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ORIGINAL RESEARCH

Harmonic mean Sylow numbers of nonsolvable groups

C. S. Anabanti · A. K. Asboei

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Abstract Let G be a finite group. We write $hsn(G)$ for the harmonic mean Sylow number of G. The Fitting subgroup of G is denoted by $F(G)$. It is known that if $hsn(G) < \frac{45}{7}$, then G is solvable. In this paper, we extend the study to nonsolvable groups. We prove that if G is a finite nonsolvable group, then the following holds:

(a) if $hsn(G) < \frac{360}{47}$ and $hsn(G) \neq \frac{480}{71}$, then $G/F(G) \cong A_5$;
(b) if $hsn(G) = \frac{480}{71}$, then $G/N \cong A_5$, where N is the largest normal solvable subgroup of G.

Keywords Finite groups · Nonsolvable groups · Sylow number

Mathematics Subject Classification 05E16 · 20D10 · 20D20 · 20D60

1 Introduction

In this paper, all groups are finite. Let G be a finite group. The set of all prime divisors of $|G|$ and the number of Sylow p -subgroups of G are denoted by $\pi(G)$ and $n_p(G)$ respectively. The largest nilpotent normal subgroup of G is called the Fitting subgroup of G, and is denoted by $F(G)$.

Let $S(G) = \{p \in \pi(G) |\ n_p(G) > 1\}$. Adopting the notation of [4], we denote the harmonic mean Sylow number of G by $hsn(G)$, and defined it as follows:

$$hsn(G) := \frac{|S(G)|}{\sum\limits_{p \in S(G)} \frac{1}{n_p(G)}}.$$

Other notations are standard, and can be found in [1,2,6]. We recall the following result of [4].

Theorem 1 *[4] Let G be a finite group. If* $hsn(G) < \frac{45}{7}$*, then G is solvable.*

We recently observed that $hsn(A_5 \times C_n) = \frac{45}{7}$ for all $n \in \{0, 1, 2, \dots\}$; so there are infinitely many finite nonsolvable groups that do not meet the tight bound of Theorem 1. The goal of this paper is to extend the study of $hsn(G)$ to finite nonsolvable groups G. We give our main result as follows:

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C. S. Anabanti (✉)
Department of Mathematics and Applied Mathematics, University of Pretoria, Hatfield 0028, Pretoria, South Africa
E-mail: chimere.anabanti@up.ac.za; chimere.anabanti@unn.edu.ng

A. K. Asboei
Department of Mathematics Education, Farhangian University, P.O. Box 14665-889, Tehran, Iran
E-mail: a.khalili@cfu.ac.ir

Theorem 2 *Let G be a finite nonsolvable group with $hsn(G) < \frac{360}{47}$. If $hsn(G) \neq \frac{480}{71}$, then $G/F(G) \cong A_5$. If $hsn(G) = \frac{480}{71}$, then $G/N \cong A_5$, where N is the largest normal solvable subgroup of G.*

Remark 3 (i) Observe that $\frac{45}{7} \approx 6.42857$, $\frac{480}{71} \approx 6.76056$ and $\frac{360}{47} \approx 7.65957$.

(ii) We now give an example of Theorem 2. Let G be a group that can be realised from $(C_2^3 \rtimes C_7) \times A_5$. We can use GAP [3] to observe that $hsn(G) = \frac{45}{7}$ or $\frac{480}{71}$. When $hsn(G) = \frac{45}{7}$, we have that $F(G) \cong C_{14} \times C_2^2$ and $G/F(G) \cong A_5$. But when $hsn(G) = \frac{480}{71}$, we have that $F(G) \cong C_2^3$ and $G/F(G) \cong C_7 \times A_5 \ncong A_5$; so we can take N to be realised from $C_2^3 \rtimes C_7$ such that $G/N \cong A_5$.

(iii) Here, we show that the bound of Theorem 2 is tight. Let $G = A_5 \times A_4$. We can use GAP [3] to see immediately that $hsn(G) = \frac{360}{47} \neq \frac{480}{71}$, $F(G) \cong C_2 \times C_2$ and $G/F(G) \cong GL(2,4) \ncong A_5$.

(iv) There are infinitely many nonsolvable groups G that satisfy $hsn(G) = \frac{360}{47}$. Some of them are

$$A_5 \times A_4 \times C_2^n, \quad SL(2,5) \times A_4 \times C_2^n \quad \text{and} \quad SL(2,3) \times A_5 \times C_2^n,$$

where $n \in \{0, 1, 2, \ldots\}$.

(v) We shall consider the geometric mean Sylow number of nonsolvable groups in another paper.

For the rest of this section, we recall two results.

Lemma 4 *Let G be a group and N be a normal subgroup of G. Then $n_p(N)n_p(G/N) \mid n_p(G)$ for every prime p. In particular, if $S_1, \ldots, S_t$ are the composition factors of G including repetitions, then $n_p(S_1)\cdots n_p(S_t) \mid n_p(G)$.*

Proof See [6, Theorem 2.1] for the first part. The second part is an immediate consequence.

Lemma 5 *[5, Theorem 9.3.1] Let G be a finite solvable group and $|G| = m \cdot n$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m,n) = 1$. Let $\pi = \{p_1, \ldots, p_r\}$ and h_m be the number of π-Hall subgroups of G. Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ satisfies the following condition for all $i \in \{1, 2, \ldots, s\}$: $q_i^{\beta_i} \equiv 1 \pmod{p_j}$, for some p_j.*

2 A program for the harmonic mean Sylow number of *G*

```
###This program prints hsn(G) for a finite group G.
hsn:=function(G)
local prime_div, S_G, A, a, p;
prime_div:=PrimeDivisors(Size(G)); #List of prime divisors of |G|.
S_G:=[]; A:=[]; #A will be used to collect n_p(G) for each p.
for p in prime_div do
a:=Size(ConjugacyClassSubgroups(G,SylowSubgroup(G,p))); #a=n_p(G).
if a>1 then Add(S_G,p); Add(A,Float(1/a)); fi; od;
return Float(Size(S_G)/Sum(A));
end;
```

We give an example of how to use the above program by computing $hsn(A_5 \times A_4)$ below.

```
gap> A5:=AlternatingGroup(5);; A4:=AlternatingGroup(4);;
gap> b:=hsn(DirectProduct(A5,A4)); Rat(b);
7.65957
360/47
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3 Proof of Theorem 2

Proof of Theorem 2 First, we show that $|S(G)| \leq 9$. Suppose for contradiction that $|S(G)| \geq 10$. Then

$$hsn(G) \geq \frac{10}{\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{12}+\frac{1}{14}+\frac{1}{18}+\frac{1}{20}+\frac{1}{24}+\frac{1}{30}} = \frac{504}{61} \approx 8.2623 > 7.65957 \approx \frac{360}{47};$$

contradiction. Hence, $|S(G)| \leq 9$.

Let M be a nonabelian composition factor of G. First, let $n_p(M) \geq 8$ for every prime $p \mid |M|$. Since $2 \in \pi(S)$ and $n_2(M) \mid n_2(G)$, we have $n_2(G) \geq 9$. Since M is a nonabelian simple group, at least one of the primes 3 or 5 is in $\pi(M)$. By the third Sylow's theorem, either $n_3(G) \geq 10$ or $n_5(G) \geq 11$. Let $3 \in \pi(M)$.

If $|S(G)| = 3$, then $hsn(G) \geq \frac{3}{\frac{1}{9}+\frac{1}{10}+\frac{1}{6}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 4$, then $hsn(G) \geq \frac{4}{\frac{1}{9}+\frac{1}{10}+\frac{1}{6}+\frac{1}{8}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 5$, then $hsn(G) \geq \frac{5}{\frac{1}{9}+\frac{1}{10}+\frac{1}{6}+\frac{1}{8}+\frac{1}{12}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 6$, then $hsn(G) \geq \frac{6}{\frac{1}{9}+\frac{1}{10}+\frac{1}{6}+\frac{1}{8}+\frac{1}{12}+\frac{1}{14}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 7$, then $hsn(G) \geq \frac{7}{\frac{1}{9}+\frac{1}{10}+\frac{1}{6}+\frac{1}{8}+\frac{1}{12}+\frac{1}{14}+\frac{1}{18}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 8$, then $hsn(G) \geq \frac{8}{\frac{1}{9}+\frac{1}{10}+\frac{1}{6}+\frac{1}{8}+\frac{1}{12}+\frac{1}{14}+\frac{1}{18}+\frac{1}{20}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 9$, then $hsn(G) \geq \frac{9}{\frac{1}{9}+\frac{1}{10}+\frac{1}{6}+\frac{1}{8}+\frac{1}{12}+\frac{1}{14}+\frac{1}{18}+\frac{1}{20}+\frac{1}{24}} > \frac{360}{47}$, a contradiction.

Similarly, if $5 \in \pi(M)$ or $3, 5 \in \pi(M)$, then we get a contradiction. Therefore, $n_p(M) < 8$ for some prime divisor p of $|M|$. Then M has a proper subgroup of index 7 and we deduce that M is isomorphic to a simple subgroup of S_7. Then $M \cong A_5$, A_6, A_7 or $\mathrm{PSL}(2, 7)$.

Let $M \cong A_6$. Then $n_2(M) = 45$, $n_3(M) = 10$ and $n_5(M) = 36$. By Lemma 4, $n_2(M) \mid n_2(G)$, $n_3(M) \mid n_3(G)$ and $n_5(M) \mid n_5(G)$. Since $|S(G)| \leq 9$, we consider the following cases. If $|S(G)| = 3$, then $hsn(G) \geq \frac{3}{\frac{1}{45}+\frac{1}{10}+\frac{1}{36}} > \frac{360}{47}$, a contradiction.

If $|S(G)| = 4$, then $hsn(G) \geq \frac{4}{\frac{1}{45}+\frac{1}{10}+\frac{1}{36}+\frac{1}{8}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 5$, then $hsn(G) \geq \frac{5}{\frac{1}{45}+\frac{1}{10}+\frac{1}{36}+\frac{1}{8}+\frac{1}{12}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 6$, then $hsn(G) \geq \frac{6}{\frac{1}{45}+\frac{1}{10}+\frac{1}{36}+\frac{1}{8}+\frac{1}{12}+\frac{1}{14}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 7$, then $hsn(G) \geq \frac{7}{\frac{1}{45}+\frac{1}{10}+\frac{1}{36}+\frac{1}{8}+\frac{1}{12}+\frac{1}{14}+\frac{1}{18}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 8$, then $hsn(G) \geq \frac{8}{\frac{1}{45}+\frac{1}{10}+\frac{1}{36}+\frac{1}{8}+\frac{1}{12}+\frac{1}{14}+\frac{1}{18}+\frac{1}{20}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 9$, then $hsn(G) \geq \frac{9}{\frac{1}{45}+\frac{1}{10}+\frac{1}{36}+\frac{1}{8}+\frac{1}{12}+\frac{1}{14}+\frac{1}{18}+\frac{1}{20}+\frac{1}{24}} > \frac{360}{47}$, a contradiction.

Now, let $M \cong \mathrm{PSL}(2, 7)$. Then $n_2(M) = 21$, $n_3(M) = 28$ and $n_7(M) = 8$. By Lemma 4, we have that $n_2(M) \mid n_2(G)$, $n_3(M) \mid n_3(G)$ and $n_7(M) \mid n_7(G)$. Since $|S(G)| \leq 9$, we consider the following cases.

If $|S(G)| = 3$, then $hsn(G) \geq \frac{3}{\frac{1}{21}+\frac{1}{28}+\frac{1}{8}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 4$, then $hsn(G) \geq \frac{4}{\frac{1}{21}+\frac{1}{28}+\frac{1}{8}+\frac{1}{6}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 5$, then $hsn(G) \geq \frac{5}{\frac{1}{21}+\frac{1}{28}+\frac{1}{8}+\frac{1}{6}+\frac{1}{12}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 6$, then $hsn(G) \geq \frac{6}{\frac{1}{21}+\frac{1}{28}+\frac{1}{8}+\frac{1}{6}+\frac{1}{12}+\frac{1}{14}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 7$, then $hsn(G) \geq \frac{7}{\frac{1}{21}+\frac{1}{28}+\frac{1}{8}+\frac{1}{6}+\frac{1}{12}+\frac{1}{14}+\frac{1}{18}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 8$, then $hsn(G) \geq \frac{8}{\frac{1}{21}+\frac{1}{28}+\frac{1}{8}+\frac{1}{6}+\frac{1}{12}+\frac{1}{14}+\frac{1}{18}+\frac{1}{20}} > \frac{360}{47}$, a contradiction.
If $|S(G)| = 9$, then $hsn(G) \geq \frac{8}{\frac{1}{21}+\frac{1}{28}+\frac{1}{8}+\frac{1}{6}+\frac{1}{12}+\frac{1}{14}+\frac{1}{18}+\frac{1}{20}+\frac{1}{24}} > \frac{360}{47}$, a contradiction.

Similarly, it is easy to check that $hsn(G) > \frac{360}{47}$ when $M \cong A_7$. So $M \cong A_5$. We know that $n_2(M) = 5$, $n_3(M) = 10$ and $n_5(M) = 6$. On the other hand, by Lemma 4, $n_p(M) \mid n_p(G)$ for every prime p. Since $hsn(G) < \frac{360}{47}$, we conclude that $n_2(G) = 5$, $n_3(G) = 10$ and $n_5(G) = 6$.

Let N be the largest normal solvable subgroup of G and let K/N be a chief factor of G. We know that K/N is isomorphic to A_5, or a direct product of many copies of A_5. If K/N is isomorphic to a direct product of two or more copies of A_5, then by Lemma 4, $hsn(G) > \frac{360}{47}$, a contradiction. Next, let $C/N = C_{G/N}(K/N)$. Then, $C/N \times K/N \trianglelefteq G/N$. By Lemma 4, we have that $n_p(C/N) \times n_p(K/N) \mid n_p(G)$ for every prime p. So, if C/N is a nonsolvable group, then $n_2(G) > 5$ or $n_3(G) > 10$ or $n_5(G) > 6$, a contradiction. Thus, C/N is solvable;

whence $C = N$. Hence, $G/N \lesssim \mathrm{Aut}(A_5) = S_5$. If $G/N = S_5$, then since $n_2(G/N) = 15$ and $n_2(G) = 5$, we get a contradiction. Hence, $G = K$.

Suppose that there exists a prime $r \geq 7$ such that $r \mid |G|$. If $n_r(G) > 1$, then $n_r(G) \geq 8$. Assume that $7 \mid |G|$ and $n_7(G) = 8$. So, $hsn(G) = \frac{480}{71}$, a contradiction by our assumtion. If $11 \mid |G|$ and $n_{11}(G) = 12$, then $hsn(G) = \frac{80}{11} < \frac{360}{47}$. Since $G/N = K/N$ is isomorphic to A_5, we have $P_{11} \leq N$, where P_{11} is a Sylow 11-subgroup of G. Hence, $n_{11}(N) = n_{11}(G) = 12$. As N is a solvable subgroup, by Lemma 5, we have that $3 \equiv 1 \bmod 11$, a contradiction. Thus, $r \geq 13$, and then $hsn(G) > \frac{360}{47}$, a contradiction. Therefore, $n_r(G) = 1$ for every $r \geq 7$. By Lemma 4, we have that $n_p(N) = 1$ for every prime p. So N is nilpotent; whence $N = F(G)$.

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