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ORIGINAL RESEARCH

Injectivity in the category of $C(X)$-algebras

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Abstract Let X be a compact Hausdorff space. We show the existence and uniqueness of injective envelope in the category of unital $C(X)$-algebras. As an application, we characterize injective objects in the category of upper semi-continuous C*-bundles.

Keywords $C_0(X)$-algebra · C*-bundle · Completely positive map · Injective envelope

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1 Introduction

Characterizing injective objects is a central problem in the study of categories in functional analysis. This is done in the categories of Banach spaces [3,14,20], Banach modules [9,10,15], operator systems [2,12] and C*-algebras [11]. Injective objects in the category of Banach spaces with contractive linear maps as morphisms are studied in [20]. In this category, injective objects prove to be quite exceptional: they are Banach spaces which are isometrically isomorphic to $C(X)$, for a compact and extremely disconnected space X.

For a Banach algebra A, the injective objects in the category of Banach A-modules are discussed in [9]. Same is done for the subcategory of non-degenerate locally $C_0(X)$-convex $C_0(X)$-Banach modules in [15]. Such modules do not have an algebra structure and could not be considered as generalizations of C*-algebras. A related category is the category of $C_0(X)$-algebras which are used to define the crossed products of C*-algebras by groupoids [21]. In this paper we characterize injective objects in the category of $C(X)$-algebras for a compact Hausdorff space X. As an application, we prove the existence of injective objects in the category of upper semi-continuous C*-bundles.

Injectivity is widely studied in the category of von Neumann algebras (and more generally in that of C*-algebras, operator systems, or operator spaces). Arveson's celebrated extension theorem [1] (which generalizes Hahn-Banach theorem) could be stated as the injectivity of the algebra $B(H)$ of bounded operators on a Hilbert space H in the category of von Neumann algebras. From this it follows that an operator system $R \subseteq B(H)$ is injective if and only if there is a completely positive projection from $B(H)$ onto R [2]. Hakeda and Tomiyama were the first to notice that for von Neumann algebras, the latter property is the same as the Schwartz's Property

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P [7]. This observation could be used to construct non injective von Neumann algebras. Moreover, Effros and Lance observed in [6] that semidiscrete von Neumann algebras (i.e., $R \subseteq B(H)$ for which the product map: $R\overline{\otimes}R \to B(H)$ is norm-bounded) must be injective. In his seminal work on injective factors [4], Connes proved that if R is a factor acting on a separable Hilbert space H, then R is injective if and only if R is semidiscrete [4].

Ruan studied the structure of injective operator spaces and showed the existence (and uniqueness) of injective envelopes of arbitrary operator spaces [19]. He also gave an example of an injective operator space which is not completely isometric to any C*-algebra [19, Theorem 4.3] (giving a negative answer to an earlier question of Wittstock [22]). In general, an injective operator space is a triple subsystem of a C*- algebra, that is, an operator space E is injective if and only if there exists an injective C*-algebra A and projections p and q such that E is completely isometric to pAq [19, Theorem 4.5].

Injective envelopes of C*-algebras [11] and operator systems [12] were constructed by Hamana, and the latter is used to prove existence of the Silov boundary in the sense of Arveson [1]. Hamana also constructed injective envelopes of Banach modules [10] and C*-dynamical systems [13].

2 C(X)-algebras

In this section, we recall basic definitions in the theory of $C_0(X)$-algebras. For more details on $C_0(X)$-algebras and multiplier algebras of C*-algebras, see [21, Appendix C].

For a locally compact Hausdorff space X, a $C_0(X)$-algebra is a C*-algebra A endowed with a non-degenerate *-homomorphism from the C*-algebra $C_0(X)$ of continuous functions vanishing at infinity on X into center of the multiplier algebra of A. It is not hard to see that any $C_0(X)$-algebra A is a non-degenerate $C_0(X)$-Banach module satisfying

$$f \cdot (ab) = (f \cdot a)b = a(f \cdot b),$$

for $f \in C_0(X)$ and $a, b \in A$. A $C_0(X)$-Banach module A is called locally $C_0(X)$-convex if

$$||f \cdot a + g \cdot b|| \leq \min\{||a||, ||b||\}$$

for $a, b \in A$ and $f, g \in C_0(X)$ with $f, g \geq 0$ and $f + g \leq 1$. A non-degenerate locally $C_0(X)$-convex $C_0(X)$-Banach module is determined by its fibres. For $x \in X$, the fibre $A(x)$ is defined as the quotient $A/(C_0(X\setminus\{x\})\cdot A)$. For $a \in A$, $a(x)$ denotes the corresponding coset in the fibre $A(x)$ (see [17,18] for more details).

It is known [5, Theorem 2.5] that a Banach $C_0(X)$-module A is locally $C_0(X)$-convex if and only if

$$||a|| = \sup_{x \in x} ||a(x)||.$$

The above condition holds in $C_0(X)$-algebras [21, Proposition C.10]. In particular, a $C_0(X)$-algebra is a locally $C_0(X)$-convex $C_0(X)$-Banach module. As far as we know, there is no proof of the existence of injective envelops of $C(X)$ to $C_0(X)$ in the literature, and this is the main objective of the current paper.

Definition 1 Let A be a C*-algebra with multiplier algebra $M(A)$ and let X be a locally compact Hausdorff space. Then A is a $C_0(X)$ algebra if there is a homomorphism Φ_A from $C_0(X)$ into the center $ZM(A)$ of the multiplier algebra of A which is non-degenerate, i.e.,

$$\Phi_A(C_0(X)) \cdot A := \text{span}\{\Phi_A(f) \cdot a : f \in C_0(X), a \in A\}$$

is dense in A.

In what follows, we often omit Φ_A and use the $C_0(X)$-module notation: we write $f \cdot a$ instead of $\Phi_A(f)a$. In particular, we regard A as a $C_0(X)$-module satisfying:

(i) $f \cdot (ab) = (f \cdot a)b = a(f \cdot b) \quad (a, b \in A,\ f \in C_0(X))$,
(ii) $\overline{\text{span}}\{f \cdot a : f \in C_0(X), a \in A\} = A$.

Definition 2 A $C_0(X)$-morphism ϕ between $C_0(X)$-algebras A and B is a *-homomorphism satisfying

$$\phi(f \cdot a) = f \cdot \phi(a), \tag{1}$$

for $a \in A$ and $f \in C_0(X)$. A map $\phi : A \longrightarrow B$ is called a $C_0(X)$-map if it is a completely positive linear map and satisfies in 1.

Let A be a $C_0(X)$-algebra. Given $x \in X$, let J_x be the maximal ideal in $C_0(X)$ consisting of functions vanishing at x. We denote the ideal $\overline{J_x \cdot A}$ by I_x^A and the quotient A/I_x^A by $A(x)$. For $a \in A$ we write $a(x)$ for the image $a + I_x^A$ of a in I_x^A. One may think of $A(x)$ as the fiber of A over x and $a(x)$ as the image of a in $A(x)$.

Remark 1 Let us make the following simple observations:

(i) If A is a C*-algebra, we may regard A as a $C_0(X)$-algebra. Fix $x \in X$, the module action of $f \in C_0(X)$ is defined by $f \cdot_x a = f(x)a$, for $a \in A$.

(ii) If $\{A_x : x \in X\}$ is a family of unital C*-algebras indexed by a compact Hausdorff space X, then $\Pi_{x\in X} A_x$ is a $C(X)$-algebra. We define

$$f \cdot (a_x)_{x\in X} = (f \cdot_x a)_{x\in X} = (f(x)a_x)_{x\in X}$$

for $f \in C(X)$ and $(a_x)_{x\in X} \in \Pi_{x\in X} A_x$. Then for each $x \in X$, A_x is a $C(X)$-subalgebra of $\Pi_{x\in X} A_x$.

(iii) If A is a $C_0(X)$-algebra, then so is $\mathbb{M}_n(A)$ with the module action $f \cdot [a_{i,j}] = [f \cdot a_{i,j}]$. Also $\overline{J_x.\mathbb{M}_n(A)} = \mathbb{M}_n(\overline{J_x.A})$ and $\mathbb{M}_n(A)(x) \simeq \mathbb{M}_n(A(x))$. In particular, $\kappa_n : \mathbb{M}_n(A)(x) \longrightarrow \mathbb{M}_n(A(x))$ defined by $\kappa_n([a_{i,j}] + I_x^{\mathbb{M}_n(A)}) = [a_{i,j} + I_x^A]$ is an isometric isomorphism.

For the rest of the paper, we deal with $C(X)$-algebras, for a compact Hausdorff space X. Also, unless otherwise specified, C*-algebras are unital, their C*-subalgebras have the same units as the C*-algebras containing them, and the maps between C*-algebras preserve units.

Let A be a $C(X)$-algebra. We may consider A as $C(X)$-subalgebra of $\Pi_{x\in X} A(x)$. The following lemma is probably part of the literature. We sketch a proof for the sake of completeness.

Lemma 1 *Let A be a $C(X)$-algebra. Then the map $a \mapsto (a(x))_{x\in X}$ from A into $\Pi_{x\in X} A(x)$ is an isometric isomorphism, where $A(x)$ is regarded as a $C(X)$-algebra with the module multiplication $\cdot_x$ as above.*

Proof The map $a \mapsto (a(x))_{x\in X}$ is clearly a *-homomorphism and an isometry by [21, Proposition C.10]. For $f \in C(X)$, $a \in A$ and $x \in X$,

$$f \cdot a + I_x^A = f(x)a + I_x^A,$$

Thus,

$$\begin{aligned} ((f \cdot a)(x))_{x\in X} &= (f \cdot a + I_x^A)_{x\in X} \\ &= (f(x)a + I_x^A)_{x\in X} \\ &= f \cdot (a(x))_{x\in X}. \end{aligned}$$

Hence, the map $a \mapsto (a(x))_{x\in X}$ is a $C(X)$-morphism. □

A C*-algebra A is injective if for any unital C*-algebra B and any C*-subalgebra C of B, containing the unit of B, any completely positive linear map of C into A extends to a completely positive linear map of B into A . An extension of A is a pair (B, i) of a C*-algebra B and a $*$-monomorphism i from A into B. The extension (B, i) is called injective if B is injective, and it is called an injective envelope of A if it is an injective extension of A such that the identity map id_B on B is the unique completely positive linear map of B into itself which fixes each element of $i(A)$, i.e., the extension (B, i) is rigid (essential). The existence and uniqueness of injective envelopes of C*-algebras is proved in [11].

Definition 3 Let A be a $C(X)$-algebra.

(i) A $C(X)$-extension of A is a pair (B, i) consisting of a $C(X)$-algebra B and an isometric $C(X)$-morphism $i : A \to B$.

(ii) A $C(X)$-extension (B, i) of A is essential if for any $C(X)$-map ϕ of B into a $C(X)$-algebra C, ϕ is completely isometric whenever $\phi \circ i$ is so.

(iii) An $C(X)$-extension (B, i) of A is called rigid if for any $C(X)$-map $\phi : B \longrightarrow B$, $\phi \circ i = i$ implies $\phi = \mathrm{id}_B$.

Definition 4 Let A be a $C(X)$-algebra.

(i) A is injective if for any $C(X)$-extension (B, i) of $C(X)$-algebra C and $C(X)$-map $\phi : C \longrightarrow A$, there exists a $C(X)$-map $\widehat{\phi} : B \longrightarrow A$ such that $\widehat{\phi} \circ i = \phi$.

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[d, "\phi"'] & B \arrow[dl, dotted, "\widehat{\phi}"] \\ A. & \end{tikzcd}$$

(ii) A $C(X)$-extension (D, i) of $C(X)$-algebra A is an injective $C(X)$-envelope of A if it is injective and essential.

The next lemma is proved similar to [12, Lemma 3.7].

Lemma 2 *Let (B, i) be an injective $C(X)$-extension of a $C(X)$-algebra A. Then (B, i) is rigid if and only if it is essential.*

Lemma 3 *Let A and B be C*-algebras and $\phi : A \longrightarrow B$ be a completely positive map. Let I and J be closed ideals respectively in A and B, such that $\phi(I) \subseteq J$. Then $\widetilde{\phi} : A/I \longrightarrow B/J$ defined by $\widetilde{\phi}(a+I) = \phi(a)+J$; $a \in A$, is a completely positive map.*

Proof Let us identify $M_n(A/I)$ with $M_n(A)/M_n(I)$. Then

$$\widetilde{\phi}^{(n)}([a_{i,j}][a_{i,j}]^* + M_n(I)) = \phi^{(n)}([a_{i,j}][a_{i,j}]^*) + M_n(J) \geq 0.$$

for each $[a_{i,j}] \in M_n(A)$. □

Lemma 4 *Let A and B be $C(X)$-algebras, and let ϕ and ψ be bounded linear maps from A to B that preserve the involution and satisfy condition 1. Then*

(i) $a \in A$ is positive if and only if $a(x)$ is positive in $A(x)$ for all $x \in X$,
(ii) ϕ is completely positive if and only if $\phi_x : A(x) \longrightarrow B(x)$ is completely positive for all $x \in X$,
(iii) $\phi = \psi$ if and only if $\phi_x = \psi_x$ for all $x \in X$.

Proof (i) Suppose that $a \geq 0$ and choose $b \in A$ with $a = b^*b$. Then $a(x) = b(x)^*b(x) \geq 0$ in $A(x)$. Conversely, suppose that $a(x) \geq 0$ for all $x \in X$. Then

$$||a - a^*|| = \sup_{x \in X} ||(a - a^*)(x)|| = 0.$$

Therefore, a is hermitian. Let $0 \leq t \leq ||a|| = \sup_{x \in X} ||a(x)||$. Since $a(x)$ is positive, $||a(x) - t|| \leq t$ for each x, thus $||a - t|| \leq t$ and a is positive.

(ii) Suppose that $\phi : A \longrightarrow B$ is completely positive. Then $\phi(I_x^A) \subseteq I_x^B$ and $\phi : A(x) \longrightarrow B(x)$ is completely positive by Lemma 3. Conversely, Suppose that $\phi_x : A(x) \longrightarrow B(x)$ is completely positive for each $x \in X$. Let $[a_{i,j}]$ be a positive element in $\mathbb{M}_n(A)$. Then $[a_{i,j}(x)]$ is positive in $\mathbb{M}_n(A)(x) \simeq \mathbb{M}_n(A(x))$ and so is $[\phi(a_{i,j})(x)]$ in $\mathbb{M}_n(B(x)) \simeq \mathbb{M}_n(B)(x)$, for all $x \in X$. Therefore, $[\phi(a_{i,j})]$ is positive in $\mathbb{M}_n(B)$ by part (i).

(iii) Suppose that $\phi_x = \psi_x$ for any $x \in X$. Let $a \in A$, then

$$||\phi(a) - \psi(a)|| = \sup_{x \in X} ||\phi(a)(x) - \psi(a)(x)|| = \sup_{x \in X} ||\phi_x(a) - \psi_x(a)|| = 0,$$

that is, $\phi(a) = \psi(a)$. The converse is trivial. □

Lemma 5 *Let A be a C*-algebra. Then A is an injective C*-algebra if and only if A is an injective $C(X)$-algebra with the module multiplication $\cdot_{x_0}$, for some (any) $x_0 \in X$.*

Proof Suppose that A is an injective C*-algebra, V and B are $C(X)$-algebras, $i : V \longrightarrow B$ is an injective (isometric) $C(X)$-map, and $\phi : V \longrightarrow A$ is a completely positive $C(X)$-map. The map i induces fiber maps $i_x : V(x) \longrightarrow B(x)$; $i_x(v(x)) = i(v)(x)$. Since i is isometric, so are i_x for all $x \in X$ [17, Proposition A.1.7]. Since $I_{x_0}^A \subseteq \ker \phi$, the map $\phi : V \longrightarrow A$ induces a map $\phi_{x_0} : V(x_0) \longrightarrow A$ defined by $\phi_{x_0}(v(x_0)) = \phi(v)$. By

Lemma 3, ϕ_{x_0} is completely positive. Since A is injective, there exists a completely positive map $\widehat{\phi_{x_0}} : B(x_0) \longrightarrow A$ such that $\widehat{\phi_{x_0}} \circ i_{x_0} = \phi_{x_0}$.

$$\begin{array}{ccc} V(x_0) & \xrightarrow{i_{x_0}} & B(x_0) \\ \phi_{x_0}\downarrow & \swarrow \widehat{\phi_{x_0}} & \\ A. & & \end{array}$$

Define $\widehat{\phi}(b) = \widehat{\phi_{x_0}}(b(x_0))$. Then $\widehat{\phi}$ is completely positive as the composition of two completely positive maps. For $f \in C(X)$, $b \in B$ and $v \in V$,

$$\widehat{\phi}(f.b) = \widehat{\phi_{x_0}}((f.b)(x_0)) = \widehat{\phi_{x_0}}(f(x_0).b(x_0)) = f(x_0)\widehat{\phi_{x_0}}(b(x_0)) = f.\widehat{\phi}(b),$$

and

$$\widehat{\phi}(i(v)) = \widehat{\phi_{x_0}}(i(v)(x_0)) = \widehat{\phi_{x_0}}(i_{x_0}(v(x_0))) = \phi_{x_0}(v(x_0)) = \phi(v).$$

Therefore, A is an injective $C(X)$-algebra. The converse is trivial. □

The proof of the next lemma is straightforward.

Lemma 6 *If $\{A_\alpha : \alpha \in \Gamma\}$ is a family $C(X)$-algebras, then $\Pi_{\alpha\in\Gamma} A_\alpha$ is injective if and only if A_α is injective for all $\alpha \in \Gamma$.*

It is shown in [9, Theorem 3.8] (and independently in [12, Lemma 1]) that any Banach A-module V is isomorphic to a submodule of an injective A-module W. In fact one may choose W such that $W = \Pi_{j\in J}\tilde{A}^* = \ell^\infty(J, \tilde{A}^*)$ for some set J, where $\tilde{A}$ is the minimal unitization of A. In this case $\tilde{A}^*$ is canonically an A-module and is injective [9, Lemma 1]. A result of Arveson [1, Theorem 1.2.3] states that the C*-algebra $B(H)$ of all bounded linear operators on a Hilbert space H is injective, in particular, each C*–algebra has an injective extension. The same holds for $C(X)$-algebras.

Proposition 1 *Every $C(X)$-algebra has an injective $C(X)$-extension.*

Proof Let A be a $C(X)$-algebra. For any $x \in X$, let H_x be a Hilbert space such that $A(x) \subseteq B(H_x)$. By [1, Theorem 1.2.3], $B(H_x)$ is an injective C*-algebra and so is an injective $C(X)$-algebra with action $\cdot_x$ as above, by Lemma 5 . Therefore, $\Pi_{x\in X} B(H_x)$ is an injective $C(X)$-algebra, and the map $a \mapsto (a(x))_{x\in X}$ is an isometric morphism from A into $\Pi_{x\in X} A(x) \subseteq \Pi_{x\in X} B(H_x)$ by Lemma 1. □

Injectivity of C*-algebras is characterized by being the range of a conditional expectation from the C*-algebra $B(H)$, for some Hilbert space H [11]. Recall that, a conditional expectation from a C*-algebra A into a C*-subalgebra B of A is a completely positive projection of norm one from A onto B.

Theorem 2 *A $C(X)$-algebra $A \subseteq \Pi_{x\in X} B(H_x)$ is injective if and only if there exists a $C(X)$-map $\phi : \Pi_{x\in X} B(H_x) \longrightarrow A$ such that $\phi|_A = id_A$.*

Proof Suppose that A is injective. Since $\mathrm{id}_A : A \longrightarrow A$ is a $C(X)$-map, there exists a $C(X)$-map $\phi : \Pi_{x\in X} B(H_x) \longrightarrow A$ such that $\phi|_A = \mathrm{id}_A$. Conversely, suppose that there exists a $C(X)$-map $\phi : \Pi_{x\in X} B(H_x) \longrightarrow A$ such that $\phi|_A = \mathrm{id}_A$. Let B be a $C(X)$-algebra and (c, i) be an extension of B and let $\psi : B \longrightarrow A \subseteq \Pi_{x\in X} B(H_x)$ be a $C(X)$-map. Since $\Pi_{x\in X} B(H_x)$ is an injective $C(X)$-algebra, there exists a $C(X)$-map $\theta : C \longrightarrow \Pi_{x\in X} B(H_x)$ such that $\theta \circ i = \psi$. Put $\widetilde{\phi} = \phi \circ \theta$. Thus $\widetilde{\phi}$ is a $C(X)$-map and $\widetilde{\phi} \circ i = \psi$. □

Definition 5 Let $A \subseteq B$ be $C(X)$-algebras .

(i) A $C(X)$-map $\phi : B \longrightarrow B$ is an A-projection on B if it is idempotent ($\phi^2 = \phi$) and $\phi|_A = \mathrm{id}_A$.
(ii) A seminorm p on B is an A-seminorm if $p(\cdot) = ||\phi(\cdot)||$ for some completely positive $C(X)$-map $\phi : B \longrightarrow B$ with $\phi|_A = \mathrm{id}_A$.
(iii) Let ϕ and ψ be A-projections on B (resp., p and q be A-seminorms on B). Define a partial ordering $\preceq$ (resp., $\leq$) by $\phi \preceq \psi$ (resp., $p \leq q$) if $\phi \circ \psi = \psi \circ \phi = \phi$ ($p(b) \leq q(b)$ for all $b \in B$).
(iv) An A -projection (resp., A-seminorm) which is minimal with respect to the partial ordering $\preceq$ (resp.,$\leq$) is called a minimal A-projection (resp., minimal A-seminorm).

Theorem 3 *Any $C(X)$-algebra A has an injective $C(X)$-envelope (D, i), which is unique in the sense that if (D_1, i_1) is another injective $C(X)$-envelope, there exists a $C(X)$-isomorphism $\phi : D \longrightarrow D_1$ such that $\phi \circ i = i_1$.*

Proof Let $x \in X$ and let H_x be a Hilbert space with $A(x) \subseteq B(H_x)$. If we regard $B(H_x)$ as a $C(X)$-algebra with module multiplication $._x$, then by Lemma 5, $B(H_x)$ is an injective $C(X)$-algebra and $\prod_{x\in X} B(H_x)$ is an injective $C(X)$-algebra , by Lemma 6. Since we may regard A as a sub-$C(X)$-algebra of $\prod_{x\in X} B(H_x)$ (see Proposition 1), it is sufficient to prove that that there exists a minimal A- projection on $\prod_{x\in X} B(H_x)$, by Theorem 2, see [11,12]. We prove this by the following steps:

Step 1. Any decreasing net $\{p_i\}_{i\in I}$ of A-seminorms on $\prod_{x\in X} B(H_x)$ has a lower bound.

First note that since the unit ball of $B(H)$ is compact in the σ-weak topology ([16, Theorem 4.2.4]), the unit ball of $B(A, B(H)))$, the Banach space of all bounded linear maps of A into $B(H)$, is compact in the point-σ-weak topology, for any Hilbert space H. Since $\prod_{x\in X} B(H_x)$ is σ-weak closed in $B(\prod_{x\in X} H_x)$, the unit ball of $B(A, \prod_{x\in X} B(H_x))$ is compact in the point-σ-weak topology. For $\xi = (\xi_x)_{x\in X}$ in $\prod_{x\in X} H_x$ and $f \in C(X)$, define $f \cdot \xi = (f(x)\xi_x)_{x\in X}$. Then $f \cdot \xi \in \prod_{x\in X} H_x$ and for $T = (T_x)_{x\in X} \in \prod_{x\in X} B(H_x)$ we have $(f \cdot T)\xi = T(f \cdot \xi)$.

This implies that the module multiplication on $\prod_{x\in X} B(H_x)$ is σ-weak continuous.

Suppose that $\phi_i : \prod_{x\in X} B(H_x) \longrightarrow \prod_{x\in X} B(H_x)$ is a completely positive $C(X)$-map such that $p_i(\cdot) = ||\phi_i(\cdot)||$ and $\phi_i|_A = id_A$. Regarding $\{\phi_i\}_{i\in I}$ as a subset of the unit ball of $B(A, \prod_{x\in X} B(H_x))$, the above remark shows that there are a subnet $\{\phi_j\}_{j\in J}$ and element $\phi_0 \in B(A, \prod_{x\in X} B(H_x))$ such that $\phi_j(a) \longrightarrow \phi_0(a)$, σ-weakly for all a in A. Then ϕ_0 is a completely positive map in $B(A, \prod_{x\in X} B(H_x))$. Since the module multiplication is σ-weakly continuous, ϕ_0 is an $C(X)$-map and hence $p_0(\cdot) = ||\phi_0(\cdot)||$ is an A-seminorm on $\prod_{x\in X} B(H_x)$. For $T \in \prod_{x\in X} B(H_x)$,

$$p_0(T) = ||\phi_0(T)|| \le \limsup_j ||\phi_j(T)|| = \lim_i p_i(T) = \inf_i p_i(T).$$

Thus ϕ_0 is a lower bound of $\{\phi_i\}_{i\in I}$.

Step 2. There exists a minimal A-seminorm on $\prod_{x\in X} B(H_x)$.

Step 1 and Zorn's lemma imply the existence of a minimal A-seminorm p on $\prod_{x\in X} B(H_x)$.

Step 3. Let p be a minimal A -seminorm on $\prod_{x\in X} B(H_x)$ and ϕ be a completely positive $C(X)$-map such that $p(\cdot) = ||\phi(\cdot)||$ and $\phi|_A = \mathrm{id}_A$. Then ϕ is a A-projection on $\prod_{x\in X} B(H_x)$. The proof of the minimality of ϕ is verbatim to that of the proof in [12, Theorem 3.5].

The map $\phi : \prod_{x\in X} B(H_x) \longrightarrow \prod_{x\in X} B(H_x)$ is an idempotent completely positive $C(X)$-map, and under the Choi-Effros product " $\circ$ " on Imϕ, defined by $x \circ y = \phi(xy)$, $D = \mathrm{Im}\phi$ is a C*-algebra, see [11, Theorem 2.3].

Step 4. The extension $(D = Im\phi, i)$ is an injective envelope of A. Since ϕ is a $C(X)$-map, D is a $C(X)$-algebra. Note that, $(D, \circ)$ is not a C*-subalgebra of $\prod_{x\in X} B(H_x)$, but D is an operator system in $\prod_{x\in X} B(H_x)$, i.e., the inclusion map $j : D \longrightarrow \prod_{x\in X} B(H_x)$ is a completely positive $C(X)$-map, [2, Theorem 3.1]. Also, the restriction ϕ to D is the identity map id_D. This implies that D is an injective $C(X)$-algebra. The proof of the rigidity of (D, i) is exactly the same as of the proof in [12, Lemma 3.6]. If (D_1, i_1) is another injective envelope of A, then there exist $C(X)$-maps $\theta : D \longrightarrow D_1$ and $\theta_1 : D_1 \longrightarrow D$ such that $\theta \circ i = i_1$ and $\theta_1 \circ i_1 = i$. Thus,

$$\theta \circ \theta_1 \circ i_1 = i_1 \quad \theta_1 \circ \theta \circ i = i,$$

hence,

$$\theta \circ \theta_1 = \mathrm{id}_{D_1}, \qquad \theta_1 \circ \theta = \mathrm{id}_D.$$

Therefore, ϕ is a complete order isomorphism and hence an *-isomorphism by [8, Corollary 5.2.3]. □

3 Upper semi-continuous C*-bundles

In this section we assume that X is a compact Hausdorff space and consider the canonical functor from the category of $C(X)$-algebras to the category of upper semi-continuous bundles over X, and show that it is an equivalence of categories (by providing a weak inverse functor). We use this to find injective objects in the latter category.

Definition 6 An upper semi-continuous C*-bundle over X is a continuous, open, surjection $p : \mathcal{A} \longrightarrow X$ together with operations and norms making each fibre $\mathcal{A}_x = p^{-1}(x)$ into a C*-algebra such that:

(i) The map $a \mapsto ||a||$ is upper semi-continuous from $\mathcal{A}$ into $\mathbb{R}$.
(ii) The involution $a \mapsto a^*$ is continuous from $\mathcal{A}$ into $\mathcal{A}$.
(iii) The maps $(a, b) \mapsto ab$ and $(a, b) \mapsto a + b$ are continuous from $\mathcal{A}^{(2)} = \{(a, b) \in \mathcal{A} \times \mathcal{A} : p(a) = p(b)\}$ into $\mathcal{A}$.
(iv) For each $\lambda \in \mathbb{C}$, the map $a \mapsto \lambda a$ is continuous form $\mathcal{A}$ into $\mathcal{A}$.
(v) If $\{a_i\}$ is a net in $\mathcal{A}$ such that $||a_i|| \longrightarrow 0$ and $p(a_i) \longrightarrow x$, then $a_i \longrightarrow 0_x$, where 0_x is the zero element in $\mathcal{A}_x$.

Let $p_{\mathcal{A}} : \mathcal{A} \longrightarrow X$ and $p_{\mathcal{B}} : \mathcal{B} \longrightarrow X$ be upper semi-continuous C*- bundles over X. A continuous map $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ is called a bundle morphism (resp., bundle map) if $p_{\mathcal{B}} \circ \Phi = p_{\mathcal{A}}$ and for each $x \in X$, the restriction map $\Phi : \mathcal{A}_x \longrightarrow \mathcal{B}_x$ is a *-homomorphism (resp., completely positive) i.e., the following diagram is commutative:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\Phi"] \arrow[dr, "p_{\mathcal{A}}"'] & & \mathcal{B} \arrow[dl, "p_{\mathcal{B}}"] \\ & X & \end{tikzcd} .$$

If $p : \mathcal{A} \longrightarrow X$ is an upper semicontinuous C*-bundle, a function $\xi : X \longrightarrow \mathcal{A}$ is called a section if $p(\xi(x)) = x$ for all $x \in X$. The set of continuous sections of an upper semi-continuous C*-bundle $p : \mathcal{A} \longrightarrow X$ is denoted by $\Gamma(X, \mathcal{A})$.

It is known that for any upper semi-continuous bundle $p : \mathcal{A} \longrightarrow X$, $\Gamma_0(X, \mathcal{A})$ is a $C(X)$-algebra and for each $x \in X$, $\Gamma(X, \mathcal{A})(x) = \mathcal{A}_x$ [21, Proposition C.23]. One could define a functor Γ from the category of upper semi-continuous bundles over X into the category of $C(X)$-algebras. The functor Γ assigns to each upper semi-continuous bundle $p : \mathcal{A} \longrightarrow X$ the $C(X)$ algebra $\Gamma(X, \mathcal{A})$. If $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a bundle morphism, then $\Gamma(\Phi) : \Gamma(X, \mathcal{A}) \longrightarrow \Gamma(X, \mathcal{B})$ defined by

$$\Gamma(\Phi)(\xi)(x) = \Phi(\xi(x)), (x \in X),$$

is a $C(X)$-morphism. If Φ is a bundle map the $\Gamma(\Phi)$ is a $C(X)$-map by Lemma 4.

Next, we define a functor $\mathcal{C}$ from the category of $C(X)$-algebras into the category of upper semi-continuous C*- bundles over X and show that it is a weak inverse to Γ. Let A be a $C(X)$-algebra and define $\mathcal{C}(A) = \coprod_{x \in X} A(x)$ and let $p : \mathcal{C}(A) \longrightarrow X$ be the canonical map. For each $a \in A$, the map $x \mapsto a(x)$ is a section. For $\varepsilon > 0$ and open subset $U \subseteq X$, define

$$W(a, U; \varepsilon) = \{b(x) \in \mathcal{C}(A) : x \in U, ||a(x) - b(x)|| < \varepsilon\}.$$

The collection of all unions of the sets of the form $W(a, U, \varepsilon)$ is a topology τ on $\mathcal{A}$ such that $p : \mathcal{C}(A) \longrightarrow X$ is an upper semi-continuous C*-bundle over X and there is an isomorphism from A into $\Gamma(X, \mathcal{C}(A))$, that is,

$$\mu_A : A \longrightarrow \Gamma(X, \mathcal{C}(A)); \quad \mu_A(a)(x) = a(x), (x \in X),$$

cf., [21, Theorems C.25 and C.26].

If $\phi : A \longrightarrow B$ is a $C(X)$-map, define $\mathcal{C}(\phi) : \mathcal{C}(A) \longrightarrow \mathcal{C}(B)$ by

$$\mathcal{C}(\phi)(a(x)) = \phi(a)(x), (x \in X, a \in A)$$

Lemma 7 *$\mathcal{C}(\phi) : \mathcal{C}(A) \longrightarrow \mathcal{C}(B)$ is continuous.*

Proof Suppose that $b_0 \in B$, $\varepsilon > 0$ and U is an open subset of X. We show that $\mathcal{C}(\phi)^{-1}(W(b_0, U, \varepsilon))$ is open in $\mathcal{C}(A)$. Given any $a_0(x_0) \in \mathcal{C}(\phi)^{-1}(W(b_0, U, \varepsilon))$, we have $x_0 \in U$ and $||b_0(x_0) - \phi(a_0)(x_0)|| < \varepsilon$. By upper semi-continuity, the set

$$V = \{x \in X : ||b_0(x) - \phi(a)(x)|| < \delta\}$$

is open and contains x_0, where $||b_0(x_0) - \phi(a_0)(x_0)|| < \delta < \varepsilon$. We claim that $W(a_0, V \cap U, \varepsilon - \delta) \subseteq \mathcal{C}(\phi)^{-1}(W(b_0, U, \varepsilon))$. Given $a(x) \in W(a_0, V \cap U, \varepsilon - \delta)$, we have $x \in V \cap U$ and $||a_0(x) - a(x)|| < \varepsilon - \delta$. Thus

$$\begin{aligned} ||\phi(a)(x) - b_0(x)|| &\le ||\phi(a)(x) - \phi(a_0)(x)|| + ||\phi(a_0)(x) - b_0(x)|| \\ &< ||a(x) - a_0(x)|| + \delta < \varepsilon - \delta + \delta = \varepsilon. \end{aligned}$$

□

In particular, if $\phi : A \longrightarrow B$ is a $C(X)$-map, then $\mathcal{C}(\phi) : \mathcal{C}(A) \longrightarrow \mathcal{C}(B)$ is a bundle map.

Theorem 4 *The functors Γ and $\mathcal{C}$ give an equivalence of categories between the above two categories.*

Proof Suppose that $\phi : A \longrightarrow B$ is a $C(X)$-map. Then the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\mu_A"] & B \arrow[d, "\mu_B"] \\ \Gamma(X, \mathcal{C}(A)) \arrow[r, "\Gamma(\mathcal{C}(\phi))"] & \Gamma(X, \mathcal{C}(B)) \end{tikzcd}$$

where μ_A and μ_B are defined as above. Thus $\Gamma\mathcal{C}$ is naturally isomorphic to the identity.

If $p_{\mathcal{A}} : \mathcal{A} \longrightarrow X$ is an upper semi-continuous C*-bundle, then $\mathcal{C}\Gamma(A) = C(\Gamma(X, \mathcal{A})) = \coprod_{x \in X} \mathcal{A}(x)$. By [21, Theorems C.25-C.26], there is a unique topology on $\coprod_{x \in X} \mathcal{A}(x)$ such that $\mathcal{A} = \coprod_{x \in X} \mathcal{A}(x)$ is an upper semi-continuous C*-bundles. Therefore, $\mathcal{C}\Gamma$ is the identity functor. □

Let $\mathcal{A}$ be an upper semicontinuous C*-bundle over X. We say that $(\mathcal{B}, \iota)$ is a bundle extension of $\mathcal{A}$ if $\mathcal{B}$ is also an upper semicontinuous C*-bundle over X, and $\iota : \mathcal{A} \to \mathcal{B}$ is an injective bundle morphism (monomorphism). This means that $p_{\mathcal{B}} \circ \iota = p_{\mathcal{A}}$, and for any $x \in X$, the restriction map $\iota : \mathcal{A}_x \to \mathcal{B}_x$ is an injective *-homomorphism. Injectivity in the category of upper semicontinuous C*-bundles can be defined in the same way as in any category. Specifically, an object $\mathcal{A}$ is an injective object in the category of upper semicontinuous C*-bundles over X if for any bundle extension $(\mathcal{B}, \iota)$ of an upper semicontinuous C*-bundle $\mathcal{C}$ and bundle map $\Phi : \mathcal{C} \to \mathcal{A}$, there exists bundle map $\widehat{\Phi} : \mathcal{B} \to \mathcal{A}$ such that $\widehat{\Phi} \circ \iota = \Phi$. The above functors Γ and $\mathcal{C}$ preserve extensions. Therefore, for a compact Hausdorff space X, $\mathcal{A}$ is an injective object in the category of upper semicontinuous C*-bundles over X if and only if $\Gamma(X, \mathcal{A})$ is an injective object in the category of $C(X)$-algebras. One can also define essentiality in the category of upper semicontinuous C*-bundles over X as follows: a bundle extension $(\mathcal{B}, \iota)$ of $\mathcal{A}$ is essential if for any bundle map Φ from $\mathcal{B}$ to an upper semicontinuous C*-bundle $\mathcal{C}$, whenever $\Phi \circ \iota$ is a completely isometric bundle map (i.e., $x \in X$, the restriction map $\Phi : \mathcal{A}_x \to \mathcal{B}_x$ is completely isometric), then Φ itself is a completely isometric bundle map. A bundle extension $(\mathcal{B}, \iota)$ of an upper semicontinuous C*-bundle $\mathcal{A}$ is an injective envelope of $\mathcal{A}$ if it is both injective and essential. Since the functors Γ and $\mathcal{C}$ preserve essentiality, we have the following result.

Corollary 1 *Any object $\mathcal{A}$ in the category of upper semicontinuous bundle over X has an injective envelope $(\mathcal{D}, \iota)$, which is unique in the sense that if $(\mathcal{D}_1, \iota_1)$ is another injective envelope, there exists a bundle isomorphism $\Phi : \mathcal{D} \longrightarrow \mathcal{D}_1$ such that $\Phi \circ \iota = \iota_1$.*

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