ORIGINAL RESEARCH

On the eigenvalues of zero divisor graphs associated with commutative rings

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Abstract
For a commutative ring R, with non-zero zero divisors $Z^*(R)$. The zero divisor graph $\Gamma(R)$ is a simple graph with vertex set $Z^*(R)$, and two distinct vertices $x, y \in V(\Gamma(R))$ are adjacent if and only if $x \cdot y = 0$. This article presents counter examples for the energy, the second Zagreb index, and the eigenvalues associated with zero divisor graphs of rings that were found in [Johnson and Sankar, J. Appl. Math. Comp. (2023)]. For the zero divisor graph $\mathbb{Z}_p[x]/\langle x^4\rangle$, we rectify the eigenvalues (energy) and the Zagreb index results. We show that for any prime p, $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ is non-hyperenergetic and for prime $p \geq 3$, $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ is hypoenergetic. We give a formulae for the topological indices of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ and show that its Zagreb indices satisfy Zagreb Conjecture [Hansen and Vukičcević, Croatica Chem. Acta, (2007)].

Keywords Adjacency matrix · Commutative rings · Zero divisor graphs · Energy · Topological indices

Mathematics Subject Classification 05C50 · 05C25 · 15A18 · 13A70

1 Introduction

A finite simple graph $G = G(V(G), E(G))$ consists of a vertex set $V(G)$ and an edge set $E(G)$. The number $|V(G)| = n$ is *order* and the number $|E(G)| = m$ is *size* of G. By $u \sim v$, we denote u and v are adjacent. The set of vertices $N_G(v)$ adjacent to v is the *neighbourhood* (open neighbourhood) of v. The number $|N_G(v_i)|$ is *degree* d_i (or d_{v_i}) of v_i. An *independent* set is a collection of non-adjacent vertices and a *clique* is a subset of V such that any two vertices in it are adjacent. A graph G is regular if d_i is same for every i. The *adjacency matrix* $A(G)$ is a matrix with (0, 1) entries, such that its (i, j)-th entry is 1 if $v_i \sim v_j$, and 0, otherwise. The matrix $A(G)$ is real symmetric matrix, we order its eigenvalues as

$$\lambda_n(G) \leq \lambda_{n-1}(G) \leq \cdots \leq \lambda_2(G) \leq \lambda_1(G),$$

and the multiset of $\lambda_i(G)$'s is the *spectrum* of G (or $A(G)$). The *energy* of G is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

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The energy $\mathcal{E}(G)$ of a graph G is very well studied and was defined by Gutman in 1978. The characteristic polynomial of a matrix M is $\Phi(M, \lambda) = \det(\lambda I - M)$, where det is determinant of M and I is the identity matrix. However the intension of energy was already there, since in 1930's Huckel's Molecular Orbital Theory allows us to predict the approximate values of energies associated with orbitals of electrons in hydrocarbons. The method works with the process that the Hamiltonian operator can be written as a linear combination of certain orbitals, and it provides us required energies with the help of the time-independent Schrödinger equation. But Gunthard and Primes discovered in 1956 that the matric method used by Huckel is a first degree characteristic polynomial of $A(G)$ for a graph structure G associated to molecule. There is a vast literature about $A(G)$ and its energy, see [6–8, 17]. Apart from the chemical importance of energy, for a general matrix M, the energy is the trace norm of M, the largest eigenvalue of $A(G)$ is same as the spectral norm. A detailed analysis of norms of $A(G)$ is given by Nikiforov [21].

For a commutative ring R with non-zero identity. An element $0 \neq x \in R$ is known as the zero divisor of R if there is $0 \neq y \in R$ such that $x \cdot y = 0$. Beck [2] introduced the concept of the zero divisor graphs of commutative rings and included 0 in the definition. While Anderson and Livingston [1] excluded 0 and altered the definition of zero divisor graphs, they defined edges between two non-zero zero divisors x and y if and only if $x \cdot y = 0$. The adjacency eigenvalues of zero divisor graphs were first discussed by Young [34]. The spectral theory was further elaborated in [20, 25]. More spectral invariants of zero divisor of other matrices can be seen in [15, 22, 23].

Motivated by the work of authors [16], we further investigate the zero divisor graphs and correct some already published results in [16]. We use standard notations, complete graph is denoted by K_n, and for other notations and terminology, the readers are referred to [4]. An eigenvalue λ with multiplicity m is denoted by $\lambda^{[m]}$.

The rest of the paper is organized as follows. In Section 2, we give the eigenvalues, the eigenvectors and the energy of the zero divisor graphs of $\mathbb{Z}_p[x]/\langle x^4\rangle$, thereby correcting the earlier published results by Johnson and Sankar in [16]. We prove that for any prime p, $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ is non-hyperenergetic and for prime $p \geq 3$, $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ is hypoenergetic. In Section 3, we give the topological indices of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$, which in turn corrects the result related to second Zagreb index recently published in [16]. We also show that the Zagreb indices of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ satisfy the Hansen and Vukičcević conjecture [14]. We end up the article with a conclusion.

2 Eigenvalues and energy of zero divisor graphs

For the sake of completeness and the typos in the construction of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$, we discuss its structural process. For a prime p, the structure of zero divisor graph $G \cong \Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ (see, [16]) of $\mathbb{Z}_p[x]/\langle x^4\rangle$ can be constructed as given below:

We partition the vertex set $V(G)$ of G into following sets

$$\begin{aligned} A &= \Big\{kx^3 | k = 1, 2, \ldots, (p-1) \text{ and } p \nmid k\Big\}, \\ B &= \Big\{lx^3 + kx^2 | l = 1, 2, \ldots, (p-1) \text{ and } p \nmid l\Big\}, \\ C &= \Big\{mx^3 + lx^2 + kx | m = 1, 2, \ldots, (p-1) \text{ and } p \nmid m\Big\}. \end{aligned}$$

The elements of A are $\Big\{x^3, 2x^3, 3x^3, \ldots, (p-2)x^3, (p-1)x^3\Big\}$ and its cardinality is $|A| = p - 1$. Similarly, $|B| = p(p-1)$ and that of C is $p(p^2 - p)$. Further, for arbitrary $k_i x^3$ and $k_j x^3$ in A, we have

$$(k_i x^3) \cdot (k_j x^3) = k_i k_j x^6 \equiv 0 \mod x^4.$$

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Thus, it follows that each element of A is adjacent to the remaining elements contained inside A. Again for any two elements say $l_i x^3 + k_j x^2$ and $l_j x^3 + k_i x^2$ in B, we have

$$(l_i x^3 + k_j x^2) \cdot l_j x^3 + k_i x^2 = l_i l_j x^6 + (l_i k_i + l_j k_j)x^5 + (k_i k_j)x^4 \equiv 0 \mod x^4.$$

It implies that each vertex in B is adjacent to every other vertex of B, that is, the vertices of B form a clique of size $p^2 - p$. Again for $m_i x^3 + l_i x^2 + k_i x \in C$ and $m_j x^3 + l_j x^2 + k_j x \in C$, then we have

$$\begin{aligned}
&(m_i x^3 + l_i x^2 + k_i x) \cdot (m_j x^3 + l_j x^2 + k_j x) \\
&\quad = m_i m_j x^6 + (l_i m_j + l_j m_i)x^5 + (k_i m_j + l_i l_j + m_i k_j)x^4 + (k_i l_j + l_i k_j)x^3 + k_i k_j x^2 \\
&\quad \not\equiv 0 \mod x^4.
\end{aligned}$$

Thus, the vertices in C are not adjacent to themselves, so they form an independent set of cardinality $p^3 - p^2$. Again, for $k_i x^3 \in A$ and $l_i x^3 + k_j x^2 \in B$, we see that

$$(k_i x^3) \cdot (l_i x^3 + k_j x^2) = l_i k_i x^6 + k_i k_j x^5 \equiv 0 \mod x^4.$$

So, it implies that each vertex of A is adjacent to every vertex in B. Likewise for any $k_i x^3 \in A$ and $m_i x_3 + l_i x_2 + k_j x \in C$, we see that

$$(k_i x^3) \cdot (m_i x_3 + l_i x_2 + k_j x) = m_i k_i x^6 + l_i k_i x^5 + k_i k_j x^4 \equiv 0 \mod x^4.$$

Thus, each vertex in A is adjacent to every vertex in C. Lastly, if $l_i x^3 + k_i x^2 \in B$ and $m_j x_3 + l_j x_2 + k_j x \in C$, then

$$(l_i x^3 + k_i x^2) \cdot (m_j x_3 + l_j x_2 + k_j x) = l_i m_j x^6 + (k_i m_j + l_i l_j)x^5 + (k_i l_j + l_i k_j)x^4 + k_i k^j x^3 \not\equiv 0 \mod x^4.$$

Therefore, it follows that the vertices in B are not adjacent to any vertex in C (while authors in [16] have written opposite of it, may be a typo). This gives us the structure of G completely. We illustrate this process with the help of an example (same as given in [16]).

For $R \cong \mathbb{Z}_3[x]/\langle x^4 \rangle$, with $p = 3$. the vertex set of $G \cong \Gamma\big(\mathbb{Z}_3[x]/\langle x^4 \rangle\big)$ are given as

$$\begin{aligned}
&A = \{x^3, 2x^3\}, \\
&B = \{x^2, 2x^2, x^3 + x^2, x^3 + 2x^2, 2x^3 + x^2, 2x^3 + 2x^2\}, \\
&C = \{x, 2x, x^2 + x, x^2 + 2x, 2x^2 + x, 2x^2 + 2x, x^3 + x, x^3 + 2x, 2x^3 + x, 2x^3 + 2x, \\
&\quad x^3 + x^2 + x, x^3 + x^2 + 2x, x^3 + 2x^2 + x, x^3 + 2x^2 + 2x, 2x^3 + x^2 + x, \\
&\quad 2x^3 + x^2 + 2x, 2x^3 + 2x^2 + x, 2x^3 + 2x^2 + 2x\}.
\end{aligned} \tag{1}$$

The pictorial representation of G is shown in Figure 1, where degree of each vertex in C is 2, degree of each vertex in B is 7 and the two vertices of A are dominating vertices of degree 25 (adjacent to all vertices).

Remark 2.1 From the structure of $G \cong \Gamma\big(\mathbb{Z}_p[x]/\langle x^4 \rangle\big)$, it follows that there are $p^3 - p^2$ independent vertices, $p - 1$ dominating vertices and $p^2 - p$ vertices whose induced subgraph is a clique. Overall, there the clique size of G is $p^2 - 1$.

Now, we consider the results about the energy of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$ and the adjacency matrix of $\Gamma(\mathbb{Z}_3[x]/\langle x^4 \rangle)$.

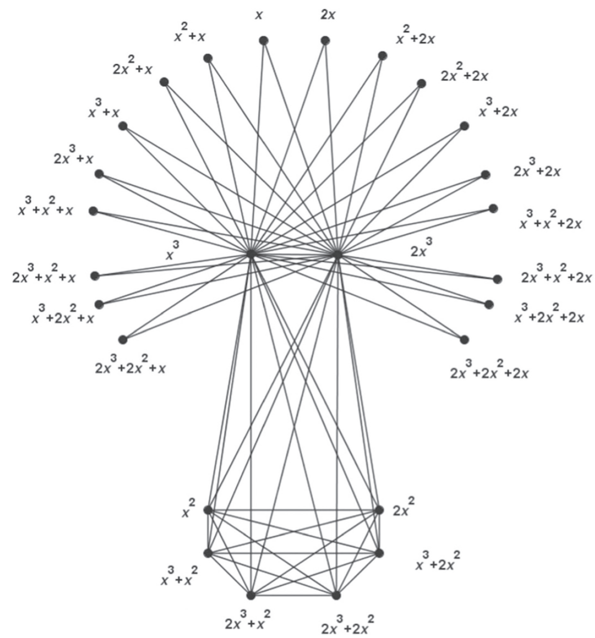

Fig. 1 Zero divisor graph of $\Gamma\big(\mathbb{Z}_3[x]/\langle x^4\rangle\big)$

Theorem 2.2 (Theorem 3.2, [16]) *Let $R \cong \mathbb{Z}_p[x]/\langle x^4\rangle$ be a zero divisor graph with prime p. Then energy of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ is*

$$\mathcal{E}(\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)) \geq \frac{1}{2}\Big(p^5 + p^4 - 4p^3 + 3p^2 - 5p + 4\Big).$$

We outline some lines of the proof here.

In the proof of the theorem, the characteristic polynomial of $G \cong \Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ and other steps are written as below:

$$\begin{aligned}\Phi(g,\lambda) &= \Big(\lambda - p^3 + 1\Big)^{(p-2)}\Big(\lambda - p + 1\Big)^{p^2(p-1)-1}\left(\lambda - \frac{1}{2}\big(p^4 - 2p^3 + 3p\big)\right)^{p(p-1)-1} \\ &\quad \times \left(\lambda^3 - \frac{1}{2}(p^4 - 2p^2 + 5p - 2)\lambda^2 + \frac{1}{2}p(p-1)(p^2+p+1)(p^3 - 2p^2 - 2p + 5)\lambda \right. \\ &\quad \left. - \frac{1}{2}p(p-1)^2(p^4 - 2p^3 - 2p^2 + 3p + 2)\right).\end{aligned} \tag{2}$$

We have $\alpha + \beta + \gamma = \frac{1}{2}(p^4 - 2p^2 + 5p - 2)$, the authors did not mention the entities α, β and γ, but it seems they are the zeros of of the polynomial $\lambda^3 - \dfrac{1}{2}(p^4 - 2p^2 + 5p - 2)\lambda^2 + \frac{1}{2}p(p-1)(p^2+p+1)(p^3 - 2p^2 - 2p + 5)\lambda - \frac{1}{2}p(p-1)^2(p^4 - 2p^3 - 2p^2 + 3p + 2)$. Next, the energy of G is given as

$$\begin{aligned}\mathcal{E}(\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)) = \sum_{i=1}^{n} |\lambda_i| &= (p-2)|p^3 - 1| + (p^2(p-1) - 1)|p - 1| \\ &\quad + (p(p-1) - 1)\left|\frac{1}{2}\big(p^4 - 2p^3 + 3p\big)\right| + |\alpha| + |\beta| + |\gamma| \\ &\geq \frac{1}{2}\big(p^5 - 4p^3 + 5p^2 - 10p + 6\big) + \alpha + \beta + \gamma \qquad (3) \\ &\geq \frac{1}{2}\big(p^5 + p^4 - 4p^3 + 3p^2 - 5p + 4\big). \qquad (4)\end{aligned}$$

□

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We will illustrate Theorem 2.2 and the characteristic polynomial given above with the help of an example. Let $G \cong \Gamma(\mathbb{Z}_3[x]/\langle x^4\rangle)$ be the zero divisor graph as given in Figure 1. We will first write its adjacency matrix, since the matrix is given in [16], but that is not correct (may be typo error), calculate its characteristic polynomial and deduce its corresponding energy. We index the vertices from A, to B and then C, where A, B and C are as in (1).

$$M = \begin{pmatrix}
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}. \tag{5}$$

By using a computer software Wolfram Mathematica, the characteristic polynomial of M is

$$\Phi(M,\lambda) = (\lambda^3 - 6\lambda^2 - 43\lambda + 180)(\lambda+1)^6(\lambda)^{17},$$

and its approximated zeros are 8.56673, 3.47673, -6.04346, -1 and 0. Thus, the spectrum of G is

$$\Big\{8.56673, 3.47673, (-1)^{[6]}, 0^{[17]}, -6.04346\Big\}.$$

Now, the energy of G is

$$\mathcal{E}(G) = \sum_{i=1}^{26} |\lambda_i| = 8.56673 + 3.47673 + 6 + 6.04346 = 24.087806. \tag{6}$$

Next, we calculate the characteristic polynomial and the energy of G with the help of Theorem 2.2 (Theorem 3.2, [16]). With $p = 3$ in (2), the characteristic polynomial of G is

$$\begin{aligned}
\Phi(G,\lambda) &= \Big(\lambda - 3^3 + 1\Big)^{(3-2)}\Big(\lambda - 3 + 1\Big)^{3^2(3-1)-1}\left(\lambda - \frac{1}{2}\Big(3^4 - 2\cdot 3^3 + 3\cdot 3\Big)\right)^{3(3-1)-1}\\
&\times \Big(\lambda^3 - \frac{1}{2}(3^4 - 2\cdot 3^2 + 5\cdot 3 - 2)\lambda^2 + \frac{1}{2}3(3-1)(3^2+3+1)(3^3 - 2\cdot 3^2 - 2\cdot 3 + 5)\lambda\\
&- \frac{1}{2}3(3-1)^2(3^4 - 2\cdot 3^3 - 2\cdot 3^2 + 3\cdot 3 + 2)\Big),\\
&= (\lambda - 26)(\lambda - 2)^{17}(\lambda - 18)^5(\lambda^3 - 38\lambda^2 + 312\lambda - 120).
\end{aligned}$$

The zeros of the above polynomial are: 26 with multiplicity one (simple zero), 2 with order 17, 18 with order 5 and the simple zeros 26.3184, 11.2772 and 0.404313. The spectrum of G is

$$\left\{26.3184, 26, 18^{[5]}, 11.2772, 2^{[17]}, 0.404313\right\}. \tag{7}$$

The spectrum of G given in (7) is not correct, we recall two facts of the adjacency matrix:

1. The spectral radius of G cannot exceed maximum degree, so in this case $\lambda_1(G) \leq 26$, while in (7), $\lambda_1(G)$ exceeds 26.
2. The trace of the adjacency matrix is zero, while in (7), there are only positive eigenvalues, so their sum cannot be zero.

Thus the spectrum obtained by Theorem 2.2 for $G \cong \Gamma(\mathbb{Z}_3[x]/\langle x^4\rangle)$ is not correct. Further, by Theorem 2.2, the energy of G is

$$\begin{aligned}\mathcal{E}(G) &= 26.3184 + 26 + 18 \times 5 + 11.2772 + 2 \times 17 + 0.404313\\ &= 187.999913.\end{aligned}$$

Which is different from the computer calculations given in (6), and hence is incorrect.

Besides, by the lower bound for the energy given by Theorem 2.2 with $p = 3$, we have

$$\mathcal{E}(G) \geq \frac{1}{2}\left(3^5 + 3^4 - 4 \cdot 3^3 + 3 \cdot 3^2 - 5 \cdot 3 + 4\right) = 116,$$

which is very high than the actual energy $\mathcal{E}(G) = 24.087806$. Therefore, with this counter example, neither the characteristic polynomial is correct nor the energy bound is correct. So, Theorem 2.2 (Theorem 3.2, [16]) is not valid. In the rest of the section, we give the correct form of the characteristic polynomial of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$, correct its spectrum and present new lower and upper bounds for the energy of the zero divisor family $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$.

We are now in a position to study the spectrum and the energy of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ and for that we need some already known results.

Lemma 2.3 ([5]) *Let G be a graph with independent set of vertices $\{v_1, v_2, \ldots, v_k\}$ such that each v_i have same set of neighbors. Then 0 is the eigenvalues of $A(G)$ with multiplicity at least $k - 1$.*

For $\alpha = 0$ in Theorem 2.1 of [24], we have the very next result.

Lemma 2.4 ([24]) *Suppose G is a graph of order n with vertices $\{v_1, v_2, \ldots, v_k\}$, $(k < n)$ satisfying $N(v_i)\setminus\{v_i\} = N(v_j) \setminus \{v_j\}$ for all $i, j \in \{1, 2, \ldots, k\}$. Then -1 is the eigenvalue of $A(G)$ with multiplicity at least $k - 1$.*

The following theorem gives the spectrum of the zero divisor graph of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$.

Theorem 2.5 *Let $G \cong \Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ be the zero divisor graph of $\mathbb{Z}_p[x]/\langle x^4\rangle$ with p being prime. Then the spectrum of G comprises of the eigenvalues -1 and 0 with multiplicities $p^2 - 3$ and $p^3 - p^2 - 1$, respectively. The other eigenvalues of G are the zeros of the polynomial given below*

$$\lambda^3 - \lambda^2(p^2 - 3) + \lambda(2 - 2p^2 + 2p^3 - p^4) + p^6 - 3p^5 + 2p^4 + p^3 - p^2. \tag{8}$$

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Proof We start labelling of vertices form the vertices in A, then in B and finally in C. Under this vertex labelling, the adjacency matrix of G can be written as

$$A(G) = \begin{pmatrix} J_{p-1} - I_{p-1} & J_{(p-1)\times(p^2-p)} & J_{(p-1)\times(p^3-p^2)} \\ J_{(p^2-p)\times(p-1)} & J_{p^2-p} - I_{p^2-p} & \mathbf{0}_{(p^2-p)\times(p^3-p^2)} \\ J_{(p^3-p^2)\times(p-1)} & \mathbf{0}_{(p^3-p^2)\times(p^2-p)} & \mathbf{0}_{(p^3-p^2)} \end{pmatrix}, \tag{9}$$

where J is the matrix whose each entry is 1, $\mathbf{0}$ is the zero matrix, and I is the identity matrix. Since, the vertices of A in G form a clique of size $p-1$ and each such vertex of A have the same neighbors satisfying the hypothesis of Lemma 2.4. Thus, G has eigenvalue -1 with multiplicity $p-2$ and its associated eigenvectors are

$$\Big(-1, 1, \underbrace{0, 0, \ldots, 0}_{p-3}, \underbrace{0, 0, \ldots, 0}_{p^3-p}\Big), \Big(-1, 0, 1, \underbrace{0, 0, \ldots, 0}_{p-4}, \underbrace{0, 0, \ldots, 0}_{p^3-p^2}\Big),$$
$$\vdots$$
$$\Big(-1, \underbrace{0, 0, \ldots, 0}_{p-4}, -1, 0, \underbrace{0, 0, \ldots, 0}_{p^3-p}\Big), \Big(-1, \underbrace{0, 0, \ldots, 0}_{p-3}, 1, \underbrace{0, 0, \ldots, 0}_{p^3-p}\Big).$$

Again the vertices of B form a clique of size p^2-p and each vertex B share the common neighbors in A, so by Lemma 2.4, -1 is the eigenvalue of G with multiplicity p^2-p-1. The corresponding eigenvectors of the eigenvalue -1 are

$$\Big(\underbrace{0, 0, \ldots, 0}_{p-1}, -1, 1, \underbrace{0, 0, \ldots, 0}_{p^2-p-2}, \underbrace{0, 0, \ldots, 0}_{p^3-p^2}\Big), \Big(\underbrace{0, 0, \ldots, 0}_{p-1}, -1, 0, 1, \underbrace{0, 0, \ldots, 0}_{p^2-p-3}, \underbrace{0, 0, \ldots, 0}_{p^3-p^2}\Big),$$
$$\vdots$$
$$\Big(\underbrace{0, 0, \ldots, 0}_{p-1}, -1, \underbrace{0, 0, \ldots, 0}_{p^2-p-3}, 1, 0, \underbrace{0, 0, \ldots, 0}_{p^3-p^2}\Big), \Big(\underbrace{0, 0, \ldots, 0}_{p-1}, -1, \underbrace{0, 0, \ldots, 0}_{p^2-p-2}, 1, \underbrace{0, 0, \ldots, 0}_{p^3-p^2}\Big).$$

Thus -1 is the eigenvalue of G with multiplicity $p^2 - p - 1 + p - 2 = p^2 - 3$. Further, from the construction of zero divisor graph, the elements in C form an independent set of size $p^3 - p^2$, where each vertex of C have a same neighbourhood. So, by Lemma 2.3, it follows that 0 is the eigenvalue of $A(G)$ with multiplicity $p^3 - p^2 - 1$. The corresponding eigenvectors of the eigenvalue 0 are

$$\Big(\underbrace{0, 0, \ldots, 0}_{p-1}, \underbrace{0, 0, \ldots, 0}_{p^2-p}, -1, 1, \underbrace{0, 0, \ldots, 0}_{p^3-p^2-2}\Big), \Big(\underbrace{0, 0, \ldots, 0}_{p-1}, \underbrace{0, 0, \ldots, 0}_{p^2-p}, -1, 0, 1, \underbrace{0, 0, \ldots, 0}_{p^3-p^2-3}\Big),$$
$$\vdots$$
$$\Big(\underbrace{0, 0, \ldots, 0}_{p-1}, \underbrace{0, 0, \ldots, 0}_{p^2-p}, -1, \underbrace{0, 0, \ldots, 0}_{p^3-p^2-3}, 1, 0\Big), \Big(\underbrace{0, 0, \ldots, 0}_{p-1}, \underbrace{0, 0, \ldots, 0}_{p^2-p}, -1, \underbrace{0, 0, \ldots, 0}_{p^3-p^2-2}, 1\Big).$$

Thus, with this method, we have obtained $p^3 - p^2 - 1 + p^2 - 3 = p^3 - 4$ eigenvalues. Next, we are required to find the remaining three eigenvalues of $A(G)$. Let X be the eigenvector of $A(G)$ with $x_i = X(v_i)$, for $i = 1, 2, 3, \ldots, p^3 - 1$. Then, it follows that (see, [4]) every component of X that corresponds to every vertex of A is equal to x_1, components of X corresponding to the vertices of B is x_2, and the components of X that corresponds to the vertices C is taken as x_3. Thus, with $X = \Big(\underbrace{x_1, x_1, \ldots, x_1}_{p-1}, \underbrace{x_2, x_2, \ldots, x_2}_{p^2-p}, \underbrace{x_3, x_3, \ldots, x_3}_{p^3-p^2}\Big)$, the eigenequation $A(G)X = \lambda X$ gives us

$$\begin{aligned}
\lambda x_1 &= \underbrace{x_1 + x_1 + \cdots + x_1}_{p-2} + \underbrace{x_2 + x_2 + \cdots + x_2}_{p^2-p} + \underbrace{x_3 + x_3 + \cdots + x_3}_{p^3-p^2}\\
\lambda x_2 &= \underbrace{x_1 + x_1 + \cdots + x_1}_{p-1} + \underbrace{x_2 + x_2 + \cdots + x_2}_{p^2-p-1} + 0 \cdot x_3\\
\lambda x_3 &= \underbrace{x_1 + x_1 + \cdots + x_1}_{p-1} + 0 \cdot x_2 + 0 \cdot x_3
\end{aligned}$$

The coefficient matrix of the right hand side of the above system of equations is

$$Q = \begin{pmatrix} p-2 & p^2-p & p^3-p^2 \\ p-1 & p^2-p-1 & 0 \\ p-1 & 0 & 0 \end{pmatrix}. \tag{10}$$

The characteristic polynomial of the above matrix Q given in (10) is

$$\lambda^3 - \lambda^2(p^2-3) + \lambda(2 - 2p^2 + 2p^3 - p^4) + p^6 - 3p^5 + 2p^4 + p^3 - p^2.$$

The remaining three eigenvalues of G are the zeros of the above polynomial. □

Remark 2.6 We attempted to compute the eigenvalues of the matrix Q using computer software, but we were unsuccessful because they are difficult to find. Although we may find the interval where these eigenvalues fall using the intermediate value theorem, it is currently useless, particularly when attempting to set bounds for the energy of G.

A graph G is said to be non-singular if $det(A(G)) \neq 0$. A graph is said to be without repeated eigenvalues if all its eigenvalues are simple (that is, with multiplicity one). Very often there are papers published related to properties of non-singular graphs and graphs with simple eigenvalues, see [6]. The following corollary is immediate from Theorem 2.5.

Corollary 2.7 *For prime p, the zero divisor graph $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ have the following properties.*

(i) $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ *is always singular.*
(ii) $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ *is never with simple eigenvalues for $p \geq 3$.*

Proof For $p \geq 2$, the cardinality of C is at least 4, so Lemma 2.3 implies that 0 is always the eigenvalues of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$. For the second case with $p = 2$, Lemma 2.4 implies that -1 is simple eigenvalue of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$. But Lemma 2.3 implies that 0 is the eigenvalue of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ with multiplicity 3. For $p \geq 3$, the eigenvalues -1 and 0 are always repeated zeros of the characteristic polynomial of the adjacency matrix of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$. Thus, for any prime the zero divisor graph $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ is always with repeated eigenvalues. □

Let $\xi_1 \geq \xi_2 \geq \xi_3$ be the zeros of polynomial (8) or the eigenvalues of the matrix Q given in (10). The energy of G equals

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{p^2-3} |-1| + |\xi_1| + |\xi_2| + |\xi_3| = p^2 - 3 + \mathcal{E}(Q), \tag{11}$$

where $\mathcal{E}(Q)$ is the energy of Q. We note that ξ_i are actually some λ_i's.

One trivial lower bound from (11) is

$$\mathcal{E}(G) \geq p^2 - 3.$$

For non-negative matrices ξ_1 is same as λ_1 and is always positive, so from (11), we have

$$\mathcal{E}(G) = p^2 - 3 + \xi_1 + |\xi_2| + |\xi_3| \geq p^2 - 3 + \xi_1 = p^2 - 3 + \lambda_1,$$

with equality holding if and only if $\xi_2 = \xi_3 = 0$ or ξ_2 and ξ_3 are symmetric towards origin, that is $\xi_2 = -\xi_3$. By noting that the spectral λ_1 of any graph is bounded above by maximum degree Δ, that is, $\lambda_1 \leq \Delta$, with equality holding if and only G is Δ regular. With this observation, we get

$$\mathcal{E}(G) \leq p^2 - 3 + p^3 - 2 + |\xi_2| + |\xi_3| = p^3 + p^2 - 5 + |\xi_2| + |\xi_3|,$$

since the maximum degree of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ is $n - 1 = p^3 - 2$. Equality occurs if and only if G is regular, and in that case Q is rank one matrix along with $\xi_2 = \xi_3 = 0$. Now, we establish the lower and the upper bounds for the energy of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$.

Let $\{q_1, q_2, q_3, \ldots, q_s\}$ be the set of positive real numbers and let Π_k be the average of products of k-element subset of $\{q_1, q_2, q_3, \ldots, q_s\}$, that is

$$\begin{aligned}
\Pi_1 &= \frac{q_1 + q_2 + q_3 + \cdots + q_s}{s},\\
\Pi_2 &= \frac{1}{\frac{s(s-1)}{2}}\Big(q_1q_2 + q_1q_3 + \cdots + q_1q_s + q_2q_3 + \cdots + q_{s-1}q_s\Big),\\
&\vdots\\
\Pi_s &= q_1q_2\ldots q_s.
\end{aligned}$$

The following Maclaurin symmetric mean inequality [3] relates Π_i's among themselves.

$$\Pi_1 \geq \Pi_2^{\frac{1}{2}} \geq \Pi_3^{\frac{1}{3}} \geq \cdots \geq \Pi_s^{\frac{1}{s}}, \tag{12}$$

with equalities holding if and only if $q_1 = q_2 = \cdots = q_s$.

Using these relations, in the next very result, we establish bounds for the energy of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$.

Theorem 2.8 *Let $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ be the zero divisor graph of $\mathbb{Z}_p[x]/\langle x^4\rangle$. Then*

$$\mathcal{E}(G) \geq p^2 - 3 + \sqrt{3p^4 - 4p^3 - 2p^2 + 5 + 6|p^2 - p^3 - 2p^4 + 3p^5 - p^6|^{\frac{2}{3}}}$$

and

$$\mathcal{E}(G) \leq p^2 - 3 + \sqrt{9p^4 - 12p^3 - 6p^2 + 15},$$

with equalities holding if and only if $|\xi_1(Q)| = |\xi_2(Q)| = |\xi_3(Q)|$.

Proof By (11), we have

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| = p^2 - 3 + \mathcal{E}(Q). \tag{13}$$

We will establish bounds for the entity $\mathcal{E}(Q)$. By applying Maclaurin (12) to the set

$$\Big\{|\xi_1(Q)|, |\xi_2(Q)|, |\xi_3(Q)|\Big\},$$

we have

$$\left(\frac{|\xi_1(Q)|+|\xi_2(Q)|+|\xi_3(Q)|}{3}\right)^2 \geq \frac{1}{\frac{3(3-1)}{2}}\Big(|\xi_1(Q)||\xi_2(Q)|+|\xi_1(Q)||\xi_3(Q)|+|\xi_2(Q)||\xi_3(Q)|\Big), \tag{14}$$

that is,

$$\begin{aligned}\Big(|\xi_1(Q)|+|\xi_2(Q)|+|\xi_3(Q)|\Big)^2 &\geq 3\Big(|\xi_1(Q)||\xi_2(Q)|+|\xi_1(Q)||\xi_3(Q)|+|\xi_2(Q)||\xi_3(Q)|\Big)\\ &= \frac{3}{2}\left(\Big(\sum_{i=1}^{3}|\xi_i(Q)|\Big)^2-\sum_{i=1}^{3}\xi_i^2(Q)\right),\end{aligned}$$

that is,

$$\Big(\mathcal{E}(Q)\Big)^2 = \left(\sum_{i=1}^{3}|\xi_i(Q)|\right)^2 \leq 3\sum_{i=1}^{3}\xi_i^2 = 3\cdot tr(Q^2),$$

where $tr(Q^2)$ is the trace of Q^2. From (10), the trace of Q^2 is

$$\begin{aligned}tr(Q^2) &= (p-2)^2+(p^2-p-1)^2+2(p-1)(p^2-p)+2(p-1)(p^3-p^2)\\ &= 3p^4-4p^3-2p^2+5.\end{aligned}$$

Substituting it in above expression, we obtain

$$\mathcal{E}(Q) \leq \sqrt{3\cdot tr(Q^2)} = \sqrt{3(3p^4-4p^3-2p^2+5)}.$$

Thus by (13), we get

$$\mathcal{E}(G) = p^2-3+\mathcal{E}(Q) \leq p^2-3+\sqrt{9p^4-12p^3-6p^2+15}.$$

Equality holds if and only if equality holds in (14), that is, $|\xi_1(Q)| = |\xi_2(Q)| = |\xi_3(Q)|$. Again from the second inequality of (12), we have

$$\frac{1}{\frac{3(3-1)}{2}}\Big(|\xi_1(Q)||\xi_2(Q)|+|\xi_1(Q)||\xi_3(Q)|+|\xi_2(Q)||\xi_3(Q)|\Big) \geq \Big(|\xi_1(Q)||\xi_2(Q)||\xi_1(Q)|\Big)^{\frac{2}{3}},$$

that is, equivalent to

$$2\Big(|\xi_1(Q)||\xi_2(Q)|+|\xi_1(Q)||\xi_3(Q)|+|\xi_2(Q)||\xi_3(Q)|\Big) \geq 6\Big(|\xi_1(Q)||\xi_2(Q)||\xi_1(Q)|\Big)^{\frac{2}{3}},$$

that is,

$$(|\xi_1(Q)|+|\xi_2(Q)|+|\xi_1(Q)|)^2-\Big(|\xi_1(Q)|^2+|\xi_2(Q)|^2+|\xi_3(Q)|^2\Big) \geq 6|det(Q)|^{\frac{2}{3}},$$

that is,

$$\mathcal{E}(Q) = \sum_{i=1}^{3} |\xi_i(Q)| \geq \sqrt{tr(Q)^2 + 6|det(Q)|^{\frac{2}{3}}}.$$

From (10), the determinant of Q is $det(Q) = -(p-1)(p^2-p-1)(p^3-p^2) = p^2 - p^3 - 2p^4 + 3p^5 - p^6$. Therefore, from (13), we have

$$\mathcal{E}(G) \geq p^2 - 3 + \sqrt{3p^4 - 4p^3 - 2p^2 + 5 + 6|p^2 - p^3 - 2p^4 + 3p^5 - p^6|^{\frac{2}{3}}},$$

with equality holding if and only if $|\xi_1(Q)| = |\xi_2(Q)| = |\xi_3(Q)|$. □

Remark 2.9 By substituting $p = 3$, in Theorem 2.8, the energy of $G \cong \Gamma(\mathbb{Z}_3[x]/\langle x^4\rangle)$ satisfies

$$23.6997 \leq \mathcal{E}(G) \leq 25.1311.$$

The actual energy of G is $\mathcal{E}(G) = 24.0869$, which is close to the values given by Theorem 2.8.

Since the energy of K_n is $2(n-1)$, and a graph G of order n is said to be hyperenergetic if $\mathcal{E}(G) > 2(n-1)$ and non-hyperenergetic if $E(G) < 2(n-1)$. A graph G with $E(G) = 2(n-1)$ is referred to as borderenergetic. A graph G is called hypoenergetic if $\mathcal{E}(G) < n$. Over the years many papers were written for borderenergetic, hyperenergetic, hypoenergetic and non-hyperenergetic, for details see [7, 8, 13, 30]. We have the following result in this direction.

Proposition 2.10 *For any prime p, the following holds for the zero divisor graph* $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ *of ring* $\mathbb{Z}_p[x]/\langle x^4\rangle$.

(i) *For* $p \geq 2$, $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ *is non-hyperenergetic.*
(ii) *For* $p \geq 3$, $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ *is hypoenergetic.*

Proof Let $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ be the zero divisor graph of order $n = p^3 - 1$. Also, it is well known that the energy of the complete graph of order $n = p^3 - 1$ is $\mathcal{E}(K_n) = 2(p^3 - 2)$. By Theorem 2.8, the energy of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ satisfies

$$\mathcal{E}(\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)) \leq p^2 - 3 + \sqrt{9p^4 - 12p^3 - 6p^2 + 15}.$$

Suppose to the contrary $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ is hyperenergetic, that is,

$$\mathcal{E}(\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)) > 2(p^3 - 2),$$

that is,

$$p^2 - 3 + \sqrt{9p^4 - 12p^3 - 6p^2 + 15} > 2(p^3 - 2),$$

which after simplification gives

$$2p^6 - 2p^5 - 4p^4 + 4p^3 + 4p^2 - 7 < 0. \tag{15}$$

Clearly Inequality (15) is not true, since $f(p) = 2p^6 - 2p^5 - 4p^4 + 4p^3 + 4p^2 - 7$ is an increasing function for primes $p \geq 2$. In fact the global minimum attained by $f(p)$ is 41 for $p \geq 2$. The graphical representation of Inequality (15) is shown in Figure 2

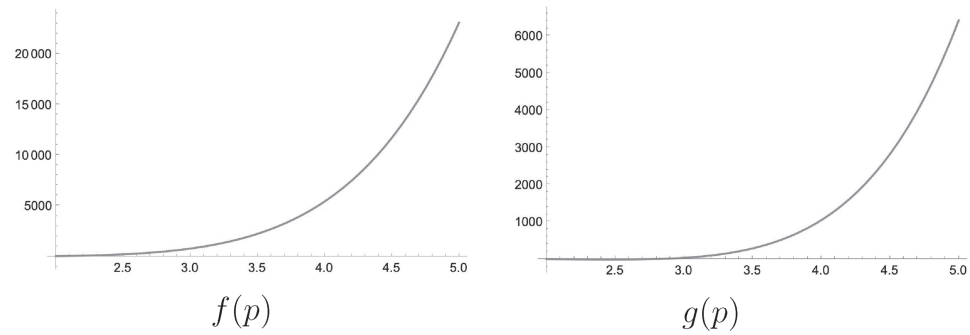

Fig. 2 Picture showing the increasing behaviour of $f(p)$ and $g(p)$

Therefore, $\mathcal{E}(\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)) < 2p^3 - 4$ and hence is non-hyperenergetic for any prime p.

(b). Comparing $\mathcal{E}(\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)) \le p^2 - 3 + \sqrt{9p^4 - 12p^3 - 6p^2 + 15} < p^3 - 1$, we obtain

$$g(p) = p^6 - 2p^5 - 8p^4 + 16p^3 + 2p^2 - 11 > 0,$$

which is always true for any prime $p \ge 3$ and is negative for $p \in [2, 3)$. The plot of $g(p)$ is shown in Figure 2. □

3 Topological indices of zero divisor graphs of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$

For a graph G, a general vertex-degree-based (VDB) topological index ϕ [10] is defined as

$$\phi(G) = \sum_{uv\in E(G)} \phi_{d_u d_v},$$

$\phi_{d_u d_v}$ is a function with the property $\phi_{d_u d_v} = \phi_{d_v d_u}$. For particular values of $\phi_{d_u d_v}$, we get well know topological indices like arithmetic-geometric index [26] $\phi_{d_u d_v} = \frac{d_u + d_v}{2\sqrt{d_u d_v}}$, the general Randić index [28] $\phi_{d_u d_v} = (d_u d_v)^\alpha$, $\Big($for $\alpha = -\frac{1}{2}$, we obtain the ordinary Randić index [29] $R = \displaystyle\sum_{uv\in E(G)} \frac{1}{\sqrt{d_u d_v}}$, and for $\alpha = 1$ we get the second Zagreb index [12], $M_2(G) = \sum_{uv\in E(G)} (d_u \cdot d_v)\Big)$, the general Sombor index $\phi_{d_u d_v} = \left(d_u^2 + d_v^2\right)^\alpha$, the general sum connectivity index [35] with $\phi_{d_u d_v} = (d_u + d_v)^\alpha$ and for $\alpha = 1$, we get the first Zagreb index [12], $M_1(G) = \sum_{uv\in E(G)} (d_u + d_v)$, the general inverse sum indeg index [11] $\phi_{d_u d_v} = \left(\frac{d_u d_v}{d_u + d_v}\right)^\alpha$ and for $\alpha = 1$, we get the inverse inverse sum indeg index. Similarly, all the degree based topological indices can be generated by the function $\phi_{d_u d_v}$. The exponential of the topological indices as given as $e^{\phi}(G) = \sum_{uv\in E(G)} e^{\phi_{d_u d_u}}$ [27].

A topological index of G is an entity (numeric) related to a graph structure that links chemical structure with certain physico-chemical relation, like biological activity or chemical reactivity. With molecular modelling, the relationship between the structural properties and activity of chemical compounds can be recorded/observed [9]. The aim of descriptors were discovered to play a significant role in (QSPR/QSAR) analysis of molecular structures, earlier Wiener in 1947 [33].

The following result gives a general formula for the topological indices for the zero divisor graph of $\mathbb{Z}_p[x]/\langle x^4\rangle$.

Theorem 3.1 *For $G \cong \Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$, the VDB topological index is*

$$\binom{p-1}{2}\phi_{d_u d_u} + (p-1)(p^2-p)\phi_{d_u d_v} + \binom{p^2-p}{2}\phi_{d_v d_v} + (p^3-p^2)(p-1)\phi_{d_u d_w},$$

where d_u represent the common degree of the vertices in A, d_v represent the common degree of the vertices in B and d_w represent the common degree of the vertices in C.

Proof Since the vertices of G are partitioned into three mutually disjoint sets A, B and C. The vertices in A form a clique of order $p-1$, the vertices in B form a clique of size p^2-p and each vertex of A is connected to every vertex of B. Also, C is an independent set of cardinality p^3-p and each such vertex is adjacent to every vertex of A. With this structure, the number of edges inside A are $\binom{p-1}{2}$, that of in B are $\binom{p^2-p}{2}$, between A and B, there are $(p-1)(p^2-p)$ edges and between A and C, there are $(p-1)(p^3-p^2)$ edges. Let the vertex labelling of G be

$$\{u_1, u_2, \ldots, u_{|A|}, v_1, v_2, \ldots, v_{|B|}, w_1, w_2, \ldots, w_{|C|}\},$$

where u_i's are the vertices of A, v_i's are the vertices of B and w_i's are the vertices of C. From the structure of G, it is clear that the vertices in A have common degree p^3-2 and we denote it by $d_u = p^3-2$. Similarly, the vertices in B and C have common degrees and are denoted by $d_v = p^2-2$ and $d_w = p-1$, respectively. Thus, the VDB topological index of G is

$$\begin{aligned}\phi(G) &= \sum_{u_iu_j\in E(G)} \phi_{d_{u_i}d_{u_j}} + \sum_{u_iv_j\in E(G)} \phi_{d_{u_i}d_{v_j}} + \sum_{v_iv_j\in E(G)} \phi_{d_{v_i}d_{v_j}} + \sum_{u_iw_j\in E(G)} \phi_{d_{u_i}d_{w_j}}\\ &= \binom{p-1}{2}\phi_{d_ud_u} + (p-1)(p^2-p)\phi_{d_ud_v} + \binom{p^2-p}{2}\phi_{d_vd_v} + (p^3-p^2)(p-1)\phi_{d_ud_w}.\end{aligned}$$

□

The following is an immediate consequence of Theorem 3.1, which gives the closed formula for the first and the second Zagreb index of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$.

Corollary 3.2 *For $G \cong \Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$, the following hold.*

(i) $M_1(G) = p^7 - 11p^4 + 11p^3 + 3p^2 - 4$.
(ii) $M_2(G) = \frac{1}{2}\Big(6p^8 - 15p^7 + 2p^6 + 3p^5 + 32p^4 - 28p^3 - 8p^2 + 8\Big)$.

Proof By substituting $\phi_{d_ud_v} = d_u + d_v$ and $\phi_{d_ud_v} = d_u \cdot d_v$ in Theorem 3.1, respectively. The result follows. □

We state the result of [16] for the Zagreb indices M_1 and M_2 for the zero divisors of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$.

Theorem 3.3 (Theorem 3.4, [16]) *Let $R = \mathbb{Z}_p[x]/\langle x^4\rangle$ with prime p, then*

$$\begin{aligned}M_1(\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)) &= p^7 - 11p^4 + 11p^3 + 3p^2 - 4\\ M_2(\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)) &= 12(6p^8 - 15p^7 + 2p^6 + 3p^5 + 32p^4 + 28p^3 - 8p^2 - 8).\end{aligned}$$

As we observe that the values for $M_2(G)$ are different in Theorem 3.3 than given in Corollary 3.2. We observe that there is calculation correction in Theorem 3.4 [16], in fact in the proof of Theorem 3.3, the evaluation of $M_2(G)$ in third line is written incorrectly. We verify the second Zagreb index by software and by Corollary 3.2 for the zero divisor graph $\Gamma(\mathbb{Z}_3[x]/\langle x^4\rangle)$ in the following remark.

Remark 3.4 For the graph $G \cong \Gamma(\mathbb{Z}_3[x]/\langle x^4\rangle)$ shown in Figure 1. We calculate the values of M_1 and M_2. With $p = 3$ in Corollary 3.2, the first and seconds Zagreb indices are

$$M_1(G) = 1616 \text{ and } M_2(G) = 5260.$$

While by By Theorem 3.3 (Theorem 3.4, [16]) with $p = 3$, the value of the second Zagreb index is $M_2(G) = 6008$, but by software calculation (Mathematica, SageMath or AutographiX) and by Corollary, the correct value of $M_2(G)$ is 5260. Thus, the formula for $M_2(\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle))$ given in Theorem 3.3 (Theorem 3.4, [16]) is not correct.

Hansen and Vukičcević [14] compared the first and the second Zagreb indices, and put forward the following conjecture.

Conjecture 1 *For a simple graph G,*

$$\frac{M_2(G)}{|E(G)|} \geq \frac{M_1(G)}{|V(G)|}. \tag{16}$$

The authors in [14] showed that Conjecture 1 holds for chemical graphs. In [31], it was shown that the conjecture holds for trees with equality for a star graph. In [18], the authors settled the conjecture 1 for connected unicyclic graphs with equality when the graph is a cycle. The equality case of 16 is studied in [32]. A survey on comparing Zagreb indices can be seen in [19].

For the graph $G \cong \Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$. The second Zagreb Index given in Corollary 3.2 can be written as

$$M_2(G) = \frac{1}{2}(p-1)\left(6p^7 - 9p^6 - 7p^5 - 4p^4 + 28p^3 - 8p - 8\right),$$

the edge cardinality of G is

$$|E(G)| = \frac{1}{2}(p-1)\left(3p^3 - p^2 - 2p - 2\right).$$

By Inequality 16, we have

$$\frac{6p^7 - 9p^6 - 7p^5 - 4p^4 + 28p^3 - 8p - 8}{3p^3 - p^2 - 2p - 2} \geq \frac{p^7 - 11p^4 + 11p^3 + 3p^2 - 4}{p^3 - 1}.$$

This gives us

$$(p-1)p^2\left(3p^7 - 5p^6 - 10p^5 + 15p^4 + 8p^3 - 5p^2 - 6p - 2\right) \geq 0. \tag{17}$$

Since $p - 1$ and p^2 are positive for $p \geq 2$, so we consider

$$h(p) = 3p^7 - 5p^6 - 10p^5 + 15p^4 + 8p^3 - 5p^2 - 6p - 2.$$

The global minimum value attained by $h(p)$ is 14 at $p = 2$, and the function is strictly increasing for $p \geq 3$. Thus, Inequality 17 is true and it follows that Inequality (16) is true for $G \cong \Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$. Hence, we have the following result.

Proposition 3.5 *Let $G \cong \Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ be the zero divisor graph. Then the first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ satisfy Conjecture 1.*

The last result is a consequence of Theorem 3.1 and gives some well known general topological indices of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$.

Corollary 3.6 *For $G \cong \Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$, with $d_u = p^3 - 2$, $d_v = p^2 - 2$ and $d_w = p - 1$, the following hold.*

(i) *The general Randić index of G is*

$$R_\alpha(G) = \binom{p-1}{2}(d_u^2)^\alpha + \binom{p^2-p}{2}(d_v^2)^\alpha + p(p-1)^2(d_u d_v)^\alpha + p^2(p-1)^2(d_v d_w)^\alpha.$$

(ii) *The general sum-connectivity index of G is*

$$\chi_\alpha(G) = \binom{p-1}{2}(2d_u)^\alpha + \binom{p^2-p}{2}(2d_v)^\alpha + p(p-1)^2(d_u+d_v)^\alpha + p^2(p-1)^2(d_v+d_w)^\alpha.$$

(iii) *The general inverse sum indeg index of G is*

$$S_\alpha(G) = \binom{p-1}{2}\left(\frac{d_u}{2}\right)^\alpha + \binom{p^2-p}{2}\left(\frac{d_v}{2}\right)^\alpha + p(p-1)^2\left(\frac{d_u d_v}{d_u+d_v}\right)^\alpha + p^2(p-1)^2\left(\frac{d_v d_w}{d_v+d_2}\right)^\alpha.$$

(iv) *The general Sombor index of G is*

$$SO_\alpha(G) = \binom{p-1}{2}(2d_u^2)^\alpha + \binom{p^2-p}{2}(2d_v^2)^\alpha + p(p-1)^2(d_u^2+d_v^2)^\alpha + p^2(p-1)^2(d_v^2+d_w^2)^\alpha.$$

Proof The proof follows by taking $\phi_{d_u d_u} = (d_u d_v)^\alpha$, $\phi_{d_u d_u} = (d_u + d_v)^\alpha$, $\phi_{d_u d_u} = \left(\frac{d_u d_v}{d_u+d_v}\right)^\alpha$, and $\phi_{d_u d_u} = (d_u^2 + d_v)^\alpha$ in Theorem 3.1. □

4 Conclusion

The primary findings of Johnson and Sankar [16] are corrected by the current work, which also corrects the results pertaining to the characteristic polynomial, the energy, and the Zagreb indices of the zero divisor graphs of $\mathbb{Z}_p[x]/\langle x^4\rangle$. Additionally, we demonstrate that the Hansen and Vukičcević conjecture is valid for $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ and provide a closed formula that allows all of the well-known topological indices of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ can be obtained from the aforementioned result.

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Declarations

Conflicts of Interest The authors declare that they have no competing interests.

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