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ORIGINAL RESEARCH

A linearization technique for solving some quantum optimal control problems governed by the Schrödinger equation

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Abstract In this paper, we attempted to synthesize a nonlinear optimal control procedure for the quantum equation. To this end, to analyze the quantum optimal control system derived from the time-dependent Schrödinger equation, we developed a linearization method. Using a new approach based on the embedding process, we first formulated the system in variational form. Then, by defining a positive Radon measure, we represented the problem in a space of measures. Consequently, the problem was transformed into an infinite linear one, whose solution is guaranteed. At this stage, by applying two subsequent approximation steps, the optimal solution was identified through an optimization search technique. This method effectively transforms a potentially complex, nonlinear problem into a linear one by utilizing mathematical tools and approximations. It simplifies the problem-solving process, making it more tractable and amenable to computational methods. The steps involved mathematical abstraction, transformation into a suitable space, expansion to ensure solvability, reduction to a finite-dimensional problem, and finally, determination and visualization of the optimal solutions.

Keywords Embedding process · Nonlinear optimal control problems · Quantum equations · Radon measure · Schrödinger equation · Search technique

Mathematics Subject Classification 49M25 · 35J10

1 Introduction

In recent decades, the fields of Nonlinear Physics and Mathematics have seen remarkable growth, driven not only by technological advancements and computational tools but also by extensive scientific endeavors and theoretical breakthroughs [13]. Recent studies, particularly those focusing on quantum control and water wave modeling, have made significant strides in comprehending and managing intricate physical systems [2,8]. Investigations led by researchers have illuminated novel methodologies for addressing optimal control challenges and enhancing the prediction accuracy of water wave dynamics through the application of symbolic computations [6,7]. Those include the introduction of auto-Bäcklund transformations and hetero-Bäcklund transformations, which offer valuable insights into the analysis and control of complex physical systems [8,10].

Furthermore, the inherent nonlinearity present in many models associated with quantum control and water wave modeling underscores the importance of developing effective problem-solving strategies tailored to these nonlinear models. Small alterations in control mechanisms within quantum systems can yield substantial and nonlinear impacts on the system's state, emphasizing the critical need for precise and reliable solutions. Similarly,

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the nonlinearity of water wave models can give rise to complex and previously unexplored phenomena in wave behavior, necessitating the application of suitable problem-solving techniques [6,7,9].

As we delve deeper into the exploration of these concepts, it becomes evident that the foundation of our understanding lies in the principles of quantum theory, which have dominated the discourse on the physics of the microworld since the early 20th century [18]. Following the establishment of quantum physics, the discipline of quantum control has evolved significantly, influenced by advancements across various scientific domains, including quantum chemistry, quantum information, quantum optics, and atomic and molecular physics [2,3,17,23,25]. Classical control methods, such as feedback control, Lyapunov control, optimal control, and robust control, have been extensively examined, revealing their pivotal role in facilitating the transition of quantum systems from one state to another using external control fields [1,12,14–16,19]. However, further research is needed in the area of control within quantum systems. Specifically, optimal control in quantum systems is crucial due to its widespread application. Additionally to analytical control methods in quantum systems, numerical methods have also been developed.

Let $\mathcal{H}$ denote a finite- or infinite-dimensional Hilbert space of a quantum system, and let Ψ represent the state of this system. The Schrödinger equation can be expressed as follows [11,24]:

$$ih\frac{\partial \Psi}{\partial t} = H\Psi, \tag{1}$$

where $\Psi \in \mathcal{H}$ is the state variable, h is the Planck constant, and H is a self-adjoint Hamiltonian operator in $\mathcal{H}$.

The Schrödinger equation in (1) is changed as

$$ih\frac{\partial \Psi}{\partial t} = \left(H_0 + \sum_k H_k u_k(t)\right)\Psi,$$

When an external field acts on a quantum system, where $H_0 : \mathcal{H} \mapsto \mathcal{H}$, which is the internal Hamiltonian, and the Hamiltonian linear operator $H_k : \mathcal{H} \mapsto \mathcal{H}$ represents the coupling of the system to external fields $u_k(t)$. In this study, to simplify it, we consider a quantum system with one control, and set $k = 1$. Then, we can write:

$$ih\frac{\partial \Psi}{\partial t} = (H_0 + H_1 u(t))\,\Psi.$$

The state equation $ih\dot{\Psi} = (H_0 + H_1 u(t))\Psi$ describes the time evolution of the quantum system's state vector Ψ; Here:

- i is the imaginary unit, indicating that the evolution is governed by the Schrödinger equation.
- $\dot{\Psi}$ denotes the derivative of Ψ with respect to time, showing how the state changes over time.
- H_0 represents the free Hamiltonian, which accounts for the inherent dynamics of the system without any external control.
- H_1 is the control Hamiltonian, which couples the system to the control input $u(t)$.
- $u(t)$ is the control function, which can be manipulated to influence the system's evolution.

In quantum systems, the design of controls is considered for energy-efficient population transfer. These controls operate the desired transmission and optimize a specific performance index. This specific energy performance index is displayed below:

$$\min J[u] = \int_0^T u^2(t)dt.$$

Previous research suggests that this cost function, used for control, has been widely utilized in the optimal control of quantum systems as part of various objective functions. It serves as a measure of the energy used to construct a control field. The main objective of this paper is to present an efficient linearization algorithm to solve the following optimal control problem [18]:

$$\begin{aligned} \min \quad & J[u] = \textstyle\int_0^T u^2(t)dt \\ S.\,to: \quad & i\dot{\Psi} = (H_0 + H_1 u(t))\,\Psi; \\ & \Psi(0) = \psi_0. \end{aligned} \tag{2}$$

The system described by the equations (2) is a mathematical formulation of a quantum control problem. The condition $\Psi(0) = \psi_0$ specifies the initial state of the system at time $t = 0$, where ψ_0 is the initial state vector.

The relationship $i\dot{\Psi} = (H_0 + H_1 u(t))\Psi$ indicates that changes in the quantum state Ψ occur nonlinearly relative to the control $u(t)$. This implies that small changes in control can have significant and nonlinear effects on the system. The objective function $J[u]$ is also nonlinear because the consumed energy $u^2(t)$ is directly proportional to the square of the control. This leads the optimization problem toward finding combinations of control that have the greatest effect on the system. Linearizing the problem can simplify the solution process, but it must be done with high accuracy to obtain precise and reliable results.

The embedding method is one of the newer techniques used in solving optimal control problems in recent years. Utilizing Radon measures for solving optimal control problems, based on the idea of L.C. Young, was first applied in [22], and the method was theoretically established by Rubio [20]. Wilson and Rubio used the embedding method for the first time to solve a control problem defined based on the diffusion equation in 1977, [20]. In 1986, Rubio introduced a new embedding process for solving optimal control problems [4]. This method is based on embedding the solution space of the optimal control problem into an appropriate measure space, thereby enabling the use of the strong linearity property of measures to obtain the optimal solution. Compared to other methods, this approach offers several key advantages, such as guaranteeing a solution, transforming the problem into a new linear one, and facilitating the discovery of a global solution. For those interested in delving deeper into the subject matter, reference [4] is recommended for further reading.

The novel method we have introduced has been employed for the first time in solving nonlinear quantum problems. Based on solving a linear programming problem, this method provides a straightforward computational approach to uncovering the unknowns of the problem. This approach not only simplifies the solution process but also enables obtaining accurate and reliable results.

In general, we can define the main steps of this method as the following:

Step 1: Define the classical problem as integral equations, Initially, problem (2) is formulated in terms of integral equations. This step involves setting up the mathematical model of the problem, which might include differential equations, integral equations, or other mathematical representations that describe the system's behavior.

Step 2: Transfer the problem into measure space, The next step involves converting the problem into a form that uses positive Radon measures. Defining positive Radon measures allows us to represent the problem in a space of measures, which can offer a different perspective and potentially simplify the problem.

Step 3: Expand the space to prevent issues, Expanding the space is often necessary to avoid issues such as the lack of solutions or the presence of singularities. This could involve adding constraints or increasing the dimensionality of the problem to ensure that there are enough degrees of freedom to find a solution.

Step 4: Change dimension and number of equations into finite, Through two approximation steps, the problem is transformed so that the dimension and the number of equations become finite. This is crucial because most computational methods require a finite-dimensional representation of the problem. Approximation techniques, such as discretization, are used here to reduce the complexity of the problem.

Step 5: Determine and plot path and optimal control functions, Finally, after solving the linear programming problem, the optimal control functions and paths are determined. These functions represent the best possible controls for the system under consideration. Visualizing these paths and functions can provide insights into the system's behavior and the effectiveness of the control strategies.

2 Displaying the problem in variational form

As it is mentioned, quantum optimal control is defined as follow [18]:

$$\begin{array}{ll} \min & J[u] = \int_0^T u^2(t)dt \\ S.\,to: & i\dot{\Psi} = (H_0 + H_1 u(t))\,\Psi; \\ & \Psi(0) = \psi_0. \end{array} \tag{3}$$

By multiplying the constraint (3) in imaginary number i, we have:

$$\dot{\Psi} = -i\,(H_0 + H_1 u(t))\,\Psi.$$

Let t be a real variable. Consider:

1. A real closed interval $J = [t_a, t_b]$ with $t_a < t_b$. The interior of this interval will be denoted by $J^0 = (t_a, t_b)$.

2. A bounded, closed, pathwise-connected set A in $\mathcal{H}$. The trajectory of the controlled system is constrained to stay in this set for $t \in J$.
3. A bounded, closed subset U of $\mathbb{R}^m$. This is the set in which the control functions are to take values.
4. Let $\Omega = J \times A \times U$ and consider the differential equation

$$\dot{\Psi} = g(t, \Psi(t), u(t)), \qquad t \in J.$$

where the trajectory function $\Psi(t) \in A$ is absolutely continuous and the control function $u(t) \in U$ is Lebesgue-measurable.

We shall say that a trajectory-control pair $p = (\Psi(.), u(.))$ is admissible if the following conditions hold:

1. The trajectory function $\Psi(.)$ satisfies $\Psi(t) \in A, \quad t \in J$, and is absolutely continuous on J, furthermore, $\Psi(0) = \psi_0$.
2. The control function $u(.)$ takes values in the set U, and is Lebesgue measurable on J.

We denote by W the set of admissible pairs. A classical control problem does not have a solution unless this set is nonempty. It is desired to minimize functional (3) over the set W. Let B is an open ball consist of $J \times A$ and we denote by $C'(B)$ the space of continuously differentiable functions on B such that they and their first derivatives are bounded on B. Let $\phi \in C'(B)$, and define

$$\phi^g(t, \Psi, u) = \phi_\Psi(t, \Psi)g(t, \Psi, u) + \phi_t(t, \Psi); \tag{4}$$

After integrating the above relation, we have:

$$\begin{aligned} &\int_0^T \phi^g\,(t, \Psi, u)\,dt = \int_0^T \left\{\phi_\Psi(t, \Psi)\dot{\Psi} + \phi_t(t, \Psi)\right\} dt \\ &\int_0^T \dot{\phi}[t, \Psi(t)]dt = \phi(T, \Psi(T)) - \phi(0, \Psi(0)) = \triangle\phi; \end{aligned}$$

$\triangle\phi$ is unknown, because $\Psi(T)$ is unknown. In the following parts it is demonstrated how to calculate $\triangle\phi$.

let $\mathcal{D}(J^0)$ be the space of infinitely differentiable real-valued functions with compact support in J^0. If $\eta \in \mathcal{D}(J^0)$, we have:

$$\begin{aligned} &\int_0^T \eta\,(t, \Psi, u)\,dt = \int_0^T \left\{\Psi_j(t)\eta^{'}(t)\right\} dt + \int_0^T g_j\,(t, \Psi, u)\,dt \\ &\Psi_j\eta(t)|_0^T + \int_0^T \left\{\dot{\Psi}_j(t) - g_j\,(t, \Psi, u)\right\}\eta(t)dt = 0; \end{aligned}$$

since the trajectory and control functions in an admissible pair satisfy (3) and since function η has compact support in J^0, $\eta(0) = \eta(T) = 0$.

This equality is not independent of (5); it can be derived from it if we choose the function ϕ of the form

$$\phi(t, \Psi, u) = \Psi_j\eta(t), \quad (t, \Psi, u) \in \Omega, \quad j = 1, 2, ..., n.$$

Now, we select some other desirable $C^{'}(B)$ functions which are only dependent on variable t; therefore, we call $C_1(\Omega) \subseteq C(\Omega)$, the set of functions only dependent on the variable t. We do that to know, such functions are designated and to use them for identifying optimal control in the next parts, so for $f(t) \in C_1(\Omega)$ we have:

$$\int_0^T f(t, \Psi, u)dt = a_f, \qquad f \in C_1(\Omega);$$

where a_f, is the integral of $f(., \Psi, u)$ over J, independent of Ψ and u.

3 Transferring the problem to measure space

Generally, either it is hard to find an optimal pair for $p = (\Psi, u)$ or numerical calculation is impossible. It seems that these problems can be solved if we expand set W. This change is based on the fact that each acceptable pair could be calculated as one positive Radon measure, so that there is a one-to-one correspondence between the acceptable pairs and a set of positive Radon measures. Let $F \in C(\Omega)$ and consider the mapping

$$\Lambda_p : F \in C(\Omega) \to \int_0^T F[t, \Psi(t), u(t)]dt;$$

The following aspects of this mapping are of interest:

1. It is well defined, because $F \in C(\Omega)$ and as a result there is an integral.
2. It is linear; i.e.
$$\Lambda_p(\alpha F + \beta G) = \alpha \Lambda_p(F) + \beta \Lambda_p(G).$$
3. It is positive; that is, if $F \geq 0$, then $\Lambda_p(F) \geq 0$.
4. It is continuous [20]; i. e.
$$\left\| \Lambda_p(F) \right\| \leq k \sup_{\Omega} \| F(t, \Psi, u) \| .$$

Therefore, problem (3) can be demonstrated as an equivalent, in terms of functional Λ_p for each $p \in W$:

$$\begin{aligned} \min \quad & J(p) = \Lambda_p(u^2) \\ S.\,to: \quad & \Lambda_p(\phi^g) = \triangle\phi; \\ & \Lambda_p(\eta) = 0; \\ & \Lambda_p(f) = a_f. \end{aligned} \tag{5}$$

In fact, (5) is problem (3) with its constraints defined as integrals and therefore they are equivalent.

According to Riesz representation theorem [21], there is a specific measure μ_p on $C(\Omega)$ for each $p \in W$, equivalent linear, bounded and positive functional Λ_p so that for every desirable function $F \in C(\Omega)$ we have:

$$\Lambda_p = \int_{\Omega} F d\mu_p \equiv \mu_p(F);$$

Now, we consider the wider set of all positive Radon measures which apply to the constraints of (5), instead of measures μ_p that apply to Riesz representation theorem, and call it Q. Therefore, we define problem (6) as the following:

$$\begin{aligned} \inf_{\mu \in Q} \quad & J(\mu) = \mu(u^2) \\ S.\,to: \quad & \mu(\phi^g) = \triangle\phi; \\ & \mu(\eta) = 0; \\ & \mu(f) = a_f. \end{aligned} \tag{6}$$

Using the weak* topology properties, it is proven that Q is compact in this space and the functional $J : Q \to \mathbb{R}$ which is defined in problem (6) is continuous [20]; this means optimal control problem (6) definitely has a solution. But it is not easy to determine this solution exactly; because, the space of result and also the number of constraints, are infinite. Therefore, first we select the number of constraints with proper approximation, and then using discretization we select the number of variables to become finite. This approximated solution will get more and more accurate once the conditions are increased and the discretization (number of variables) is made more exact.

3.1 First step of approximation

First, we consider the minimization of (6) not over the set Q but over a subset of it defined by requiring that only a finite number of constraints (6) be satisfied.

a) We select functions ϕ_i be such that the linear combinations of these functions are uniformly dense (i.e. they are dense in the topology of uniform convergence) in space $C'(B)$, these functions can be taken to be monomials in the components of variables t and Ψ.

1. For the next two sets of equations, we select: $\eta_r = \sin(\frac{2\pi r t}{T}),\ r = 1, 2, ..., M_{21};\ \ \eta_{r'} = 1 - \cos(\frac{2\pi r' t}{T}),\ r' = 1, 2, ..., M_{22}$ such that $M_{21} + M_{22} = M_2$. These functions are useful as bases for Fourier series, which are usually utilized in analyzing physics and engineering problems. On the other hand, because of the density of the linear combinations of these functions in different mathematical spaces, it is possible to approximate the functions in these spaces using a linear combination of these triangular functions.
2. In the third group of constraints, we select the functions as the following specific functions:
$$f_s(t) = \begin{cases} 1 & t \in J_s \\ 0 & t \notin J_s \end{cases}$$
where $J_s = [\frac{T(s-1)}{M_3}, \frac{sT}{M_3}),\ s = 1, 2, ..., M_3$. Although the functions in the last set are not continuous, the linear combinations of them can efficiently and properly approximate any functions in $C(\Omega)$ properly.

Theorem 3.1 *For integer and positive numbers M_1, M_2 and M_3, we consider minimization problem $\mu \mapsto \mu(u^2)$, so that $Q(M_1, M_2, M_3)$ is a subset of Q which satisfies the following conditions:*

$$\begin{array}{ll} \mu(\phi_i^g) = \triangle\phi_i, & i = 1, 2, ..., M_1; \\ \mu(\eta_j) = 0, & j = 1, 2, ..., M_2; \\ \mu(f_s) = a_s, & s = 1, 2, ..., M_3. \end{array} \tag{7}$$

then, $\inf\limits_{Q(M_1,M_2,M_3)} \mu(u^2)$ *tends to* $\inf\limits_{Q} \mu(u^2)$ *when* $M_1, M_2, M_3 \longrightarrow \infty$.

Proof The same as proof of proposition $III.1$ from reference [20]. □

3.2 Second step of approximation

We will attempt to determine the optimal solution through solving a finite linear programming problem, by making the number of variables to be finite in second approximation step.

Theorem 3.2 *Optimal measure μ^* in $Q(M_1, M_2, M_3)$ has the form $\mu^* = \sum_{j=1}^{M_1+M_2+M_3} \alpha_j^* \delta(z_j^*)$ with the triples z_j^* belonging to a dense subset of Ω and the coefficients $\alpha_j^* \geq 0$ for $j = 1, 2, \cdots, M_1 + M_2 + M_3$, [20].*

$\delta(t)$ is unit measure with the support of one point set $\{t\}$. In this case, the unknowns are coefficients α_j and supports z_j and this problem is non-linear. Now, a discretization by nodes on a dense subset of Ω can reduce the number of unknowns to only coefficients α_j^*'s. Thus, we will have the following linear programming problem:

$$\begin{array}{lll} \min: & \sum_{j=1}^{N} \alpha_j {u_j^*}^2 & \\ S.to: & \sum_{j=1}^{N} \alpha_j (\phi_i^g)(z_j) = \triangle\phi_i, & i = 1, 2, ..., M_1; \\ & \sum_{j=1}^{N} \alpha_j (\eta_k)(z_j) = 0, & k = 1, 2, ..., M_2; \\ & \sum_{j=1}^{N} \alpha_j f_s(z_j) = a_s, & s = 1, 2, ..., M_3; \\ & \alpha_j \geq 0, & j = 1, 2, ..., N; \end{array} \tag{8}$$

where $a_s = \int_{J_s} dx = \frac{T}{M_3}$ that $J_s = [\frac{T(s-1)}{M_3}, \frac{Ts}{M_3})$ and $\bigcup J_s = [0, T]$.

4 Determination of control function and optimal path

The nearly optimal control function u and path Ψ which are result of the above method can be determined by the following algorithm:

Step 1: Defining region Ω: In this problem, $\Omega = [0, T] \times A \times U$, where $[0, T]$ is interval for t, A interval for $\Psi(t)$ and U interval for $u(t)$. Choosing a number N, region Ω is divided into N equal parts, so that we obtain a partitioning for region Ω which includes N cells. Next, the constituting intervals of Ω, are divided into n_1, n_2 and n_3 equal parts respectively; in this manner, region Ω will have $N = n_1 \times n_2 \times n_3$ cells. Choosing $z_j = (t_j, \Psi_j, u_j)$ from each cell as a representative.

Step 2: Therefore, for given fixed numbers M_1, M_2 and M_3, we select M_1 number of ϕ_i, M_2 of η_j, and M_3 number of f_s functions from the mentioned total sets. Now we have programming problem (8) with N variables and M constraints which is dependent on $\Psi(T)$.

Step 3: We use a repeated method to solve problem (8). Function $J_1 : \mathbb{R} \to \mathbb{R}$ is defined as $J_1(\Psi(T)) = J^*(\alpha, \Psi(T))$ by assuming a fixed amount for $\Psi(T)$. Then, this function will be optimized by one of the standard optimization algorithms (such as linear search method) [19]. Each time that this function is calculated, linear programming problem (8) with the value $\Psi(T)$ defined by this function, must be solved.

Step 4: Upon finding $\alpha_1^*, \alpha_2^*, ..., \alpha_m^*$, we obtain control function and optimal path in this manner:

a) Assuming $t_0 = 0$, we set $t_i = t_{i-1} + \alpha_i^*$, $i = 1, 2, ..., m$.

b) For $t \in [t_{i-1}, t_i)$ we set $u(t) = u_i$, where u_i is a component of z_i (which is related to α_i^*)

c) The following difference equation is reached using differential equation $\dot{\Psi} = g\,(t, \Psi, u)$:

$$\Psi_i = \Psi_{i-1} + (-i\,(H_0 + H_1 u(t)))\,(t_i - t_{i-1})\Psi$$

Using the resulting information from the previous step, (t_i, Ψ_i, u_i) are defined.

5 Numerical example

We will provide numerical examples with comparison of results in order to show the efficiency of the given method in actual use. It is worth mentioning that these examples are taken from [18] as well as from other studies cited by it in order for the readers to be able to compare and contrast the two methods.

Example 5.1 We consider the two-level system $i\dot{\Psi} = (H_0 + H_1 u(t))\,\Psi$, where $\Psi \in \mathbb{C}^2$ as follows:

$$H_0 = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \qquad H_1 = \begin{bmatrix} 0 & v_{12} \\ v_{12}^* & 0 \end{bmatrix}$$

If we suppose $E_1 = 2$, $E_2 = -2$ and $v_{12} = 2 + 3i$, then, by defining $\Omega = (0, 30) \times (-2, 2) \times (-0.1, 0.1)$, $\Psi(T)$ was determined optimally in domain Ω as mentioned in step 3. Then, by selecting the following functions and setting them in (8) for $M_1 = 4$, $M_2 = 6$ and $M_3 = 5$, we set up the corresponding LP with:

$$\phi_1^g(t, \Psi, u) = 2t\Psi^2;\quad \phi_2^g(t, \Psi, u) = t\Psi;\quad \phi_3^g(t, \Psi, u) = t^2\Psi;\quad \phi_4^g(t, \Psi, u) = t^2\Psi^2;$$

$$\eta_r(t) = sin(\frac{2\pi r t}{30}),\ r = 1, 2, 3;\qquad \eta_{r'}(t) = (1 - \cos(\frac{2\pi r' t}{30})),\ r' = 1, 2, 3;$$

$$J_s = [\frac{30(s-1)}{5}, \frac{30s}{5}];\qquad a_s = \int_{J_s} d\theta = 6,\ s = 1, 2, ..., 5.$$

By dividing each of intervals [0, 30] and [−2, 2] into ten and $U = [-0.1, 0.1]$ into ten equal parts, we selected $N = 10^3$ points of z_i; thus, LP (8) was set up with N variables and 15 constraints. We obtained the nearly optimal control (see Figure 1) and state variables (see Figure 2 and 3)Figure4,56and 7.

Example 5.2 In this example, we consider the three-level system $i\dot{\Psi} = (H_0 + H_1 u(t))\,\Psi$, where $\Psi \in \mathbb{C}^3$ as follows:

$$H_0 = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix} \qquad H_1 = \begin{bmatrix} 0 & v_{12} & v_{13} \\ v_{12}^* & 0 & v_{23} \\ v_{13}^* & v_{23}^* & 0 \end{bmatrix}$$

We suppose $E_1 = 2$, $E_2 = 0$, $E_3 = 6$, $v_{12} = i + 1$, $v_{13} = 4$ and $v_{23} = 2 + 3i$,then we can obtain the numerical results as follows:

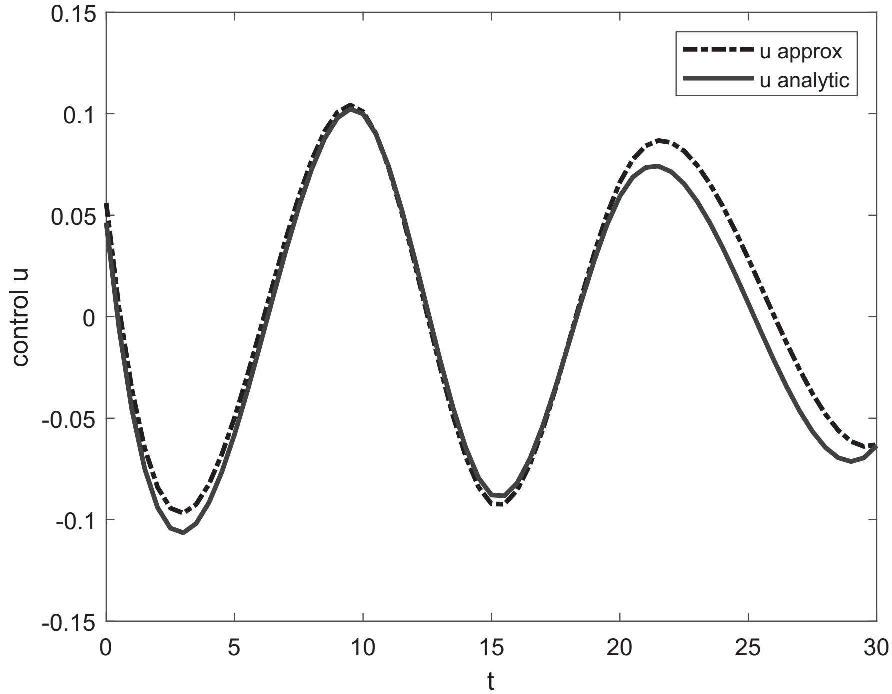

Fig. 1 Approximate and analytic control variable in Example 1

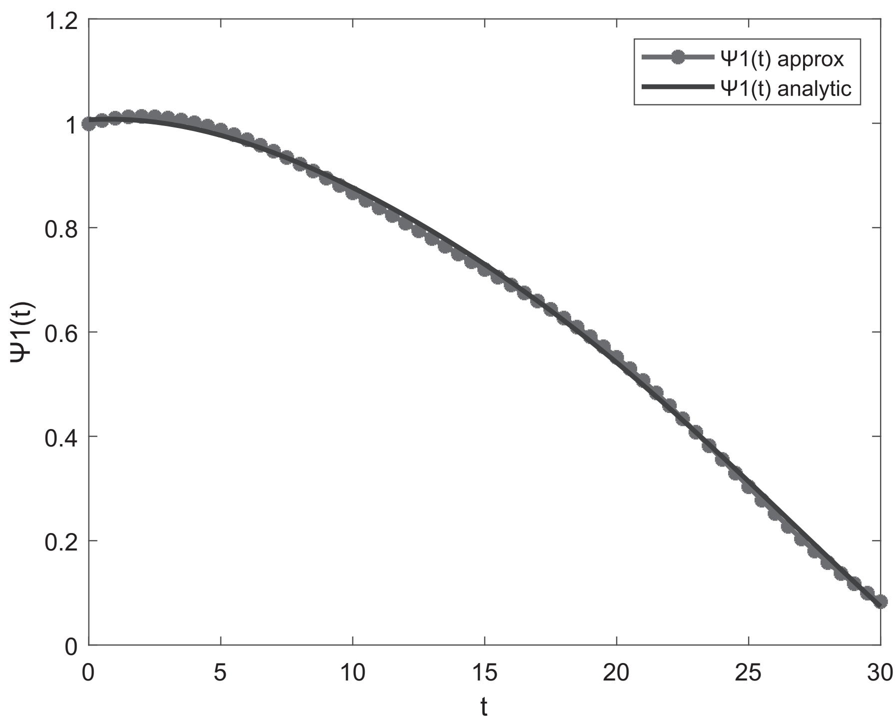

Fig. 2 Optimal state variable $\Psi_1(t)$ in Example 1

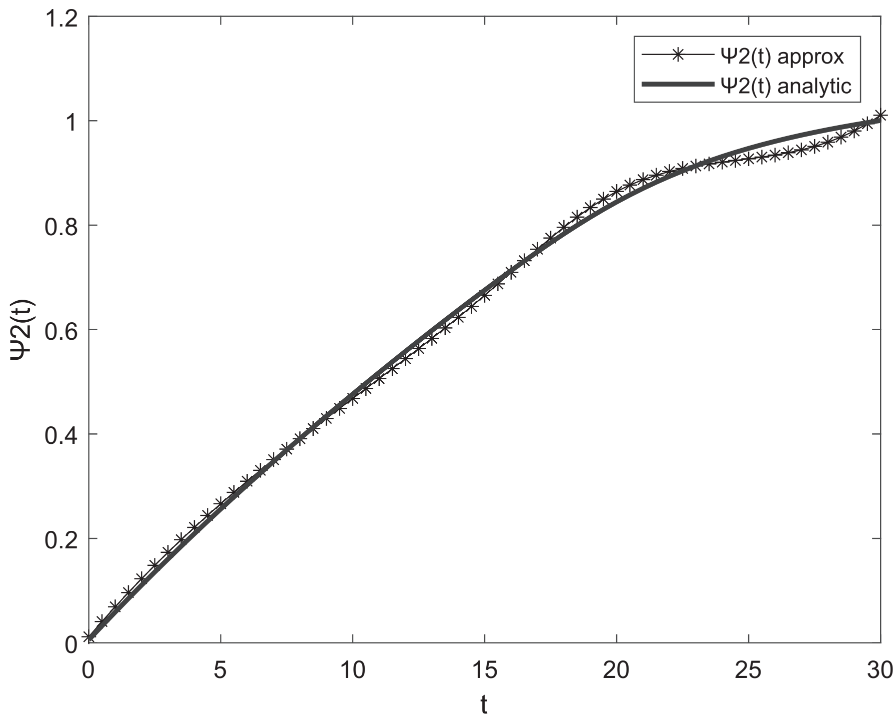

Fig. 3 Optimal state variable $\Psi_2(t)$ in Example 1

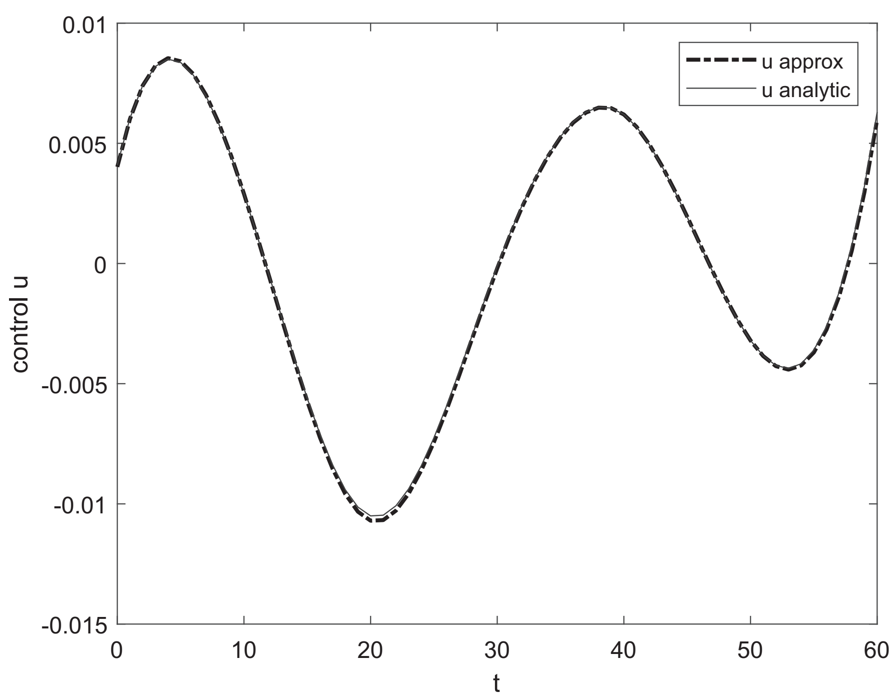

Fig. 4 Optimal control function in Example 2

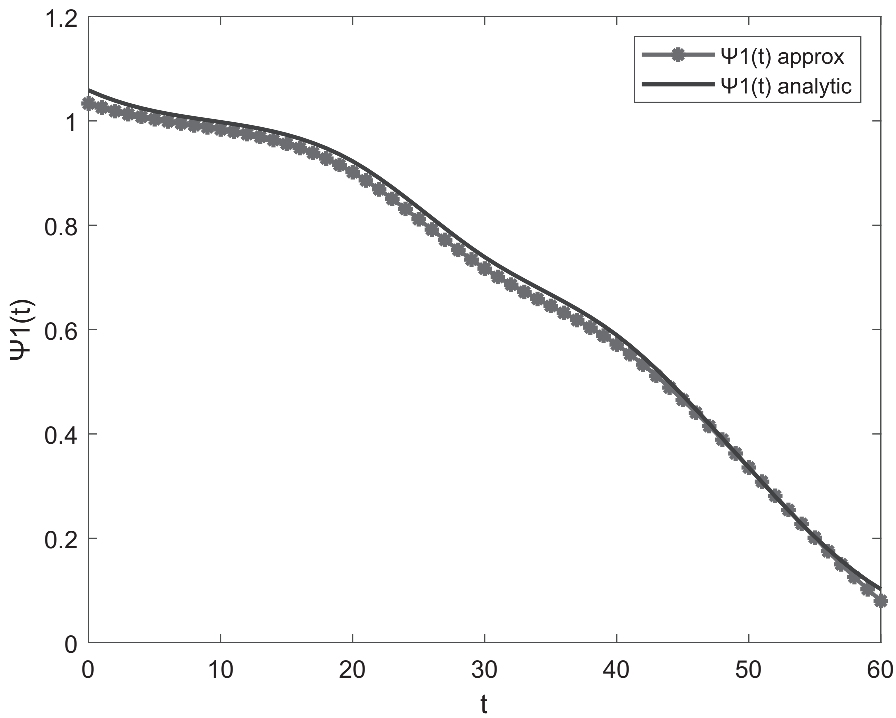

Fig. 5 Optimal state variable $\Psi_1(t)$ in Example 2

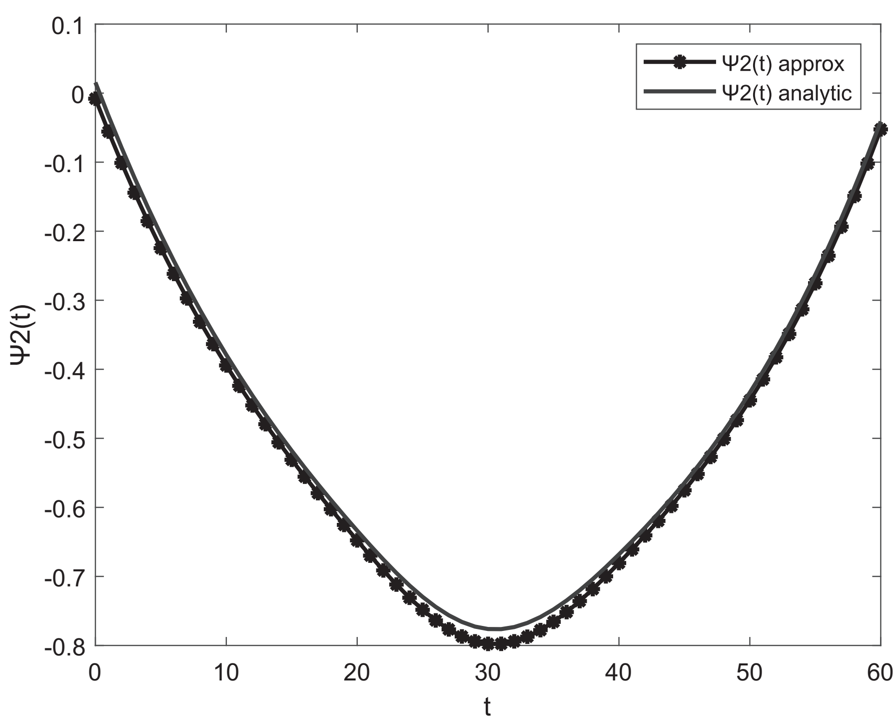

Fig. 6 Optimal state variable $\Psi_2(t)$ in Example 2

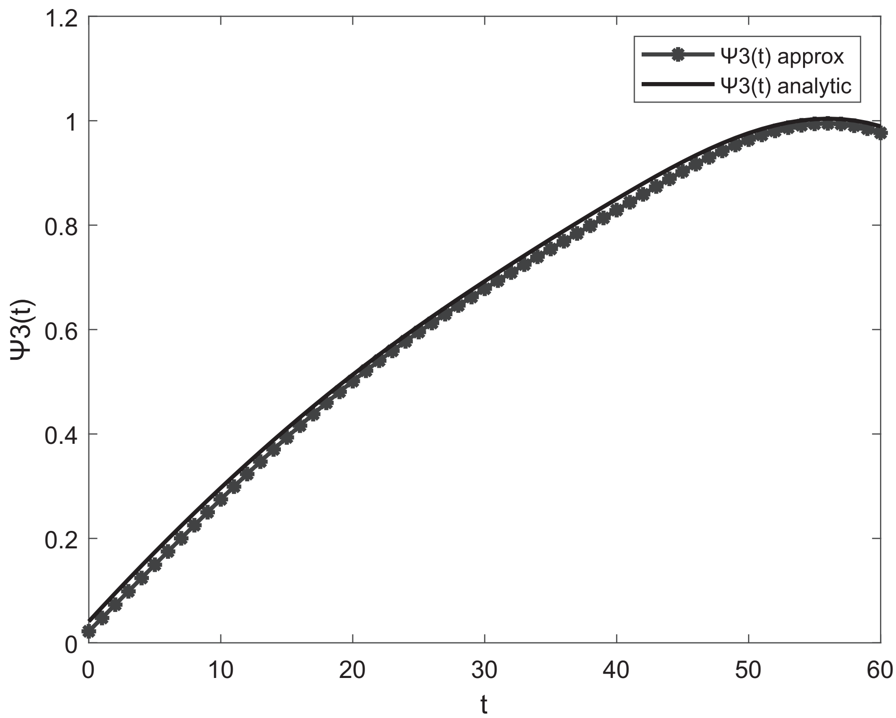

Fig. 7 Optimal state variable $\Psi_3(t)$ in Example 2

6 Conclusion

Considering the widespread application of quantum optimal control across basic sciences and engineering, we have proposed a novel linearization technique aimed at addressing the quantum optimal control problem. Our approach involves an embedding process and leverages the characteristics of positive Radon measures to transform the problem into a format conducive to optimization. Specifically, we employ an optimization search technique to identify the optimal state and control variables. What sets our method apart from others is its simplicity and straightforwardness, offering a more accessible pathway to solving complex quantum control problems.

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Declarations

Consent to Participate and Consent to Publish Not applicable.

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