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ORIGINAL RESEARCH

On the z-classes in Fuchsian groups

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Abstract We investigate the conjugacy classes of centralizers or the z-classes in a Fuchsian group. We classify the z-classes of parabolic and hyperbolic elements. As an application, we classify the z-classes in the Hecke groups.

Keywords z-Classes · Hecke groups · Fuchsian groups

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1 Introduction

Given a group G, the centralizer $Z(x)$ of an element $x \in G$ consists of all elements $y \in G$ such that x and y commute with each other. We say x and y are z-conjugate or belong to the same z-class, denoted by $x \sim_z y$, if the centralizers $Z(x)$ and $Z(y)$ are conjugate in G, i.e. there is a $g \in G$ such that $Z(x) = gZ(y)g^{-1}$. Clearly, $\sim_z$ is an equivalence relation and it partitions G into disjoint classes. The notion was implicit in classical literature in geometry and group theory. Kulkarni introduced the terminology 'z-class' in [6] and pointed out its relevance with 'dynamical types' arising in geometry. Kulkarni proved a set theoretic parametrization of a z-class in a group. cf. [6, Theorem 2.1]. Following the program suggested in [6], the z-classes in hyperbolic geometries were classified in [3,4]. For a survey on the topic and further literature in this direction, see [1].

Recently, Parsad [8] has investigated the z-classes in the mapping class group of a closed orientable surface. This has motivated us to ask for the z-classes in any Fuchsian group. Since the centralizers of elements in a Fuchsian group are known to be cyclic, two elements g and h are in the same z-class in a Fuchsian group Γ

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if and only if the cyclic subgroups $\langle g\rangle$ and $\langle h\rangle$ are conjugate in Γ. In this note, we offer a parametrization of the z-classes for non-elliptic elements of a Fuchsian group whose Dirichlet domain do not have a side at the boundary at infinity, see Sect. 3. We prove that the z-classes of hyperbolic elements in a Fuchsian group Γ corresponds to the Γ-congruence classes of fixed points of primitive hyperbolic elements of Γ. The results in this note are consequences of classical results and could have been observed by experts. However, we have not seen any literature with a viewpoint to understand the z-classes in Fuchsian groups.

We further apply our observation to classify the z-classes in the Hecke groups H_p whose notion we recall below. In particular, for $p=3$, the Hecke group H_3 is the modular group $\mathrm{PSL}(2,\mathbb{Z})$.

Definition 1 The Hecke group H_p is a Fuchsian group which is generated by the maps

$$\iota : z \mapsto -\frac{1}{z}$$

and

$$\alpha_p : z \mapsto z + 2\cos(\pi/p), \;\; p \geq 3.$$

The Hecke group H_p can be identified with the free product $\mathbb{Z}_2 * \mathbb{Z}_p$. We will denote $\iota\alpha_p$ by γ_p. This is an element of order p.

We give a complete classification of the z-classes for Hecke groups as follows. Let D be the Dirichlet domain for H_p. Let $C(u,\bar{u})$ denote the geodesic in $\mathbb{H}^2$ with endpoints u and $\bar{u}$. Consider the subset $\mathcal{H}$ of $\mathbb{R}^2$ whose points are the (unordered) tuples $(u,\bar{u})$, where u and $\bar{u}$ are either the solutions of the equation $cx^2+(d-a)x-b=0$, $a,b,c,d \in \mathbb{Z}[\lambda_p]$, $(a,c)_p=(b,d)_p=1$, $|a+d|>2$ and $C(u,\bar{u})\cap D \neq \emptyset$, or they are end-points of a uniquely chosen representative of a H_p equivalence class of sides of D. Here, the notation $(a,c)_p$ comes from the pseudo-Euclidean algorithm developed in [7, 3.3.]. The z-classes in the Hecke group H_p are given by the following theorem.

Theorem 1 *In the Hecke group* $\mathrm{H_p}$*, the conjugacy classes of centralizers, or the z-classes, are given by the following.*

(1) *There are exactly two z-classes for elliptic elements represented by* ι *and* γ_p*.*
(2) *There is exactly one z-class for parabolic elements represented by* α_p*.*
(3) *The z-classes for hyperbolic elements are parametrized by the the set* $\mathcal{H}$*.*

The proof of this theorem is noted in Sect. 4.

2 Preliminaries

We consider the upper half model of the hyperbolic plan $\mathbb{H}^2$. The boundary $\partial\mathbb{H}^2$ may be identified with the extended real line $\mathbb{R}\cup\{\infty\}$. The group $\mathrm{SL}(2,\mathbb{R})$ acts on the upper half plane by Möbius transformation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d}.$$

The orientation preserving group of isometries of $\mathbb{H}^2$ i.e., $\mathrm{Isom}^+(\mathbb{H}^2)$ can be seen as $\mathrm{PSL}(2,\mathbb{R})$. The lift of the isometry g in $\mathrm{PSL}(2,\mathbb{R})$ is denoted as $\tilde{g}$ in $\mathrm{SL}(2,\mathbb{R})$. An isometry g of $\mathbb{H}^2$ belongs to one of the following three dynamical types:

(i) *elliptic* if g has a fixed point on $\mathbb{H}^2$, equivalently if $|trace(\tilde{g})|<2$.
(ii) *parabolic* if g has exactly one fixed point on $\partial\mathbb{H}^2$, equivalently if $|trace(\tilde{g})|=2$.
(iii) *hyperbolic* if g fixes two points on $\partial\mathbb{H}^2$, equivalently if $|trace(\tilde{g})|>2$.

Let g be a hyperbolic element represented by the martix $\tilde{g} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{SL}(2, \mathbb{R})$ such that $c \neq 0$. Then, the fixed points of g are irrational conjugates: $\dfrac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c}$.

A discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ is known as a Fuchsian group. The following is well-known, cf. [5].

Lemma 1 *Let Γ be a Fuchsian group. The centralizer of an element g in Γ is cyclic. It is finite if g is elliptic, and it is an infinite cyclic group if g is either parabolic or hyperbolic.*

An immediate consequence of this lemma is the following.

Corollary 1 *Let M and N do not have the same dynamical types. Then they can not belong to the same z-class in a Fuchsian group Γ.*

Finally we note that every discrete subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ has a fundamental domain, cf. [5]. A standard example is the Dirichlet domain D_Γ or simply, D, which gives a locally finite convex fundamental domain for Γ. It is the interior of a polygon bounded by sides which may be either geodesic segments or segment of the real line. The closure of D, i.e. the interior along with its sides, is denoted by $\bar{D}$.

A standard fact is the following.

Theorem 2 [5] *If D is the fundmanetal domain for a Fuchsian group Γ then $\mathbb{H}^2/\Gamma$ is homeomorphism to $\bar{D}/\Gamma$.*

2.1 Pseudo Euclidean algorithm [7]

Let a, b be non-zero elements in $\mathbb{Z}[\lambda_p]$. There exists an integer n such that $a = b(n\lambda_p) + r$ where $-|b\lambda_p|/2 \leq r \leq |b\lambda_p|/2$. This is called the *Pseudo Euclidean algorithm*. Repeatedly applying this algorithm, if it terminates at the $(m+1)^{th}$-step, i.e. the remainder $r_{m+1} = 0$, then $(a, b)_p := |r_m|$. Also,

(1) $(a, b)_p = (b, a)_p$.
(2) $(a, b)_p = (-a, -b)_p = (-a, b)_p = (a, -b)_p$.

Proposition 1 [7, Proposition 3.7] *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a lift in $\mathrm{SL}(2, \mathbb{R})$ of an element β in $\mathrm{PSL}(2, \mathbb{Z}[\lambda_p])$. Then $\beta \in \Gamma_p$ if and only if $(a, c)_p = (b, d)_p = 1$.*

3 z-Classes of non-elliptic elements in a Fuchsian group

Next, we prove the following lemma which will help us to deal with the z-classes of non-elliptic elements.

Lemma 2 *Let Γ be a non-elementary Fuchsian group. Let $M, N \in \Gamma$ be two non-elliptic elements. Then M and N belong to the same z-class if and only if there exists $g \in \Gamma$ such that g maps the fixed point set of M to the fixed point set of N.*

Proof First assume, both M and N are hyperbolic. Without loss of generality, assume further that M, N are both primitive elements. Suppose M has fixed points $z, \bar{z}$.

Suppose $M \sim_z N$. Let $g \in \Gamma$ be such that $C(M) = g^{-1}C(N)g$. This implies that for some $n, m \in \mathbb{Z}$,

$$M^n = g^{-1}N^m g.$$

Now, M and M^n have the same fixed points. Then

$$M^n z = z$$

$$\Rightarrow g^{-1}N^m gz = z$$
$$\Rightarrow N^m(gz) = gz.$$

Thus, N fixes $\{g(z), g(\bar{z})\}$.

Conversely, suppose there is an element g in H_p such that it maps the fixed points of M to the fixed points of N. If $Fix(M) = \{z, \bar{z}\}$, then $Fix(N) = \{g(z), g(\bar{z})\}$. If g is identity then this is done. If not, then consider $\alpha = gMg^{-1}N^{-1} \in \Gamma$. If α is identity, then M and N are conjugate so $M \sim_z N$. If not then α fixes the two points $g(z)$ and $g(\bar{z})$. So, it must be some power of N. This implies,

$$gMg^{-1}N^{-1} = N^k \qquad \text{for some } k \in \mathbb{Z}.$$

So, M is conjugate to some power of N, and hence they are in same z-class.

If M and N are parabolic elements, then similar arguments follow. □

Corollary 2 *Let $\mathcal{Z}_h$ be the set of the z-classes of hyperbolic elements in Γ. Consider the set $\mathcal{F}$ that consists of the Γ-congruent classes of unordered tuples of boundary points $(u, \bar{u})$ which are the fixed points of primitive hyperbolic elements in Γ. Then $\mathcal{F}$ has a bijective correspondence with the set $\mathcal{Z}_h$.*

Let Γ be a non-elementary Fuchsian group such that the Dirichlet domain D of Γ is bounded by geodesic segments and does not contain a non-trivial segment of the boundary at infinity. Recall that two pairs of fixed points, or geodesic segments, $(u, \bar{u})$ and $(v, \bar{v})$ on $\mathbb{H}^2$ are called *congruent* if they belong to the same Γ orbit. Two line segments in $\bar{D}$ can be congruent only if they belong to the boundary of D. The congruence is an equivalence relation. Suppose, γ and γ' are two congruent sides of D. Then the end-points $(u, \bar{u})$ and $(v, \bar{v})$ are also congruent to each other. We consider a unique representative for each class.

To parametrize the set $\mathcal{Z}_h$ we shall use the bijection with $\mathcal{F}$. We choose a parameter set $\mathcal{H}$ as follows. A point on $\mathcal{H}$ is either a fixed point $(u, \bar{u})$ of a hyperbolic element f_u whose axis intersects D, or it is a unique representative from the congruence class of points which are end-points of hyperbolic axes that appear as side pairs of D.

Define a map $\Phi : \mathcal{H} \to \mathcal{Z}_h$ by $(u, \bar{u}) \to [f_u]$. Since hyperbolic elements with the same fixed points commute with each other, this map is clearly well-defined.

Theorem 3 *The map Φ is a bijection.*

Proof Let A and B be hyperbolic elements in the same z-class. Then by Lemma 2, there is an element g in Γ such that g maps the hyperbolic axis a of A to the hyperbolic axis b of B. Let $\Pi : \mathbb{H}^2 \to \mathbb{H}^2/\Gamma$ denote the projection map. Then

$$\begin{aligned}\Pi(b) &= \{\Gamma z : z \in b\}\\ &= \{\Gamma g(w) : w \in a\}\\ &= \{\Gamma w : w \in a\} = \Pi(a).\end{aligned}$$

Thus $\Pi(a)$ and $\Pi(b)$ projects to the same geodesic in $\mathbb{H}^2/\Gamma$. Since $\mathbb{H}^2/\Gamma$ is homeomorphic to $\bar{D}/\Gamma$, $\Pi(a)$ and $\Pi(b)$ either lift to the same semi-circle γ intersecting D, or they correspond to congruent sides of D. Accordingly there is a unique representative $(u, \bar{u})$ in $\mathcal{H}$, the end points of γ that corresponds to an element of $\mathcal{F}$. Thus By Corollary 2, Φ is injective.

We now show that Φ is surjective. Suppose, $[A] \in \mathcal{Z}_h$. If the hyperbolic axis I_A of A intersects D, then we are done, and if the axis of A corresponds to a side of D, then by congruence there is a pre-image. Otherwise, A has to intersect $\psi\bar{D}$ for some $\psi \in \Gamma$ since, the closure of the fundamental domain tessellates the entire hyperbolic plane by Γ orbits. So there exist an element $\psi \in \Gamma$ such that

$$\psi(\bar{D}) \cap C(u, \bar{u}) \neq \emptyset$$

$$\Rightarrow \bar{D} \cap \psi^{-1} I_B \neq \emptyset$$
$$\Rightarrow \bar{D} \cap I_{\psi B \psi^{-1}} \neq \emptyset.$$

If $I_{\psi B \psi^{-1}}$ intersects D, we are done. If it contains the side of the boundary of D then by congruence there is a unique representative of the fixed point class in $\mathcal{H}$ as pre-image. Thus, Φ is surjection. □

Lemma 3 *The z-classes for the parabolic elements in Γ have one-to-one correspondence to the congruence classes of vertices at infinity of D.*

Proof This is direct application of Lemma 2. For any point at infinity there is a cyclic group of parabolic elements i.e., the centraliser of the parabolic elements associate with it. If two parabolic element are in same z-class then there is an element in the Fuchsian group that maps one fixed point to another one. So conjugate centralizers of parabolic elements correspond to a unique class of congruent vertices of $\bar{D} \cap \partial\mathbb{H}^2$. This proves the theorem. □

4 z-Classes in Hecke groups

The Dirichlet domain for the Hecke group is given by the set, cf. [2],

$$D = \left\{ z \in \mathbb{H}^2 : |Re(z)| < \cos\frac{\pi}{p}, |z| > 1 \right\}.$$

Let g be a hyperbolic element in H_p whose lift is given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then g in H_p can not have ∞ as a fixed point since there are the parabolic element which fixes ∞ in the generating set. Otherwise, it will contradict the discreteness in H_p. Thus $c \neq 0$, and the fixed points of a hyperbolic element in H_p are irrational conjugates $u, \bar{u}$ on $\mathbb{R}$. The fixed points $u, \bar{u}$ are precisely solutions of the equation $cx^2 + (d - a)x - b = 0$ where a, b, c, d satisfy the conditions in Proposition 1. The set $\mathcal{H}$ contains such fixed-point tuples $(u, \bar{u})$ with either the condition $C(u, \bar{u}) \cap D = \emptyset$ or, one of the side of D is a part of $C(u, \bar{u})$.

From Corollary 2 we see the following.

Corollary 3 *Let $\mathcal{F}_h$ be the set of the H_p orbits of the fixed points $(u, \bar{u})$ of hyperbolic elements in H_p. Then $\mathcal{F}_h$ has one to one correspondence with the set of the z-classes of hyperbolic elements in H_p.*

By Theorem 3, we have:

Corollary 4 *The map Φ gives a bijection of $\mathcal{H}$ with the set of the z-classes of hyperbolic elements in H_p.*

Also we have the following from Lemma 3.

Corollary 5 *There is only one z-class for all parabolic elements in the Hecke group Γ_p.*

Finally we show the following.

Lemma 4 *There are two z-classes for elliptic elements in $\mathrm{H_p}$. These two classes are represented by the order two element ι and the order p element γ_p respectively.*

Proof The Hecke group $\mathrm{H_p}$ is identified with $\mathbb{Z}_2 * \mathbb{Z}_p$ where $p \geq 3$. At first we choose a elliptic element S in H_p. If T is an element in $C(S)$, then $ST = TS$, and accordingly, S and T fix the same point on $\mathbb{H}^2$. Thus, $C(S)$ consists of only elliptic elements beside identity, and hence this is a finite cyclic group.

If the order of S is m, we consider S be the order of m, then m is either p or 2. If order of S divides p, then

$$C(S) = \left\{1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{p-1}\right\},$$

where, α is a primitive element of S. If the order of S is 2, then

$$C(S) = \{1, S\}.$$

If h is another elliptic element in H_p, then h is conjugate to either ι or a power of γ_p. If $h = g\gamma_p^r g^{-1}$ where $r \in \{1, 2, \ldots, p-1\}$ and $g \in \mathrm{H}_p$. Then, the centralizer of h is conjugate to the centralizer γ_p. That means h and γ_p are in the same z-class. If not, then h is conjugate to ι. Thus h and ι are in same z-class. □

4.1 Proof of Theorem 1

The proof follows from Lemma 4, Corollaries 4 and 5. □

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