4.90997 * 10^{-7}$	$4.44089 * 10^{-16}$

Example 2 Let's say that $W_1 = W_2 = \Lambda = \mathbb{R}$. $J(u) = \sin u$ is defined. We obtain $J'(u) = \cos u$. As a result, we have $\xi = 0$. If $\eta_0(q) = \eta(q) = q$, the conditions (P_1) through (P_5) are satisfied. Following that, the radii are $\kappa_1 = 0.4$, $\kappa_2 = 0.330719$ and $\kappa_3 = 0.268837$. Consequently, we obtain $\kappa = 0.268837$ by (6).

Example 3 The following simultaneous differential equations govern an object's motion as defined by

$$J_1'(u_1) = e^{u_1},\ J_2'(u_2) = (e-1)u_2 + 1,\ J_3'(u_3) = 1$$

depending on initial conditions $J_1(0) = J_2(0) = J_3(0) = 0$. Suppose $J = (J_1, J_2, J_3)$. Let's say that $W_1 = W_2 = \mathbb{R}^3$, $\Lambda = U[0, 1]$ and we obtain $\xi = (0, 0, 0)^T$. Create a mapping for $u = (u_1, u_2, u_3)^T$ using J on Λ.

$$J(u) = (e^{u_1} - 1, \frac{e-1}{2}{u_2}^2 + u_2, u_3)^T.$$

It turns out that the operator J' is

$$J'(u) = \begin{bmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $\eta_0(q)$ and $\eta(q)$, respectively, are equal to $(e-1)q$ and $e^{\frac{1}{e-1}}q$, respectively, conditions (P_1)–(P_5) are met. As a result, the radii are $\kappa_1 = 0.229929$, $\kappa_2 = 0.215426$ and $\kappa_3 = 0.171403$. Finally, we arrive to $\kappa = 0.171403$ using (6).

Example 4 By taking into account the quartic equation for fractional conversion, which indicates the portion of the nitrogen-hydrogen feed that is transformed to ammonia, the usefulness of our findings in the real-world context may be successfully established. This equation can be written as at $500°C$ and $250atm$

$$J(u) = u^4 - 7.79075u^3 + 14.7445u^2 + 2.511u - 1.674.$$

Solving the above equation yields the value $\xi = 0.27776$. Choose $t_0 = 0.3$ and set $\Lambda = (0.3, 0.4)$. As illustrated below, it is possible to verify the conditions (Q_1)–(Q_5).

$$\|J'(t_0)^{-1}J(t_0)\| = 0.0217956 = \tau_0.$$

We achieve $\vartheta_0(q) = \vartheta(q) = 1.56036q$, $\mu_1 = 0.640878$, and $\Lambda_2 = \Lambda \cap U(t_0, \mu_1)$ through computation. Table 1 demonstrates convergence of $\{\sigma_k\}$.

Example 5 Take for instance the non-linear equation $J(u) = u^3 - 20$ defined on $W_1 = W_2 = \mathbb{R}$ where $\Lambda = U[2, 2]$. We obtain $\xi = 2.714417616594906572$. Let $t_0 = 2.6$. The error estimates are then listed in Table 2.

Example 6 We take into account the subsequent nonlinear system:

$$e^{u_1} + u_1u_2 - 1 = 0$$
$$\sin(u_1 + u_2) + u_1 + u_2 - 1 = 0.$$

Choose the starting approximation as $t_0 = (0.2, 0.9)^T$. Utilizing (2), we solve this system and obtain the solution $\xi = (0, 1)^T$ after four iterations.

Table 3 Estimates for Example (7)

$\|t_1 - \xi\|$	$\|t_2 - \xi\|$	$\|t_3 - \xi\|$	$\|t_4 - \xi\|$	$\|t_5 - \xi\|$
$5.339679334803259 * 10^{-4}$	$4.692778866425354* 10^{-6}$	$4.124041780553433* 10^{-8}$	$3.624225581270254* 10^{-10}$	$2.763362727283444* 10^{-14}$

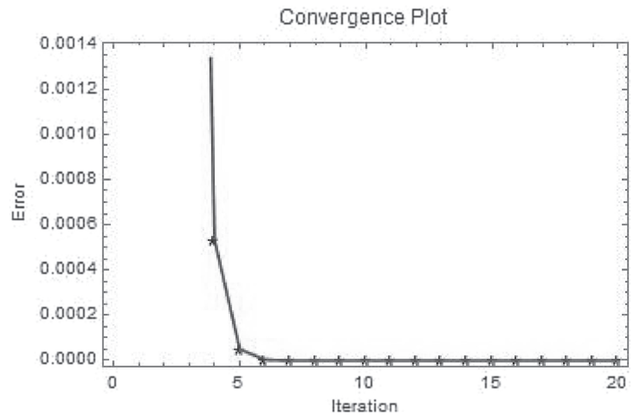

Fig. 1 Error estimates for Example 7

Example 7 We examine the system of equations described by the function $J(u)$, which is defined as a vector consisting of 8 components, $J(u) = (j_1(u), \dots, j_8(u))$, where

$$j_i(u) = u_{(i)} + 1 - 2\ln\left(1 + \sum_{j=1, j\neq i}^{8} u_{(j)}\right), \quad 1 \le i \le 8,$$

.

The initial conditions for the system are given by the vector t_0, with $t_0 = \{5.35, \dots, 5.35\}^T$. The solution vector, denoted as $\xi = \{6.753932311935358594..., \dots, 6.753932311935358594...\}^T$.

To assess the performance of the method (2), we provide error estimates in Table 3. These estimates quantify the discrepancy between the computed solutions obtained using the method (2) and the actual solution. Additionally, a convergence plot is also given showing the behavior of the error over iterations (See Fig 1).

6 Conclusion

The focus of the current study is on novel local and semi-local convergence analysis for higher order methods. It is based on generalised conditions that were created using only the first derivative and are present in the methods under examination. Older theories of local convergence rely on higher derivatives that are absent from the method. Additionally, they provide no details on the error distances that can be calculated, particularly a priori. The same is true for the convergence region. By taking into account the semi-local case—which is thought to be more interesting than the local—the methodologies are further expanded. As a result, these strategies are more widely applicable in various ways. Also, our technique is independent of the method. Therefore, it can be used with the same advantages on other similar approaches. The focus of our research in the near future will be in this direction.

Declarations

Conflict of interest The authors have no competing interests to declare.

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