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ORIGINAL RESEARCH

Positive solutions to nonhomogeneous quasilinear problems with singular and supercritical nonlinearities

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Abstract In this article, we establish the existence of nonnegative solutions to the following quasilinear and singular elliptic problems with supercritical nonlinearity:

$$\begin{cases} -\Delta_p z - \Delta_q z = \lambda \dfrac{h(x)}{z^\gamma} + z^\theta, \ z > 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open, bounded subset of $\mathbb{R}^N (N \geq 3)$ with C^2 boundary, h is a positive real-valued function, $1 < p < q < \infty$ and λ, θ, γ are positive parameters. The main contribution of this paper is the treatment of singular and supercritical nonlinearities on the right-hand side in the presence of the nonhomogeneous $(p+q)$-Laplace operator. To demonstrate the existence of a weak solution, we utilise the method of sub and supersolution.

Keywords $(p+q)$-Laplace equation · Supercritical Dirichlet problem · Singular elliptic equations · Quasilinear elliptic equations

Mathematics Subject Classification 35J92 · 35J75 · 35J62

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 3)$ be a bounded open subset with C^2 boundary and consider the following quasilinear and singular elliptic equation

$$\begin{cases} -\Delta_p z - \Delta_q z = \lambda \dfrac{h(x)}{z^\gamma} + z^\theta, \ z > 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{Q}_\lambda$}$$

where $1 < p < q < \infty$ with $0 < \gamma < 1$ and $\theta, \lambda > 0$ are real constants. The real valued nonnegative function $h(x) \in L^\infty_{loc}(\Omega)$ and satisfies the following:

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(H_γ) : For given $0 < \gamma < 1$, there exists $\delta \in (0, 1-\gamma)$ and $C > 0$, such that

$$0 \leq h(x) \leq Cd(x)^{-\delta} \text{ in } \Omega,$$

here, $d(x) := dist(x, \partial\Omega)$. The operator $-\Delta_p$ is the p-Laplacian, defined as $-\Delta_p z := -\text{div}\left(|\nabla z|^{p-2}\nabla z\right)$. If $p = q$, the operator $-\Delta_p z - \Delta_q z$ reduces to the classical p-Laplacian.

Problems related to $(\mathcal{Q}_\lambda)$ are linked to the minimisation of well-defined energy functionals, which can be described as follows:

$$z \mapsto \int_\Omega |\nabla z|^p + a(x)|\nabla z|^q \, dz,$$

where $a(x)$ is a real valued nonnegative measurable function. These functionals introduced by Zhikov [1] are known as double phase functionals, and they were initially proposed to describe strongly anisotropic materials. The double phase operator has wide applications in various physical problems, such as biophysics [2], elementary particles [3] and transonic flows [4], among others. A comprehensive overview of the latest developments on such equations can be found in Marano and Masconi [5] and references therein.

There exists an extensive body of literature concerning problems of the type

$$\begin{cases} -\Delta_p z - \Delta_q z = f(x)g(z), \ z > 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Therefore, we have limited our discussion to a brief overview of a few papers that have had the most significant impact on this paper.

In the case $p = q = 2$, problems related to (1) have received much attention in the past. For Laplacian type operators, we mention the pioneering works [6–8], in which authors considered (1) with $g(z) = z^{-\gamma}$. They proved the existence and uniqueness of the solution in the classical sense for smooth enough datum f. In [9], Boccardo and Orsina considered f in suitable Lebesgue space and proved the existence and regularity of solutions depending on the summability of f and the exponent γ. To find results on the existence, regularity, and multiplicity of the related singular problems, one can refer to [10–12] and the references cited therein. For nonhomogeneous singular problems, the interested reader can refer to [13,14].

In [15], Boccardo et al. considered the problem

$$\begin{cases} -\text{div}\,(M(x)\nabla z) = \lambda z^\gamma + z^\theta, \ z \geq 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $0 < \gamma < 1 < \theta$; λ is small and $M(x)$ is a bounded matrix which satisfies the ellipticity condition. The existence of solutions for equation (2) was established using sub and supersolutions, which take advantage of the combined effect of a concave and a convex nonlinearity. This technique can be used even in the supercritical case $\theta + 1 \geq 2^*$ (for which solutions do not generally exist if $\lambda = 0$). Furthermore, this method of sub and supersolutions also works for nonlinear principal parts like $-\Delta_p$, as demonstrated by Boccardo et al. in [15]. Subsequent important developments of the results from [15] can be found in [16], where semilinear elliptic problems with lower-order nonlinearities are studied in detail, including the presence of both sublinear and superlinear terms and the possibility of multiple solutions. Later, Boccardo [17] considered (2), with the right-hand side having the from $\lambda z^{-\gamma} + z^\theta$. They used the method of sub and supersolutions and classical Amann-Sattinger method to prove the existence of solution for all $\theta > 0$. In [18,19], authors proved the multiplicity of solutions for problems considered in [17] using variational methods. They proved the multiplicity results under the assumption $\theta \leq 2^* - 1$.

Finally, it is worth mentioning some papers that have addressed existence and multiplicity results for problems related to $(\mathcal{Q}_\lambda)$ when $\theta \leq q^* - 1$. Papageorgiou et al. [20] examined the subcritical case $(\theta < q^* - 1)$ and proved the existence of at least two solutions using the Nehari method. Subsequently, Liu [21] extended this result to Musielak-Orlicz Sobolev spaces. On the other hand, Kumar et al. [22] considered $(\mathcal{Q}_\lambda)$ with critical growth $(\theta = q^* - 1)$ and $h = 1$. By splitting the Nehari manifold, the authors established the existence of at least two positive solutions and derived L^∞ estimates.

It is important to note that the two nonlinearities on the right-hand side of the differential equation in (2) are neither concave and convex, nor are they sublinear and superlinear. Instead, they are convex and convex.

In this article, our objective is to investigate the problem $(\mathcal{Q}_\lambda)$ and the effects of the terms θ, λ and γ on the existence of the solutions. In particular, for supercritical cases, i.e., $\theta \geq q^* - 1$, we prove the existence of solutions in a certain weak sense. To the best of our knowledge, the main results of this paper are novel for supercritical nonlinearity in the framework of nonhomogeneous $(p+q)$-Laplace operator. Our motivation for this problem is taken from [17], where the authors considered the problem similar to (2) with a nonlinearity that includes both a singular and a convex term. In [17], author pointed out that the used method also works for nonlinear principle parts like $-\Delta_p$. In contrast to p-Laplace operator case, the nonhomogeneous nature of the $p+q$-Laplace operator presents challenges in understanding certain problems like the qualitative properties of solutions to nonlinear $p+q$-Laplace equations, eigenvalue problem of $p+q$-Laplace operator etc. We observe that to adopt the method used in [17], we require the comparison principle for the nonhomogeneous $(p+q)$-Laplace operator and the L^∞ estimate obtained in Section 2 (see (10)). These two results are established in [13] and [23], respectively. Moreover, when $\theta \geq q^*-1$, the problem $(\mathcal{Q}_\lambda)$ becomes more delicate because the Sobolev embedding $W_0^{1,q}(\Omega) \hookrightarrow L^{q^*}(\Omega)$ is not compact. To find a solution to the problem $(\mathcal{Q}_\lambda)$, we use sub and supersolutions along with the classical Sattinger method [24]. Additionally, we prove that a nonexistence phenomenon will occur for λ large enough.

We use the following notations throughout this paper. For a function z in the Sobolev space $W_0^{1,q}(\Omega)$, we denote the norm $\|z\|^q_{W_0^{1,q}(\Omega)} := \int_\Omega |\nabla z|^q \, dx$. The conjugate exponent of a number $q \geq 1$ is denoted by $q' = \frac{q}{q-1}$. For a function z in the Lebesgue space $L^p(\Omega)$, we use the usual $\|\cdot\|_{L^p}$ norm. The symbol $|X|$ denotes the Lebesgue measure of a set X in $\mathbb{R}^N$. For $1 < q < N$, the critical Sobolev exponent is given by $q^* = \frac{Nq}{N-q}$. To simplify notation, we use $\int_\Omega g$ to represent the integral $\int_\Omega g(x)\, dx$. We use the notation c and C to represent constants. These constants may vary from line to line but remain independent of n unless stated otherwise. We denote by $\varphi_1 > 0$ the normalized eigenfunction corresponding to the first eigenvalue λ_1 of $-\Delta_q$ with homogeneous Dirichlet boundary condition:

$$-\Delta_q \varphi_1 = \lambda_1 |\phi_1|^{q-2}\varphi_1 \quad \text{in } \Omega; \qquad \varphi_1 = 0 \quad \text{on } \partial\Omega. \tag{3}$$

We prove the existence of solution $z \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ of $(\mathcal{Q}_\lambda)$ depending on the parameters γ, θ and λ in the following weak sense

Definition 1 We say that the function $z \in W_0^{1,q}(\Omega)$ is a weak solution to the problem $(\mathcal{Q}_\lambda)$ if for all $\omega \subset\subset \Omega$ there exists $C_\omega > 0$ such that $z \geq C_\omega$ in ω and z satisfies

$$\int_\Omega |\nabla z|^{p-2}\nabla z \nabla\varphi + \int_\Omega |\nabla z|^{q-2}\nabla z \nabla\varphi = \lambda \int_\Omega \frac{h(x)}{z^\gamma}\varphi + \int_\Omega z^\theta \varphi \tag{4}$$

for all $\varphi \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$.

In this paper, we present the following main results, which we will prove in the next section.

Theorem 1 *For $\theta > q-1$, suppose that $h(x)$ satisfies the hypothesis $(H\gamma)$ then there exists $0 < \Lambda < \infty$ such that for every $\lambda \in (0, \Lambda)$, there exists a positive solution z in $W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ to the problem $(\mathcal{Q}_\lambda)$ in the sense of Definition 1.*

Theorem 2 *Assume $0 < \theta \leq q-1$ then for every $\lambda > 0$, there exists a positive weak solution of the problem $(\mathcal{Q}_\lambda)$ in the sense of Definition 1.*

2 Existence Results

In this section, we prove Theorem 1 and 2. We begin by defining weak sub and supersolution. We say that $z \in W_0^{1,q}(\Omega)$ is a weak subsolution (respectively supersolution) to the problem $(\mathcal{Q}_\lambda)$ if for all $\omega \subset\subset \Omega$ there exists $C_\omega > 0$ such that $z \geq C_\omega$ in ω and z satisfies

$$\int_\Omega |\nabla z|^{p-2}\nabla z \nabla\varphi + \int_\Omega |\nabla z|^{q-2}\nabla z \nabla\varphi \leq (\geq)\ \lambda \int_\Omega \frac{h(x)}{z^\gamma}\varphi + \int_\Omega z^\theta \varphi \tag{5}$$

for all $\varphi \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$.

Proof of Theorem 1 We prove the existence of solution through several steps

Step 1: We consider the following perturbed problem

$$\begin{cases} -\Delta_p z_n - \Delta_q z_n = \lambda \dfrac{h_n(x)}{\left(|z_n| + \frac{1}{n}\right)^\gamma} + (z_n^+)^\theta \text{ in } \Omega, \\ z_n = 0 \text{ on } \partial\Omega, \end{cases} \tag{6}$$

where $h_n(x) = \min\{h(x), n\}$. We prove the existence of weak solutions using the method of sub and supersolutions.

Step 2: We obtain a subsolution as follows. Let $\underline{z_n} \in W_0^{1,q}(\Omega)$ be a weak solution to the following problem

$$\begin{cases} -\Delta_p \underline{z_n} - \Delta_q \underline{z_n} = \lambda \dfrac{h_n(x)}{\left(|\underline{z_n}| + \frac{1}{n}\right)^\gamma} \text{ in } \Omega, \\ \underline{z_n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{7}$$

Using Schauder's fixed point we can easily prove the existence of such $\underline{z_n}$ (see [13, Lemma 2.1]). Since, right-hand side of (7) is nonnegative, by the maximum principle we get $\underline{z_n} \geq 0$, and hence satisfies

$$\begin{cases} -\Delta_p \underline{z_n} - \Delta_q \underline{z_n} = \lambda \dfrac{h_n(x)}{\left(\underline{z_n} + \frac{1}{n}\right)^\gamma} \text{ in } \Omega, \\ \underline{z_n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{8}$$

Observe that

$$-\Delta_p \underline{z_n} - \Delta_q \underline{z_n} = \frac{\lambda h_n(x)}{\left(\underline{z_n} + \frac{1}{n}\right)^\gamma} \leq \frac{\lambda h_n(x)}{\left(\underline{z_n} + \frac{1}{n}\right)^\gamma} + (\underline{z_n})^\theta.$$

Thus, $\underline{z_n}$ is a subsolution of problem (6).

Step 3: We construct a supersolution in the following way. Let $t \geq M_\lambda$ (defined later in (13)) and let $0 < \overline{z_n} \in W_0^{1,q}(\Omega)$ be the solution to the following problem

$$\begin{cases} -\Delta_p \overline{z_n} - \Delta_q \overline{z_n} = \dfrac{t}{\left(|\overline{z_n}| + \frac{1}{n}\right)^\gamma} h_n(x) \text{ in } \Omega, \\ \overline{z_n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{9}$$

By [23, Theorem 1.1], there exists some constant C_0(independent of n) such that

$$\|\overline{z_n}\|_{L^\infty(\Omega)} \leq C_0\, t^{\frac{1}{q-1+\gamma}}. \tag{10}$$

Using maximum principle in (9), we get $\overline{z_n} \geq 0$, hence $\overline{z_n}$ solves

$$\begin{cases} -\Delta_p \overline{z_n} - \Delta_q \overline{z_n} = \dfrac{t}{\left(\overline{z_n} + \frac{1}{n}\right)^\gamma} h_n(x) \text{ in } \Omega, \\ \overline{z_n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{11}$$

Next, in order to have that $\overline{z_n}$ is a supersolution of problem (6), we claim that for some $t > 0$, we get

$$\frac{t}{\left(\overline{z_n} + \frac{1}{n}\right)^\gamma} h_n(x) \geq \frac{\lambda}{\left(\overline{z_n} + \frac{1}{n}\right)^\gamma} h_n(x) + (\overline{z_n})^\theta.$$

That is

$$t h_n(x) \geq \lambda h_n(x) + \overline{z_n}^\theta \left(\frac{1}{n} + \overline{z_n}\right)^\gamma. \tag{12}$$

Using (10) in equation (12), we obtain

$$t \geq \lambda + C_0^\theta\, t^{\frac{\theta}{q-1+\gamma}} p \left(\frac{1}{n} + C_0\, t^{\frac{1}{q-1+\gamma}}\right)^\gamma.$$

Let M_λ be such that

$$M_\lambda = \lambda + 2^\gamma\, C_0^{\theta+\gamma}\, M_\lambda^{\frac{\theta+\gamma}{q-1+\gamma}}. \tag{13}$$

Using the condition $\theta > q-1$ and $\gamma > 0$, we have $\frac{\theta+\gamma}{q-1+\gamma} > 1$, that is for sufficiently small λ there exists M_λ which satisfies (13). Therefore, there exists some $\Lambda > 0$ such that, for $\lambda \in (0, \Lambda)$, a solution to the algebraic equation (13) exists. Thus, we only need to verify that for a sufficiently large n, we have

$$\lambda + 2^\gamma\, C_0^{\theta+\gamma}\, M_\lambda^{\frac{\theta+\gamma}{q-1+\gamma}} \geq \lambda + C_0^\theta\, M_\lambda^{\frac{\theta}{q-1+\gamma}p} \left(\frac{1}{n} + C_0\, M_\lambda^{\frac{1}{q-1+\gamma}}\right)^\gamma.$$

Indeed, after simplification we get

$$\frac{1}{n} \leq C_0\, M_\lambda^{\frac{1}{q-1+\gamma}}, \tag{14}$$

which holds true for large n. Hence, we proved that $\overline{z_n}$ is a supersolution of the problem (6), if $t \geq M_\lambda$.

Step 4: We prove that $\underline{z_n} \leq \overline{z_n}$. From (8) and (11), using $(\underline{z_n} - \overline{z_n})^+$ as a test function, we obtain

$$\begin{aligned}
&\int_\Omega \Big[(|\nabla \underline{z_n}|^{p-2}\nabla \underline{z_n} + |\nabla \underline{z_n}|^{q-2}\nabla \underline{z_n}) - (|\nabla \overline{z_n}|^{p-2}\nabla \overline{z_n} + |\nabla \overline{z_n}|^{q-2}\nabla \overline{z_n})\Big]\nabla(\underline{z_n} - \overline{z_n})^+ \\
&= \int_\Omega h_n(x) \left[\frac{\lambda}{\left(\underline{z_n} + \frac{1}{n}\right)^\gamma} - \frac{t}{\left(\overline{z_n} + \frac{1}{n}\right)^\gamma}\right](\underline{z_n} - \overline{z_n})^+ \\
&\leq \|h\|_{L^\infty(\Omega)} \int_\Omega \left[\frac{\lambda}{\left(\underline{z_n} + \frac{1}{n}\right)^\gamma} - \frac{\lambda}{\left(\overline{z_n} + \frac{1}{n}\right)^\gamma}\right](\underline{z_n} - \overline{z_n})^+ \\
&\quad + \|h\|_{L^\infty}(\Omega) \int_\Omega \left[\frac{\lambda}{\left(\overline{z_n} + \frac{1}{n}\right)^\gamma} - \frac{t}{\left(\overline{z_n} + \frac{1}{n}\right)^\gamma}\right](\underline{z_n} - \overline{z_n})^+ \\
&\leq \|h\|_{L^\infty}(\Omega)\lambda \int_{\{\underline{z_n} \geq \overline{z_n}\}} \left[\frac{1}{\left(\underline{z_n} + \frac{1}{n}\right)^\gamma} - \frac{1}{\left(\overline{z_n} + \frac{1}{n}\right)^\gamma}\right](\underline{z_n} - \overline{z_n}) \\
&\quad + \|h\|_{L^\infty}(\Omega)(\lambda - t) \int_{\{\underline{z_n} \geq \overline{z_n}\}} \frac{1}{\left(\overline{z_n} + \frac{1}{n}\right)^\gamma}(\underline{z_n} - \overline{z_n}) \leq 0.
\end{aligned}$$

Note that in the last line, the first integral is negative because the real valued function $h(s) = \dfrac{1}{\left(\frac{1}{n} + s\right)^\gamma}$ is decreasing for $s > 0$, while the second integral is negative because $t \geq M_\lambda$ and by (13), $M_\lambda > \lambda$. Hence, we derive

$$\int_\Omega \Big[(|\nabla \underline{z_n}|^{p-2}\nabla \underline{z_n} + |\nabla \underline{z_n}|^{q-2}\nabla \underline{z_n}) - (|\nabla \overline{z_n}|^{p-2}\nabla \overline{z_n} + |\nabla \overline{z_n}|^{q-2}\nabla \overline{z_n})\Big]\nabla(\underline{z_n} - \overline{z_n})^+ \leq 0.$$

After rearranging the terms in previous inequality, we get

$$\begin{aligned}
&\int_\Omega \Big[|\nabla \underline{z_n}|^{p-2}\nabla \underline{z_n} - |\nabla \overline{z_n}|^{p-2}\nabla \overline{z_n}\Big]\nabla(\underline{z_n} - \overline{z_n})^+ \\
&\qquad + \int_\Omega (|\nabla \underline{z_n}|^{q-2}\nabla \underline{z_n} - |\nabla \overline{z_n}|^{q-2}\nabla \overline{z_n})\nabla(\underline{z_n} - \overline{z_n})^+ \leq 0.
\end{aligned} \tag{15}$$

Using the inequality: for $q > 1$, there exists a constant $C_0 = C(q) > 0$ such that for all $a, b \in \mathbb{R}^N$ with $|a| + |b| > 0$, the following holds

$$\left(|a|^{q-2}a - |b|^{q-2}b\right)\cdot(a-b) \geq C_0\,(|a| + |b|)^{q-2}\,|a-b|^2, \tag{16}$$

we deduce that

$$\int_\Omega \left(|\nabla \underline{z_n}| + |\nabla \overline{z_n}|\right)^{q-2} |\nabla \underline{z_n} - \nabla \overline{z_n}|^2 \leq 0. \tag{17}$$

Equation (17) implies that $(\underline{z_n} - \overline{z_n})^+ = 0$ almost everywhere in Ω and hence, $\underline{z_n} \leq \overline{z_n}$ in Ω.
Step 5: Amann-Sattinger method.

We define the real valued function

$$f(s) := \lambda \frac{h(x)}{\left(s + \frac{1}{n}\right)^\gamma} + s^\theta + \lambda n^{1+\gamma} s\gamma \qquad s \in [0, \infty).$$

We can easily verify that $f(s)$ is increasing with respect to s. Using the classical Amann-Sattinger method (see [24]), we get the existence of z_n that solves

$$\begin{cases} -\Delta_p z_n - \Delta_q z_n + \lambda n^{1+\gamma} z_n \gamma = \lambda \dfrac{h_n(x)}{\left(z_n + \frac{1}{n}\right)^\gamma} + (z_n)^\theta + \lambda n^{1+\gamma} z_n \gamma, \quad z_n > 0 \text{ in } \Omega, \\ z_n = 0 \text{ on } \partial\Omega \end{cases}$$

and satisfies

$$\underline{z_n} \leq z_n \leq \overline{z_n} \leq C_0 t^{\frac{1}{q-1+\gamma}}. \tag{18}$$

Indeed, let us consider $z_{n,1} \in W_0^{1,q}(\Omega)$ be a solution of the following problem

$$\begin{cases} -\Delta_p z_{n,1} - \Delta_q z_{n,1} + \lambda n^{1+\gamma} z_{n,1}\gamma = f(\underline{z_n}), \quad z_{n,1} > 0 \text{ in } \Omega, \\ z_{n,1} = 0 \text{ on } \partial\Omega. \end{cases}$$

Since the term $\lambda n^{1+\gamma} z_{n,1}\gamma$ is linear, we can see that the comparison principle [13, Theorem 1.5] is applicable. Hence, using the fact

$$-\Delta_p \underline{z_n} - \Delta_q \underline{z_n} + \lambda n^{1+\gamma} \underline{z_n}\gamma \leq f(\underline{z_n}) \text{ in } \Omega,$$

we conclude that $\underline{z_n} \leq z_{n,1}$. Similarly, we can prove that $z_{n,1} \leq \overline{z_n}$. For every $k \in \mathbb{N}$, let us now consider the iterated problems

$$\begin{cases} -\Delta_p z_{n,k+1} - \Delta_q z_{n,k+1} + \lambda n^{1+\gamma} z_{n,k+1}\gamma = f(z_{n,k}), \quad z_{n,k+1} > 0 \text{ in } \Omega, \\ z_{n,k+1} = 0 \text{ on } \partial\Omega. \end{cases} \tag{19}$$

Furthermore, given the increasing nature of f, the solutions of these iterated problems satisfy

$$\underline{z_n} \leq z_{n,1} \leq \cdots\cdots \leq z_{n,k} \leq z_{n,k+1} \leq \overline{z_n}.$$

Hence, we can define the pointwise limit $z_n := \lim_{k\to\infty} z_{n,k}$. Moreover, choosing $z_{n,k+1}$ as a test function in (19) and integrating by parts, we obtain

$$\begin{aligned} \int_\Omega \left(|\nabla z_{n,k+1}|^{p-2}\nabla z_{n,k+1} + |\nabla z_{n,k+1}|^{q-2}\nabla z_{n,k+1}\right)\nabla z_{n,k+1} + \int_\Omega \lambda n^{1+\gamma} z_{n,k+1}^2\gamma \\ = \lambda \int_\Omega \frac{h_n(x)}{\left(z_{n,k} + \frac{1}{n}\right)^\gamma} z_{n,k+1} + \int_\Omega (z_{n,k})^\theta z_{n,k+1} + \int_\Omega \lambda\gamma n^{1+\gamma} z_{n,k} z_{n,k+1} \\ \leq \lambda \|h\|_{L^\infty(\Omega)} \int_\Omega n^\gamma \overline{z_n} + \int_\Omega (\overline{z_n})^{\theta+1} + \lambda\gamma n^{1+\gamma} \int_\Omega \overline{z_n}^2. \end{aligned}$$

Using (18) in the right-hand side of the above inequality, we get

$$\int_\Omega |\nabla z_{n,k+1}|^p + \int_\Omega |\nabla z_{n,k+1}|^q \leq C_n,$$

here C_n represents a constant which depends on n but remains independent of k. Thus, $z_{n,k}$ is bounded in $W_0^{1,q}(\Omega)$ and consequently, up to a subsequence $z_{n,k} \rightharpoonup z_n$ in $W_0^{1,q}(\Omega)$. Hence, z_n is a solution of (6).

Furthermore, by [13, Lemma 2.1], for every $\omega \subset\subset \Omega$, there exists a constant C_ω (independent of n) such that

$$\underline{z_n} \geq \underline{z_1} \geq C_\omega > 0, \quad \forall x \in \omega \text{ and every } n \in \mathbb{N}. \tag{20}$$

Step 6: Next, we prove some regularity results for $\gamma < 1$. Choosing z_n as a test function in the weak formulation of (6), we get

$$\int_\Omega \left(|\nabla z_n|^{p-2}\nabla z_n + |\nabla z_n|^{q-2}\nabla z_n\right)\nabla z_n = \lambda \int_\Omega \frac{h_n(x)}{\left(z_n+\frac{1}{n}\right)^\gamma} z_n + \int_\Omega z_n^\theta z_n.$$

Using the conditions on $h(x)$, we have

$$\int_\Omega |\nabla z_n|^p + \int_\Omega |\nabla z_n|^q = \lambda \|h\|_{L^\infty(\Omega)} \int_\Omega z_n^{1-\gamma} + \int_\Omega z_n^{\theta+1}.$$

The equation (18) implies that

$$\int_\Omega |\nabla z_n|^p \leq C, \tag{21}$$

where C is a constant independent of n, that is z_n is uniformly bounded in $W_0^{1,q}(\Omega)$.
Step 7: We prove the existence result by passing to the limits in problem (6). Recalling (21), since z_n is uniformly bounded in $W_0^{1,q}(\Omega)$ there exists a subsequence still denoted by z_n and $z \in W_0^{1,q}(\Omega)$ such that

$$\begin{cases} z_n \rightharpoonup z \text{ in } W_0^{1,q}(\Omega), \\ z_n \to z \text{ almost everywhere in } \Omega. \end{cases} \tag{22}$$

Thus, for every $\varphi \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, we get

$$\int_\Omega \left(|\nabla z_n|^{p-2}\nabla z_n + |\nabla z_n|^{q-2}\nabla z_n\right)\nabla \varphi \longrightarrow \int_\Omega \left(|\nabla z|^{p-2}\nabla z + |\nabla z|^{q-2}\nabla z\right)\nabla \varphi. \tag{23}$$

Now, for $\varphi \in W_0^{1,q}(\Omega)$ consider the set $\omega = \{\varphi \neq 0\}$, using (20), we get

$$0 \leq \lambda \frac{h_n(x)}{\left(z_n+\frac{1}{n}\right)^\gamma}\varphi \leq \frac{\lambda C}{C_\omega^\gamma}\varphi.$$

By Lebesgue dominated convergence theorem, we have

$$\lambda \int_\Omega \frac{h_n(x)}{\left(z_n+\frac{1}{n}\right)^\gamma}\varphi \longrightarrow \lambda \int_\Omega \frac{h(x)}{z^\gamma}\varphi. \tag{24}$$

From (22), we also have

$$\int_\Omega z_n^\theta \varphi \longrightarrow \int_\Omega z^\theta \varphi \qquad \text{for all } \varphi \in W_0^{1,q}(\Omega). \tag{25}$$

Now, with the help of the equations (23)-(25), we can pass to the limit in (6) to prove the existence of solution $z \in W_0^{1,q}(\Omega)$.
Step 8: Nonexistence.

Following the methods of [22, Lemma 6.1], we prove the nonexistence of solution for large λ.

Let us define $\Lambda := \sup\{\lambda : \text{Problem}\,(\mathcal{Q}_\lambda) \text{ has a solution}\}$ From *Step 3*, it becomes evident that $\Lambda > 0$. Next, our aim is to prove that $\Lambda < \infty$. On contrary, let us assume that there exists a sequence $\lambda_n \to \infty$ such that problem (6) has a solution z_n. There exists $\lambda_* > 0$ such that

$$\frac{\lambda}{s^\gamma} + s^\theta \geq (\lambda_1 + \epsilon)s^{q-1} \text{ for all } s > 0,\ \epsilon \in (0,1) \text{ and } \lambda > \lambda_* > 0, \tag{26}$$

where λ_1 is the first eigenvalue λ_1 of $-\Delta_q$. By choosing $\lambda_n > \lambda_*$, we can verify that z_n is a supersolution of the following problem:

$$\begin{cases} -\Delta_p z - \Delta_q z = (\lambda_1 - \epsilon)z^{q-1}, \quad z > 0 \text{ in } \Omega, \\ z = 0 \text{ on } \partial\Omega, \end{cases} \tag{27}$$

that is, for all $0 \leq \varphi \in W_0^{1,q}(\Omega)$, we have

$$\int_\Omega \left(|\nabla z|^{p-2}\nabla z + |\nabla z|^{q-2}\nabla z \right) \nabla\varphi \geq \int_\Omega (\lambda_1 - \epsilon) z^{q-1}\varphi.$$

According to [22, Theorem 2.1], there exists a positive constant $\rho > 0$ such that $\rho\varphi_1 < z_n$ and $\rho\varphi_1$ is a subsolution of (27). Applying a monotone iteration procedure, we can find a solution of (27) for any $\epsilon \in (0, 1)$. This contradicts the fact that λ_1 is an isolated point in the spectrum of $-\Delta_p z - \Delta_q z$ in $W_0^{1,q}(\Omega)$ (see [25,26]). □

Proof of Theorem 2 The proof of this theorem closely follows that of Theorem 1. Specifically, when $\theta < q - 1$, we obtain $\dfrac{\theta + \gamma}{q - 1 + \gamma} < 1$. By applying the Mean Value Theorem, we can demonstrate the existence of at least one solution to the algebraic equation (13) for any $\lambda > 0$. In fact, from (13), we have

$$M_\lambda = \lambda + 2^\gamma\, C_0^{\theta+\gamma}\, M_\lambda^{\frac{\theta+\gamma}{q-1+\gamma}}.$$

Let us define $G(M_\lambda) = M_\lambda - \lambda - 2^\gamma\, C_0^{\theta+\gamma}\, M_\lambda^{\frac{\theta+\gamma}{q-1+\gamma}}$. Then

$$G(0) = -\lambda < 0. \tag{28}$$

We can write

$$G(M_\lambda) = M_\lambda^{\frac{\theta+\gamma}{q-1+\gamma}} \left(M_\lambda^{1-\frac{\theta+\gamma}{q-1+\gamma}} - \lambda M_\lambda^{-\frac{\theta+\gamma}{q-1+\gamma}} - 2^\gamma\, C_0^{\theta+\gamma} \right).$$

Let us denote by $F(M_\lambda) = M_\lambda^{1-\frac{\theta+\gamma}{q-1+\gamma}} - \lambda M_\lambda^{-\frac{\theta+\gamma}{q-1+\gamma}} - 2^\gamma\, C_0^{\theta+\gamma}$, then we have

$$G(M_\lambda) = M_\lambda^{\frac{\theta+\gamma}{q-1+\gamma}} F(M_\lambda).$$

Observe that $F(M_\lambda) \to \infty$ as $M_\lambda \to \infty$, that is;

$$\exists\ N_0 \in \mathbb{N} \text{ such that } F(M_\lambda) > 1 \text{ for all } M_\lambda \geq N_0.$$

Hence,

$$G(N_0) = N_0^{\frac{\theta+\gamma}{q-1+\gamma}} F(N_0) > 0. \tag{29}$$

From (28) and (29), using Mean Value Theorem, there exists at least one M_λ, which satisfies $0 < M_\lambda < N_0$ and $G(M_\lambda) = 0$. Therefore, for any $\lambda > 0$, there exists at least one solution of (13). If we assume $\theta = q - 1$, from the above arguments, it is easy to see that (13) has a solution for every $\lambda > 0$ if $2^\gamma\, C_0^{\theta+\gamma} < 1$. Now, using these results and arguing exactly as in the proof of Theorem 1, we can conclude the proof. □

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