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ORIGINAL RESEARCH

On the solutions of the Diophantine equation $F_n \pm F_m = 3^s \cdot y^b$

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Abstract In this paper, we will give an upper bound on n satisfying the Diophantine equation

$$F_n \pm F_m = 3^s \cdot y^b$$

in nonnegative integers $s \geq 0, y \geq 2, b \geq 2, n \geq m > 0$ and $(y, 3) = 1$. Then, we determine all solutions (n, m, s, y) of this equation for $b = 2$ and $2 \leq y \leq 10^4$.

Keywords Fibonacci and Lucas numbers · Exponential Diophantine equations · Linear forms in logarithms · Baker's method

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1 Introduction

The Fibonacci sequence (F_n) is defined by $F_0 = 0,\ F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Lucas sequence (L_n), which is similar to the Fibonacci sequence, is defined by the same recursive pattern with initial conditions $L_0 = 2,\ L_1 = 1$. The terms of the Fibonacci and Lucas sequences are called Fibonacci and Lucas numbers, respectively. The Fibonacci and Lucas numbers for negative indices are defined by $F_{-n} = (-1)^{n+1} F_n$ and $L_{-n} = (-1)^n L_n$ for $n \geq 1$. Binet formulas for these numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$, which are the roots of the characteristic equation $x^2 - x - 1 = 0$. It can be seen that $1 < \alpha < 2$, $-1 < \beta < 0$ and $\alpha\beta = -1$. For Fibonacci and Lucas numbers, the identities

$$F_{m+n} + F_{m-n} = \begin{cases} L_m F_n \text{ if } n \text{ is odd}, \\ L_n F_m \text{ if } n \text{ is even}, \end{cases} \tag{1}$$

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and

$$F_{m+n} - F_{m-n} = \begin{cases} L_m F_n \text{ if } n \text{ is even}, \\ L_n F_m \text{ if } n \text{ is odd}, \end{cases} \tag{2}$$

are well known (also see [14]). These identities can be proved by using Binet formulas. Moreover, if $n \geq 1$, then the relation between F_n and α is given by

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1}. \tag{3}$$

Inequality (3) can be proved by induction. Also, we observe that the inequality

$$\frac{F_{n-1}}{F_n} \leq \frac{2}{3}$$

holds for all $n \geq 3$. This implies that

$$\frac{F_n}{F_m} \geq \frac{3}{2} \tag{4}$$

for $m < n$ and $n \geq 3$. For a brief history of Fibonacci and Lucas sequences one can consult [10]. The Fibonacci and Lucas sequences have many interesting properties and have been studied in the literature by many researchers. They specially have interested in square terms, perfect powers in these sequences and the exponential Diophantine equations including these sequences. Firstly, square terms and later perfect powers in the Fibonacci and Lucas sequences have attracted the attention of the researchers. As related these subjects, the authors gave the following four theorems, which can be deduced from [6–8] and are useful to us.

Theorem 1 *The only perfect powers in the Fibonacci sequence are* $F_0 = 0$, $F_1 = 1, F_2 = 1$, $F_6 = 8$ *and* $F_{12} = 144$.

Theorem 2 *The only perfect powers in the Lucas sequence are* $L_1 = 1$ *and* $L_3 = 4$.

Theorem 3 *If*

$$F_n = 2^s \cdot y^b$$

for some integers $n \geq 1$, $y \geq 1$, $b \geq 2$ *and* $s \geq 0$ *then* $n \in \{1, 2, 3, 6, 12\}$. *The solutions of the similar equation with* F_n *replaced by* L_n *have* $n \in \{1, 3, 6\}$.

Theorem 4 *If*

$$F_n = 3^s \cdot y^b$$

for some integers $n \geq 1$, $y \geq 1$, $b \geq 2$ *and* $s \geq 0$ *then* $n \in \{1, 2, 4, 6, 12\}$. *The solutions of the similar equation with* F_n *replaced by* L_n *have* $n \in \{1, 2, 3\}$.

Recently, many mathematicians have dealt with exponential Diophantine equations concerning Fibonacci and Lucas numbers. For example, the Diophantine equation $L_n + L_m = 2^a$ has been tackled in [3] by Bravo and Luca. Two years later, the same authors solved Diophantine equation $F_n + F_m = 2^a$ in [4]. Besides, in [8], Bugeaud et al. showed that if

$$F_n \pm 1 = y^a \tag{5}$$

for some nonnegative integers (n, y, a) with $a \geq 2$, then $n \in \{0, 1, 2, 3, 4, 5, 6\}$. In [2], the authors proved that if

$$F_n \pm 2 = y^a \tag{6}$$

for some nonnegative integers (n, y, a) with $a \geq 2$, then $n \in \{1, 2, 3, 4, 9\}$. Later, in [15], Luca and Patel handled the equation

$$F_n \pm F_m = y^p, \quad p \geq 2, \tag{7}$$

which is general form of equations (5) and (6). They found that the solutions of equation (7) in integers (n, m, y, p) satisfying $n \equiv m (\text{mod} 2)$ are given as either $\max\{|n|, |m|\} \leq 36$ or $y = 0$ and $|n| = |m|$. This problem remain open for the case $n \not\equiv m \,(\text{mod} 2)$. But, in [13], the authors solved this equation by fixing y in the interval $[2, 1000]$. Lastly, in [12], the authors of this paper have dealt with the equations

$$F_n + F_n = 2F_n = 3^s \cdot y^b \tag{8}$$

and

$$F_n \pm 1 = 3^s \cdot y^b \tag{9}$$

in nonnegative integers $s \geq 0, y \geq 1, b \geq 2, n \geq 1$ and $(y, 3) = 1$. They have proved the following two theorems.

Theorem 5 *The only solutions of Diophantine equation* (8) *in nonnegative integers* $s \geq 0$, $y \geq 1$, $b \geq 2$, $n \geq 1$ *and* $(y, 3) = 1$,*are given by*

$$(n, s, y, b) \in \{(3, 0, 2, 2), (6, 0, 2, 4), (6, 0, 4, 2), (12, 2, 2, 5)\}.$$

Theorem 6 *Let* $s \geq 0$, $y \geq 1$, $b \geq 2$, $n \geq 2$ *and* $(y, 3) = 1$. *Then all solutions of the equation*

$$F_n \pm 1 = 3^s y^b$$

are

$$\begin{aligned} &F_4 + 1 = 4 = 3^0 \cdot 2^2, \quad F_6 + 1 = 9 = 3^2 \cdot 1^b, \quad F_3 + 1 = 3 = 3^1 \cdot 1^b, \\ &F_5 - 1 = 4 = 3^0 \cdot 2^2, \quad F_3 - 1 = 1 = 3^0 \cdot 1^b, \quad F_7 - 1 = 12 = 3^1 \cdot 2^2. \end{aligned}$$

In this paper, we will handle the Diophantine equation

$$F_n \pm F_m = 3^s \cdot y^b, \tag{10}$$

which is the more general case of (8) and (9), in nonnegative integers $s \geq 0, y \geq 2, b \geq 2, n \geq m > 0$ and $(y, 3) = 1$. We will give an upper bound on n satisfying this equation. Then, we determine all solutions (n, m, s, y) of this equation for $b = 2$ and $2 \leq y \leq 10^4$.

2 Preliminaries

Recently, mathematicians have started to use Baker's theory of lower bounds for a nonzero linear form in logarithms of algebraic numbers. Since such bounds are of crucial importance in effectively solving Diophantine equation (10), we start with recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} \left(x - \eta^{(i)}\right) \in \mathbb{Z}[x],$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right) \tag{11}$$

is called the logarithmic height of η. Particularly, if $\eta = a/b$ is a rational number with $\gcd(a, b) = 1$ and $b \geq 1$, then $h(\eta) = \log(\max\{|a|, b\})$.

The following properties of logarithmic height are found in many works stated in the references:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{12}$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{13}$$

$$h(\eta^m) = |m| h(\eta). \tag{14}$$

The following theorem is deduced from Corollary 2.3 of Matveev [16] and provides a large upper bound for the unknowns $n, m, s,$ and b in equation (10) (also see Theorem 9.4 in [6]).

Theorem 7 *Assume that $\gamma_1, \gamma_2, ..., \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, ..., b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1+\log D)(1+\log B)A_1A_2\cdots A_t\right),$$

where

$$B \geq \max\{|b_1|, ..., |b_t|\},$$

and $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ for all $i = 1, ..., t$.

In [11], Dujella and Pethò gave a result, which is a version of the reduction method based on the Baker-Davenport Lemma [1]. Then, in [5], the authors proved the following lemma, which is an immediate variation of the result due to Dujella and Pethò from [11], and will be used to reduce the upper bound for the subscript n in equation (10).

Lemma 8 *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$, where $||\cdot||$ denotes the distance from x to the nearest integer. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

The following lemma is useful in the proof of the main theorem.

Lemma 9 [9, Lemma 2.2] *Let $a, u \in \mathbb{R}$ and $0 < a < 1$. If $|u| < a$, then*

$$|\log(1+u)| < \frac{-\log(1-a)}{a} \cdot |u|$$

and

$$|u| < \frac{a}{1-e^{-a}} \cdot \left|e^u - 1\right|.$$

Finally, we give the following lemma.

Lemma 10 [17, Lemma 7] *If $m \geq 1$, $T \geq (4m^2)^m$ and $\dfrac{x}{(\log x)^m} < T$, then $x < 2^m T(\log T)^m$.*

3 Main theorems

Theorem 11 *Let $s \geq 0$, $y \geq 2$, $b \geq 2$, $0 \leq m < n$ and $(y, 3) = 1$. If the Diophantine equation*

$$F_n \mp F_m = 3^s y^b \tag{15}$$

is a solution (n, m, s, b, y), then the inequalities

$$s + b < n < (s+b)\frac{\log(y+1)}{\log \alpha} + 5 \tag{16}$$

and

$$n < 1.14 \cdot 10^{35} \cdot (\log y)^4 \tag{17}$$

hold.

Proof Assume that (15) has a solution (n, m, s, b, y). It can be seen that the case $n = 1$ or 2 is not possible. Therefore, $n \geq 3$. Now, let us prove inequality (16). Using (3), we get

$$2^{s+b} \leq 2^s 2^b \leq 3^s y^b = F_n \mp F_m \leq F_n + F_m < 2F_n < \alpha^2 \alpha^{n-1} = \alpha^{n+1} < 2^n$$

and thus

$$s + b < n, \tag{18}$$

where we used the fact that $\alpha^{n+1} < 2^n$ for $n \geq 3$. Also, using $y \geq 2$, $0 \leq m < n$, $n \geq 3$ and (4), we obtain the inequality

$$(y+1)^{s+b} \geq 3^s (y+1)^b > 3^s y^b = F_n \mp F_m \geq F_n - F_m = F_n \left(1 - \frac{F_m}{F_n}\right) \geq \frac{\alpha^{n-2}}{3} > \alpha^{n-5},$$

which implies that

$$n < (s+b) \frac{\log(y+1)}{\log \alpha} + 5. \tag{19}$$

Hence, combining (18) and (19), we have inequality (16).

From now on, to prove inequality (17), we use Theorem 7. For this, let arrange Eq. (15) as follows

$$\frac{\alpha^n}{\sqrt{5}} - 3^s y^b = \frac{\beta^n}{\sqrt{5}} \mp F_m.$$

Taking absolute value of both sides and using (3), we get

$$\left| \frac{\alpha^n}{\sqrt{5}} - 3^s y^b \right| = \left| \frac{\beta^n}{\sqrt{5}} \mp F_m \right| \leq F_m + \frac{|\beta|^n}{\sqrt{5}} \leq \alpha^{m-1} + \frac{1}{\sqrt{5}}.$$

Dividing the last inequality by $\frac{\alpha^n}{\sqrt{5}}$ and making necessary calculations, we obtain

$$\left| 1 - \alpha^{-n} 3^s y^b \sqrt{5} \right| \leq \frac{\alpha^{m-1}\sqrt{5}}{\alpha^n} + \frac{1}{\alpha^n} \leq \frac{\alpha^{-1}\sqrt{5}}{\alpha^{n-m}} + \frac{1}{\alpha^{n-m}} \leq \frac{2.4}{\alpha^{n-m}}. \tag{20}$$

Put $\Lambda_1 = 1 - {\gamma_1}^{b_1} {\gamma_2}^{b_2} {\gamma_3}^{b_3} {\gamma_4}^{b_4}$, where

$$\gamma_1 := \alpha, \quad \gamma_2 := 3, \quad \gamma_3 := y, \quad \gamma_4 := \sqrt{5}, \quad b_1 := -n, \quad b_2 := s, \quad b_3 := b, \quad b_4 := 1.$$

If $\Lambda_1 = 0$, then we get $\frac{\alpha^n}{\sqrt{5}} = 3^s y^b$, or equivalently, $\alpha^{2n} = 5 \cdot 3^{2s} y^{2b}$. But, this is impossible since α^{2n} is irrational for all positive integer n. Therefore $\Lambda_1 \neq 0$. Thus, we can apply Theorem 7 to inequality (20). Note that the numbers $\gamma_1, \gamma_2, \gamma_3$ and γ_4 are positive real numbers and elements of the field of $\mathbb{K} = \mathbb{Q}\left(\sqrt{5}\right)$. The degree of this field is 2. So $D = 2$. Moreover, since

$$h(\gamma_1) = h(\alpha) = \frac{\log \alpha}{2}, \quad h(\gamma_2) = h(3) = \log 3$$
$$h(\gamma_3) = h(y) = \log y, \quad h(\gamma_4) = h\left(\sqrt{5}\right) = \log \sqrt{5}$$

by (11), we can take $A_1 := \log \alpha$, $A_2 := \log 9$, $A_3 := 2 \log y$, $A_4 := \log 5$ in Theorem 7. Also, using (18), we see that

$$\max\{|b_1|, |b_2|, |b_3|, |b_4|\} = \max\{|-n|, |s|, |b|, |1|\} = n.$$

So, we can take $B = n$. Thus, by Theorem 7, we obtain

$$\frac{2.4}{\alpha^{n-m}} > |\Lambda_1| > exp\left(-C_1 \cdot (1 + \log n) \cdot \log \alpha \cdot \log 9 \cdot 2 \log y \cdot \log 5\right),$$

where $C_1 = 1.4 \cdot 30^7 \cdot 4^{4.5} \cdot 2^2 \cdot (1 + \log 2)$. Taking logarithms in the above inequality and using the fact that $1 + \log n < 2 \log n$ for $n \geq 3$, we get

$$(n-m) \log \alpha < 7.23 \cdot 10^{14} \cdot \log n \cdot \log y \tag{21}$$

or

$$(n-m) < 1.51 \cdot 10^{15} \cdot \log n \cdot \log y. \tag{22}$$

Now, rearranging equation (15) as

$$\frac{\alpha^n}{\sqrt{5}} \mp \frac{\alpha^m}{\sqrt{5}} - 3^s y^b = \frac{\beta^n}{\sqrt{5}} \mp \frac{\beta^m}{\sqrt{5}} \tag{23}$$

and taking absolute values in here, we obtain

$$\left| \frac{\alpha^n}{\sqrt{5}} \mp \frac{\alpha^m}{\sqrt{5}} - 3^s y^b \right| = \left| \frac{\beta^n}{\sqrt{5}} \mp \frac{\beta^m}{\sqrt{5}} \right| \leq \frac{|\beta|^n + |\beta|^m}{\sqrt{5}} < \frac{2}{\sqrt{5} \cdot \alpha^m},$$

where we used the fact that $\alpha\beta = -1$ and $m < n$. Dividing both sides of the above inequality by $\frac{\alpha^n}{\sqrt{5}} \mp \frac{\alpha^m}{\sqrt{5}}$, we get

$$\left| 1 - \alpha^{-n} 3^s y^b \left(\frac{\left(1 \mp \alpha^{m-n}\right)}{\sqrt{5}} \right)^{-1} \right| < \frac{2}{\alpha^{m+n}\left(1 \mp \alpha^{m-n}\right)}. \tag{24}$$

Also, since

$$0 < \frac{1}{1 \mp \alpha^{m-n}} < \frac{1}{1 - \alpha^{m-n}} < \frac{1}{1 - \alpha^{-1}} < 2.62$$

for $n - m \geq 1$, inequality (24) implies that

$$\left| 1 - \alpha^{-n} 3^s y^b \left(\frac{\left(1 \mp \alpha^{m-n}\right)}{\sqrt{5}} \right)^{-1} \right| < \frac{5.24}{\alpha^{m+n}}. \tag{25}$$

Put $\Lambda_2 = 1 - {\gamma_1}^{b_1} {\gamma_2}^{b_2} {\gamma_3}^{b_3} {\gamma_4}^{b_4}$, where

$$\gamma_1 := \alpha, \quad \gamma_2 := 3, \quad \gamma_3 := y, \quad \gamma_4 =: \frac{\left(1 \mp \alpha^{m-n}\right)}{\sqrt{5}}, \quad b_1 := -n, \quad b_2 := s, \quad b_3 := b, \quad b_4 := -1.$$

If $\Lambda_2 = 0$, then

$$\frac{\alpha^n}{\sqrt{5}} \mp \frac{\alpha^m}{\sqrt{5}} = 3^s y^b.$$

From (23), we get $\frac{\beta^n}{\sqrt{5}} \mp \frac{\beta^m}{\sqrt{5}} = 0$, which implies that $\beta^{n-m} = \mp 1$. But this is impossible since $n > m$. Therefore $\Lambda_2 \neq 0$. Besides, it is clear that γ_1, γ_2, γ_3 and γ_4 are positive real numbers and elements of the field of $\mathbb{K} = \mathbb{Q}\left(\sqrt{5}\right)$. The degree of the field $\mathbb{K} = \mathbb{Q}\left(\sqrt{5}\right)$ is 2. $\Lambda_2 \neq 0$. Also, a similar argument to the above shows that we can take $B := n$, $A_1 := \log \alpha$, $A_2 := \log 9$ and $A_3 := 2 \log y$. Also, since

$$\begin{aligned} h(\gamma_4) = h\left(\frac{1 \mp \alpha^{m-n}}{\sqrt{5}} \right) &\leq h\left(1 \mp \alpha^{m-n}\right) + h\left(\sqrt{5}\right) \\ &\leq h(1) + h\left(\alpha^{m-n}\right) + \log 2 + h\left(\sqrt{5}\right) \\ &\leq \frac{|m-n| \log \alpha}{2} + \log 2 + \log \sqrt{5} \\ &< \frac{(n-m) \log \alpha}{2} + \log 5 \end{aligned}$$

by (11)–(14), we can take $A_4 := (n-m) \cdot \log(25\alpha)$. Thus, applying Theorem 7 to inequality (25), we get

$$\frac{5.24}{\alpha^{m+n}} > |\Lambda_2| > exp\left(-C_2 \cdot (1 + \log n) \cdot \log \alpha \cdot \log 9 \cdot 2 \cdot \log y \cdot (n-m) \cdot \log(25\alpha)\right),$$

where $C_2 = 1.4 \cdot 30^7 \cdot 4^{4.5} \cdot 2^2 \cdot (1 + \log 2)$. Taking logarithms in the above inequality and using the fact that $1 + \log n < 2 \log n$ for $n \geq 3$, we obtain

$$(n + m) \log \alpha < 1.662 \cdot 10^{15} \cdot (n - m) \cdot \log n \cdot \log y. \tag{26}$$

If we add $(n - m) \log \alpha$ to both sides of (26), then we get

$$2n \log \alpha < (n - m) \log \alpha + 1.662 \cdot 10^{15} \cdot (n - m) \cdot \log n \cdot \log y,$$

which implies that

$$n < 1.73 \cdot 10^{15} \cdot (n - m) \cdot \log n \cdot \log y. \tag{27}$$

Considering inequality (27) together with (22), it can be seen that

$$n < 2.62 \cdot 10^{30} \cdot (\log n)^2 \cdot (\log y)^2. \tag{28}$$

Put $T = 2.62 \cdot 10^{30} \cdot (\log y)^2$ in Lemma 10. In this case, we obtain from (28) that $\frac{n}{(\log n)^2} < T$, which implies by Lemma 10 that

$$\begin{aligned} n &< 2^2 \cdot 2.62 \cdot 10^{30} \cdot (\log y)^2 \cdot \left[\log\left(2.62 \cdot 10^{30} \cdot (\log y)^2\right)\right]^2 \\ &< 1.05 \cdot 10^{31} \cdot (\log y)^2 \cdot \left[71 + \log\left((\log y)^2\right)\right]^2 \\ &< 1.05 \cdot 10^{31} \cdot (\log y)^2 \cdot \left[103 \cdot \log y + \log y\right]^2 \\ &< 1.05 \cdot 10^{31} \cdot 104^2 \cdot (\log y)^4 \\ &< 1.14 \cdot 10^{35} \cdot (\log y)^4, \end{aligned}$$

where we used the fact that $71 < 103 \cdot \log y$ and $\log\left((\log y)^2\right) < \log y$ for $y \geq 2$. This completes the proof. □

Now, in the following theorem, we will give an application of Theorem 11.

Theorem 12 *Let $s \geq 0$, $0 \leq m < n$, $2 \leq y \leq 10^4$ and $(y, 3) = 1$. All solutions (n, m, s, y) of the Diophantine equation*

$$F_n \mp F_m = 3^s \cdot y^2 \tag{29}$$

are given by

$$\left\{ \begin{array}{c} (12, 0, 2, 4), (4, 1, 0, 2), (5, 1, 0, 2), (7, 1, 1, 2), (4, 2, 0, 2), \\ (5, 2, 0, 2), (7, 2, 1, 2), (11, 10, 2, 4), (14, 13, 2, 4), (13, 11, 2, 4), \\ (7, 4, 0, 4), (9, 3, 2, 2), (12, 4, 1, 7), (14, 10, 3, 4,), (17, 4, 0, 40), \\ (8, 5, 0, 4), (13, 6, 2, 5), (15, 9, 2, 8) \end{array} \right\}.$$

Proof Assume that (n, m, s, y) is a solution of equation (29). It is obvious that $n \neq 0, 1, 2$. Besides, if $m = 0$, then we have $F_n = 3^s \cdot y^2$. This equation has solution only for $n = 12$ by Theorem 4. Thus, $(n, m, s, y) = (12, 0, 2, 4)$. If $m = 1$ or $m = 2$, then we get $F_n \mp 1 = 3^s y^2$. In this case, we can see by Theorem 6 that

$$(n, m, s, y) \in \{(4, 1, 0, 2), (5, 1, 0, 2), (7, 1, 1, 2), (4, 2, 0, 2), (5, 2, 0, 2), (7, 2, 1, 2)\}.$$

When $n - m = 1$, we obtain $F_{n+1} = 3^s y^2$ or $F_{n-2} = 3^s y^2$. These equations show by Theorem 4 that $(n, m, s, y) = (11, 10, 2, 4)$ and $(n, m, s, y) = (14, 13, 2, 4)$, respectively. When $n - m = 2$, we obtain $L_{n-1} = 3^s y^2$ or $F_{n-1} = 3^s y^2$, which implies by Theorem 4 that $(n, m, s, y) = (4, 2, 0, 2)$ and $(n, m, s, y) = (13, 11, 2, 4)$, respectively. From now on, assume that $n > m \geq 3$ and $n - m \geq 3$.

On the other hand, since (n, m, s, y) is a solution of equation (29), inequalities (16) and (17) hold. Taking $b = 2$ and $2 \leq y \leq 10^4$ in these inequalities, we get

$$s + 2 < n \tag{30}$$

and

$$n < 1.14 \cdot 10^{35} \cdot (\log y)^4 < 1.14 \cdot 10^{35} \cdot \left(\log 10^4\right)^4 < 8.21 \cdot 10^{38}. \tag{31}$$

Now, let reduce this upper bound on n. For this, let

$$z_1 = s\log 3 - n\log\alpha + \log\left(y^2\sqrt{5}\right)$$

and $\Lambda_1 = 1 - e^{z_1}$. Then, from (20), we get

$$|\Lambda_1| = \left|e^{z_1} - 1\right| < \frac{2.4}{\alpha^{n-m}} < \frac{2.4}{\alpha^3} < 0.6$$

for $n - m \geq 3$. Taking $a = 0.6$ in Lemma 9, it is seen that

$$|z_1| = |\log(\Lambda_1 + 1)| < \frac{-\log(1-0.6)}{0.6} \cdot |\Lambda_1| < \frac{3.67}{\alpha^{n-m}},$$

i.e.,

$$\left|s\log 3 - n\log\alpha + \log\left(y^2\sqrt{5}\right)\right| < \frac{3.67}{\alpha^{n-m}}.$$

Dividing this inequality by $\log\alpha$, we get

$$|s\gamma - n + \mu| < A \cdot B^{-w}, \tag{32}$$

where

$$\gamma = \frac{\log 3}{\log\alpha}, \quad \mu = \frac{\log\left(y^2\sqrt{5}\right)}{\log\alpha}, \quad A = 7.63, \quad B = \alpha \quad \text{and} \quad w = n - m.$$

It can be easily seen that $\dfrac{\log 3}{\log\alpha}$ is irrational. If we take $M := 8.21 \cdot 10^{38}$, which is an upper bound on s since $s < n < 8.21 \cdot 10^{38}$ by (30), we find that

$$q_{73} = 259756903407721281290491983941452615117 94,$$

the denominator of the 73 th convergent of γ exceeds $6M$. Also, a quick computation with Mathematica gives the inequality

$$0.00012626 < \varepsilon(y) = \|\mu q_{73}\| - M\|\gamma q_{73}\|.$$

Hence, Lemma 8 says to us that if inequality (32) has a solution, then

$$w = n - m \leq \frac{\log\left(\frac{7.63\cdot q_{73}}{\varepsilon(y)}\right)}{\log\alpha} < 216.3.$$

Substituting this bound of $n - m$ into (27), we get

$$n < 3.45 \cdot 10^{18} \cdot \log n,$$

which implies that $n < 1.61 \cdot 10^{20}$. Now, to further reduce this bound on n, we apply Lemma 8. Then, let

$$z_2 = s\log 3 - n\log\alpha + \log\left(y^2\sqrt{5}\left(1 \mp \alpha^{m-n}\right)^{-1}\right)$$

and $\Lambda_2 = 1 - e^{z_2}$. Thus, from (24), it is seen that

$$|\Lambda_2| = \left|e^{z_1} - 1\right| < \frac{5.24}{\alpha^{n+m}} < 0.2$$

since $n + m \geq 7$. Choosing $a = 0.2$ in Lemma 9, we obtain

$$|z_2| = |\log(\Lambda_2 + 1)| < \frac{-\log(1-0.2)}{0.2} \cdot |\Lambda_2| < \frac{\log\left(\frac{10}{8}\right)}{0.2} \cdot \frac{5.24}{\alpha^{n+m}} < \frac{1.4}{\alpha^n},$$

that is,

$$\left| s\log 3 - n\log\alpha + \log\left(y^2\sqrt{5}\left(1 \mp \alpha^{m-n}\right)^{-1}\right) \right| < \frac{1.4}{\alpha^n}.$$

Dividing this inequality by $\log\alpha$, we get

$$|s\gamma - n + \mu| < A \cdot B^{-w}, \tag{33}$$

where

$$\gamma = \frac{\log 3}{\log\alpha}, \quad \mu = \frac{\log\left(y^2\sqrt{5}\left(1 \mp \alpha^{m-n}\right)^{-1}\right)}{\log\alpha}, \quad A = 2.91, \quad B = \alpha \quad \text{and} \quad w = n.$$

Taking $M := 1.61 \cdot 10^{20}$, which is an upper bound on s since $s < n < 1.61 \cdot 10^{20}$ by (30), we find that

$$q_{54} = 3457280841970866815534351 87,$$

the denominator of the 54 th convergent of γ exceeds $6M$. Also, a quick computation with Mathematica gives that the inequality

$$0 < 3.99 \cdot 10^{-7} < \varepsilon(y, n-m) = \|\mu q_{54}\| - M\|\gamma q_{54}\|$$

holds for all $2 \le y \le 10^4$ and $3 \le n-m \le 216$ except for the case $\varepsilon(4, 24)$, which is always negative. Then, let $n - m \ne 24$ and $y \ne 4$. As similar to the above case, Lemma 8 says to us that if inequality (33) has a solution, then

$$w = n \le \frac{\log\left(\frac{2.91 \cdot q_{54}}{\varepsilon(y, n-m)}\right)}{\log\alpha} < 160.$$

Thus, we have $3 \le m < n \le 159$ and $n - m \ge 3$. A simple computation with Mathematica shows that Eq. (29) for $3 \le m < n \le 159$ and $n - m \ge 3$ has the solutions

$$(n, m, s, y) = (7, 4, 0, 4), (9, 3, 2, 2), (12, 4, 1, 7), (14, 10, 3, 4), (17, 4, 0, 40), (36, 12, 2, 1288)$$

and

$$(n, m, s, y) = (8, 5, 0, 4), (13, 6, 2, 5), (15, 9, 2, 8).$$

Lastly, if $n - m = 24$ and $y = 4$, then from Eq. (29) we get

$$F_{m+24} - F_m = L_{m+12}F_{12} = 16 \cdot 3^s \tag{34}$$

and

$$F_{m+24} + F_m = F_{m+12}L_{12} = 16 \cdot 3^s \tag{35}$$

by (1) and (2). Equation (35) has no solution since $16 \nmid L_{12} = 322$. Equation (34) implies that $L_{m+12} = 3^{s-2}$. But, this equation has no solution by Theorem 2 since $m \ge 3$. Thus, the proof is completed. □

Declarations

Conflict of interest The authors declare no competing interests.

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