Indian J Pure Appl Math (2026) 57:1551–1565
https://doi.org/10.1007/s13226-025-00768-8

ORIGINAL RESEARCH

On the power series related to zero-balanced hypergeometric function

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Received: 14 October 2023 / Accepted: 25 February 2025 / Published online: 28 March 2025

Abstract Let $\mathcal{K}(r)$ be the complete elliptic integral of the first kind defined on $(0, 1)$. This paper deals with the power series of $x \mapsto (1-x)^p F(a, b; a+b; x)$ on $(0, 1)$, where $F(a, b; a+b; x)$ denotes the zero-balanced hypergeometric function. This result extends the recently obtained absolutely monotonic property of $(1-x)^p\, \mathcal{K}(\sqrt{x})$. As a consequence, a rational approximation of zero-balanced hypergeometric function will be established, which is an arbitrary precise approximation near $x = 0$.

Keywords Hypergeometric function · Zero-balanced · Power series · Absolutely monotonic · Negative–Positive type series

Mathematics Subject Classification Primary 33E05 · Secondary 26E60

1 Introduction

Throughout this paper, $\mathbb{N}$ ($\mathbb{R}_+$) stands for the set of positive integers (real numbers, resp.) and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b, c \in \mathbb{R}$ and $c \neq \mathbb{N}_0$, the Gaussian hypergeometric function is defined by [4,7]

$$F(a, b; c; x) := {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad |x| < 1,$$

where $(a)_n$ denotes the *Pochhammer symbol* defined by $(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ for $n \in \mathbb{N}$ and in particular $(a)_0 = 1$. Here $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ ($\mathrm{Re}(x) > 0$) is the Euler gamma function [12,24,30]. In the case $c = a + b$, the resulting function $F(a, b; c; x)$ is said to be *zero-balanced*. The hypergeometric function $F(a, b; c; x)$ has a simple derivative formula

$$\frac{d}{dx} F(a, b; c; x) = \frac{ab}{c} F(a+1, b+1; c+1; x). \tag{1.1}$$

In the zero-balanced case, there is a singularity near at $x = 1$ and has a Ramanujan's approximation [7, 1.48, p. 18] of $F(a, b; a+b; x)$, as x tends to 1,

$$B(a, b)F(a, b; c; x) + \log(1-x) = R(a, b) + O((1-x)\log(1-x)), \tag{1.2}$$

Communicated by Apoorva Khare.

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where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function,

$$R(a,b) = -2\gamma - \psi(a) - \psi(b), \quad \psi(x) = \Gamma'(x)/\Gamma(x),$$

and $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.57721\cdots$ is the Euler–Mascheroni constant.

One of the most important special cases of $F(a,b;c;x)$ is the complete elliptic integral of the first kind $\mathcal{K}(r)$ [9,14,17,18,25,26] with the modulus $r \in (0,1)$, which given by

$$\mathcal{K}(r) = \int_0^{\pi/2}(1 - r^2\sin^2\theta)^{-1/2}d\theta = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right). \tag{1.3}$$

In particular, by (1.3), the asymptotic formula (1.2) reduces to

$$\mathcal{K}(r) \sim \log\frac{4}{r'}, \quad r \to 1^-,$$

where and in what follows we denote $r' = \sqrt{1-r^2}$. For more information on these and related functions, we refer the reader to [2,7,8] and, for recently obtained related results, to [3,10,13,16,22,27,31,33,34] and the references contained therein.

It is well-known that a real function φ is said to be absolutely monotonic on the interval I if the kth derivative of φ, $\varphi^{(k)}(x)$, is exists and is non-negative for each $k \geq 0$ and $x \in I$. In other words, if φ can be expressed as a power series on I, then all coefficients are non-negative. Recently, a class of special power series has attracted extensive research [11,21,25,32], a Negative–Positive type series, of which the name was first proposed formally in [29]. Specifically, a power series $S(x)$ given by

$$S(x) = -\sum_{k=0}^{m} a_k x^k + \sum_{k=m+1}^{\infty} a_k x^k$$

is called a "Negative–Positive type power series", or "NP type power series" for short, if its coefficients a_k for $k \geq 0$ satisfy

(i) $a_k \geq 0$ for all $k \geq 0$;
(ii) there exist at least two integers $0 \leq k_1 \leq m$ and $k_2 \geq m+1$ such that $a_{k_1}, a_{k_2} > 0$.

Correspondingly, $S(x)$ is called a "Positive–Negative type" (or "NP type" for short) power series if $-S(x)$ is a Negative–Positive type power series.

The initial thread of this paper can be traced back to a monotonic property obtained by Anderson et al. [5, Theorem 2.2] in 1990, that is, the function $r \mapsto (1-r^2)^{1/2}\mathcal{K}(r)$ is strictly decreasing from $(0,1)$ onto $(0,\pi/2)$. This was generalized to zero-balanced hypergeometric function in [1, Theorem 1.7] and also an extension of this was given in [6, Theorem 1.2], explicitly, for each $c \in [1/2, 2]$, the function $r \mapsto (r')^c\mathcal{K}(r)$ is strictly decreasing and concave from $(0,1)$ onto $(0,\pi/2)$. Further, it was pointed out by Yang [19, Proposition 1] but without the proof that the function $r \mapsto 1-(r')^c\mathcal{K}(r)$ is absolutely monotonic on $(0,1)$ for $c \in [1/2,1]$. Recently, this has been confirmed and improved by Yang and Tian [28], which is stated that the function $x \mapsto -\left[(1-x)^p\mathcal{K}(\sqrt{x})\right]'$ is absolutely monotonic on $(0,1)$ if and only $1/4 \leq p \leq 1$.

Inspired by these results, it is natural to ask the following question.

- Can this be generalized to the case of zero-balanced hypergeometric function?
- What is a suitable region of (a,b) and the range of p such that the derivative of the function we consider is absolutely monotonic or a Negative–Positive type power series?

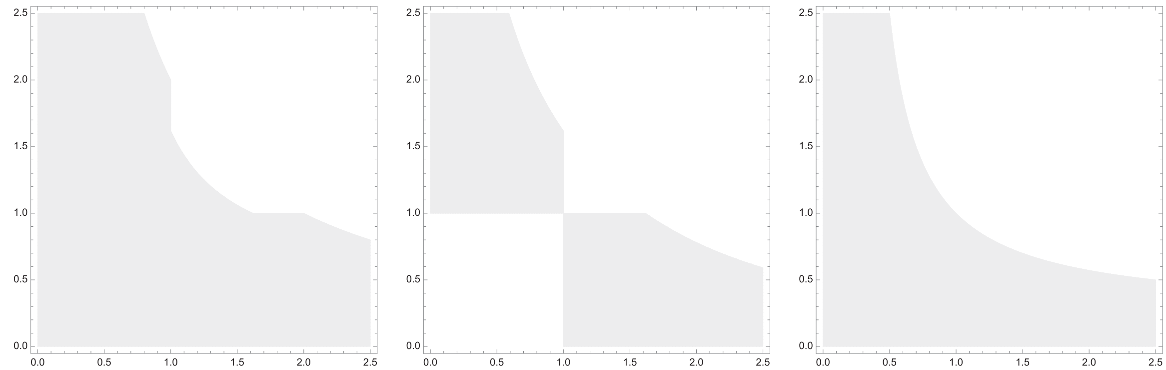
The main purpose of this paper is to answer this question. Our main result is stated as follows.

Theorem 1.1 *Let $(a,b) \in \mathbb{R}_+^2$ and $p \in \mathbb{R}$, and define the function $x \mapsto \phi_p$ on $(0,1)$ by*

$$\phi_p(x) = (1-x)^p F(a,b;a+b;x).$$

Then we have the following conclusions:

(i) *In the case of $(a,b) \in \mathcal{R}_1$, if $\max\{ab/(a+b), ab-1\} \leq p \leq 1$, then $-\phi_p'(x)$ is absolutely monotonic on $(0,1)$;*



(a) $\mathcal{R}_1$ (b) $\mathcal{R}_2$ (c) $\mathcal{R}_3$

Fig. 1 The visualized region $\mathcal{R}_j$ for $j = 1, 2, 3$

(ii) *In the case of* $(a, b) \in \mathcal{R}_2$,
if $\max\{ab - 1, 0\} \le p < ab/(a + b)$, *then* $\phi'_p(x)$ *is a Positive–Negative type power series on* $(0, 1)$;

(iii) *In the case of* $(a, b) \in \mathcal{R}_3$, *if* $\max\{ab - 1, 1\} \le p \le 1 + ab/(a + b)$ *and* $p \neq 1$, *then* $\phi'_p(x)$ *is a Negative–Positive type power series on* $(0, 1)$,

where

$$\kappa(a, b) = 2(a + b)^4 + (4 - 5ab)(a + b)^3 + ab(a + b)^2 - 4a^2b^2(a + b) - a^3b^3 \tag{1.4}$$

and

$$\mathcal{R}_1 = \Big\{(a, b) \in \mathbb{R}^2_+ |\ ab \le 2,\ \kappa(a, b) \ge 0 \text{ or } (1 - a)(b - 1) \ge 0\Big\},$$
$$\mathcal{R}_2 = \left\{(a, b) \in \mathbb{R}^2_+ |\ (1 - a)(b - 1) \ge 0,\ ab - 1 < \frac{ab}{a + b}\right\},$$
$$\mathcal{R}_3 = \left\{(a, b) \in \mathbb{R}^2_+ |\ a + b + 2 \ge 4ab,\ ab - 1 \le 1 + \frac{ab}{a + b}\right\},$$

which are illustrated in Fig. 1.

Remark 1.1 It is worth mentioning that the absolutely monotonic property of $\phi_p(x)$ on $(0, 1)$ has been proved in [15, Lemma 2.3], where our method is completely different. However, our range of p includes $p = 1$ which is not included in [15, Lemma 2.3], although our region of (a, b) is a bit smaller than that of [15, Lemma 2.3].

The rest of this paper is organized as follows. In this section, we give an introduction and highlight the relevant previous results together with the statements of main result in the form of theorem. Section 2 consists of preliminary knowledge, tools and lemmas, which will be used in the following sections. Section 3 gives the proof of the main results from Sect. 1. As an application, we provide a rational approximation of hypergeometric function in Sect. 4. In order to avoid too long proofs of some lemmas, we close the paper by putting some tedious computations in the last section.

2 Preliminaries

In this section, we collect several tools and lemmas which are used to prove the main theorem.

2.1 Tools

The first tool is to give a recurrence relation of MacLaurin's coefficients for the product of power function and hypergeometric function. In [20], it has been proved by Yang that the coefficients of the function $x \mapsto (1 - \theta x)^p F(a, b; c; x)$ satisfy a 3-order recurrence relation for $\theta \in [-1, 1]$, and in particular they satisfy a 2-order recurrence relation for $\theta = 1$.

As a special case of [20, Corollary 2], we state it in the following proposition.

Proposition 2.1 *Let* $(a,b) \in \mathbb{R}_+^2$, $p \in \mathbb{R}$ *and* $\phi_p(x)$ *be defined as in Theorem* 1.1. *Suppose that* $\phi_p(x)$ *has the MacLaurin's series* $\sum_{n=0}^{\infty} u_n x^n$, *then the coefficients* $u_n = u_n(p)$ *satisfy* $u_0 = 1$, $u_1 = ab/(a+b) - p$ *and the recurrence relation*

$$u_{n+1} = \alpha_n u_n - \beta_n u_{n-1} \tag{2.1}$$

for $n \geq 1$, *where*

$$\alpha_n = \frac{2n^2 + n(2a+2b-2p-1) + ab - (a+b)p}{(n+1)(n+a+b)}, \tag{2.2}$$

$$\beta_n = \frac{(a+n-p-1)(b+n-p-1)}{(n+1)(n+a+b)}. \tag{2.3}$$

In particular, $u_n > 0$ *for* $n \geq 0$ *in the case of* $p < 0$.

Proof By the Cauchy product of series, it follows that u_n can be written as

$$u_n = \sum_{k=0}^{n} \frac{(-p)_k}{k!} \frac{(a)_{n-k}(b)_{n-k}}{(n-k)!(a+b)_{n-k}}$$

for $n \geq 0$, which gives $u_n > 0$ in the case of $p < 0$. □

As mentioned in the introduction, for a "NP type power series" $S(x)$, if $a_l = 0$ for $l \geq n+1 \geq k_2+1$, then $S(x)$ degenerates to a polynomial, which is called a "Negative–Positive type polynomial" of degree n, or "NP type polynomial" for short. Likewise, a "PN type power series" can degenerate to a PN type polynomial (see [29]). The second tool offers a simple and efficient criterion to determine the sign of a PN type (or NP type) polynomial, which has been proved in [21,23,24].

Proposition 2.2 (See [21, Lemma 7]) *Let* $P_n(x)$ *be a Negative–Positive type polynomial of degree* n. *Then there exists* $x_0 \in (0,\infty)$ *such that* $P_n(x) < 0$ *for* $x \in (0, x_0)$ *and* $P_n(x) > 0$ *for* $x \in (x_0, \infty)$. *Consequently, for given* $\hat{x} > 0$, *if* $P_n(\hat{x}) < 0$, *then* $P_n(x) < 0$ *for all* $x \in (0, \hat{x})$ *and if* $P_n(\hat{x}) > 0$, *then* $P_n(x) > 0$ *for all* $x \in (\hat{x}, \infty)$.

2.2 Lemmas

In this section, we will present several lemmas which are used to prove Theorem 1.1. To facilitate the readability, we sometimes only give the conclusions and omit their proofs. The tedious proof of calculations will be placed in the last section.

Lemma 2.1 *Let* $(a,b) \in \mathbb{R}_+^2$ *and*

$$\mathcal{R}_{11} = \left\{(a,b) \middle| \frac{ab}{a+b} > ab-1,\ \kappa(a,b) \geq 0, \right\},$$

$$\mathcal{R}_{12} = \left\{(a,b) \middle| \frac{ab}{a+b} \leq ab-1,\ (1-a)(b-1) \geq 0 \right\},$$

$$\widehat{\mathcal{R}}_1 = \left\{(a,b) \middle| \kappa(a,b) \geq 0 \text{ or } (1-a)(b-1) \geq 0 \right\}.$$

Then $\widehat{\mathcal{R}}_1 = \mathcal{R}_{11} \cup \mathcal{R}_{12}$. *In particular,* $\mathcal{R}_1 = \widehat{\mathcal{R}}_1 \cap \left\{(a,b) \in \mathbb{R}_+^2 | \ ab \leq 2\right\}$.

Proof Clearly, $\mathcal{R}_{11} \cup \mathcal{R}_{12} \subset \widehat{\mathcal{R}}_1$, it suffices to prove that $\widehat{\mathcal{R}}_1 \setminus \mathcal{R}_{12} \subset \mathcal{R}_{11}$.

Simplifying leads to

$$\widehat{\mathcal{R}}_1 \setminus \mathcal{R}_{12} = \mathcal{R}_{11} \cup \left\{(a,b) \middle| \frac{ab}{a+b} > ab-1,\ (1-a)(b-1) \geq 0 \right\} \cup \left\{(a,b) \middle| \kappa(a,b) \geq 0,\ (1-a)(b-1) < 0 \right\}.$$

By Lemma 5.1, we conclude that

$$\widehat{\mathcal{R}}_1 \setminus \mathcal{R}_{12} \subset \mathcal{R}_{11} \Longleftrightarrow \begin{cases} \left\{(a,b)\middle| \frac{ab}{a+b} > ab-1,\ (1-a)(b-1) \geq 0\right\} \subset \mathcal{R}_{11}, \\ \left\{(a,b)\middle| \kappa(a,b) \geq 0,\ (1-a)(b-1) < 0\right\} \subset \mathcal{R}_{11}, \end{cases}$$
$$\Longleftrightarrow \begin{cases} \left\{(a,b)\middle| \frac{ab}{a+b} > ab-1,\ (1-a)(b-1) \geq 0,\ \kappa(a,b) < 0\right\} = \emptyset, \\ \left\{(a,b)\middle| \frac{ab}{a+b} \leq ab-1,\ (1-a)(b-1) < 0,\ \kappa(a,b) \geq 0\right\} = \emptyset. \end{cases}$$

This gives the desired result. □

Lemma 2.2 *Let* $(a,b) \in \mathbb{R}_+^2$, α_n, β_n *be defined by* (2.2) *and* (2.3). *Then* $\alpha_n > 0$ *and* $\beta_n > 0$ *for* $n \geq 2$ *if* $p \leq 1 + ab/(a+b)$.

Proof As in (2.2) and (2.3), we denote the numerators of α_n, β_n respectively by

$$\alpha_n^* = 2n^2 + n(2a+2b-2p-1) + ab - (a+b)p,$$
$$\beta_n^* = (a+n-p-1)(b+n-p-1).$$

If $p \leq 1 + ab/(a+b)$, then it follows that

$$\alpha_{n+1}^* - \alpha_n^* = 1 + 2a + 2b + 4n - 2p \geq 1 + 2a + 2b + 8 - 2\left(1 + \frac{ab}{a+b}\right)$$
$$= \frac{7a + 2a^2 + 7b + 2ab + 2b^2}{a+b} > 0,$$

and so α_n^* is strictly increasing for $n \geq 2$. This gives

$$\alpha_n^* \geq \alpha_2^* = 6 + 4a + 4b + ab - (4+a+b)p$$
$$\geq 6 + 4a + 4b + ab - (4+a+b)\left(1 + \frac{ab}{a+b}\right) = \frac{2a + 3a^2 + 2b + 2ab + 3b^2}{a+b} > 0.$$

On the other hand,

$$\beta_n^* \geq \left(a + 2 - 1 - \frac{ab}{a+b} - 1\right)\left(b + 2 - 1 - \frac{ab}{a+b} - 1\right) = \frac{a^2b^2}{(a+b)^2} > 0.$$

This completes the proof. □

Lemma 2.3 *Let* $\tilde{u}_{n+1} = u_{n+1} - [(n+1)/(n+p+2)]u_n$, *where* $u_n = u_n(p)$ *is given in Proposition* 2.1. *Then the following identity*

$$\tilde{u}_{n+1} = \xi_n \tilde{u}_n + \eta_n u_{n-1} \tag{2.4}$$

holds for $n \geq 1$, *where*

$$\xi_n = \frac{f_n(p)}{(1+n)(a+b+n)(p+2+n)}, \quad \eta_n = \frac{(p+1)g_n(p)}{(1+n)(a+b+n)(p+1+n)(p+2+n)}$$

and

$$f_n(p) = n^3 + (a+b+1)n^2 - \begin{bmatrix} 2p^2 - (a+b-5)p \\ -2(a+b) - ab + 3 \end{bmatrix} n + \begin{bmatrix} ab(p+2) \\ -(a+b)(p+1)^2 \end{bmatrix}, \tag{2.5}$$
$$g_n(p) = (1 - ab + p)n - (p+2)(1-a+p)(1-b+p). \tag{2.6}$$

In particular, $\xi_n > 0$ *for* $n \geq 3$ *and* $\eta_n \geq 0$ *for* $n \geq 6$ *in the range of* (a, b, p) *considered in Theorem* 1.1.

Proof By simplifying, the Relation (2.1) can be rewritten as

$$u_{n+1} - \frac{n+1}{n+p+2}u_n = \left(\alpha_n - \frac{n+1}{n+p+2}\right)\left(u_n - \frac{n}{n+p+1}u_{n-1}\right) + \left[\left(\alpha_n - \frac{n+1}{n+p+2}\right)\frac{n}{n+p+1} - \beta_n\right]u_{n-1},$$

which gives (2.4) provided that we denote

$$\xi_n = \alpha_n - \frac{n+1}{n+p+2} \quad \text{and} \quad \eta_n = \left(\alpha_n - \frac{n+1}{n+p+2}\right)\frac{n}{n+p+1} - \beta_n.$$

Moreover, f_n and g_n are respectively the numerator of the expression of ξ_n and η_n, explicitly, which are given by (2.5) and (2.6). Note that the sign of ξ_n and η_n are the same as f_n and g_n. It has been proved in Lemmas 5.3 and 5.4 that $f_n > 0$ for $n \geq 3$ and $g_n \geq 0$ for $n \geq 6$ in the range of (a, b, p) considered in Theorem 1.1. □

Lemma 2.4 *Let $(a, b) \in \mathbb{R}_+^2$ and α_n, β_n be defined as in* (2.2) *and* (2.3). *Then we have*

$$\alpha_3\alpha_2 - \beta_3 > 0$$

if $ab - 1 \leq p \leq 1 + ab/(a+b)$.

Proof By (2.2) and (2.3), we simplify as

$$\begin{aligned} h(p) &= 12(a+b+2)(a+b+3)\big(\alpha_3\alpha_2 - \beta_3\big) \\ &= \Big[(a+b)^2 + 7(a+b) + 18\Big]p^2 - \Big[7(a+b)^2 + (2ab+51)(a+b) + 10ab + 72\Big]p \\ &\quad + 18(a+b)^2 + (7ab+72)(a+b) + 15ab + (ab)^2 + 66, \end{aligned} \tag{2.7}$$

which is regarded as a quadratic function of p with the upward opening.

Note that the symmetric axis of $h(p)$ can be written by

$$p_s = \frac{7(a+b)^2 + (2ab+51)(a+b) + 10ab + 72}{2(7a + a^2 + 7b + 2ab + b^2 + 18)}$$

and

$$p_s - \left(2 + \frac{ab}{a+b}\right) = \frac{23a^2 + 3a^3 + 10ab + 5a^2b + 23b^2 + 5ab^2 + 3b^3}{2(a+b)(7a + a^2 + 7b + 2ab + b^2 + 18)} > 0.$$

This shows that $p \mapsto h(p)$ is strictly decreasing for $p \leq 2 + ab/(a+b)$.

If $ab - 1 \leq p \leq 1 + ab/(a+b)$, then $ab - 1 \leq 1 + ab/(a+b)$, that is,

$$2a + 2b + ab - a^2b - ab^2 \geq 0.$$

Combining this with the monotonicity of $h(p)$, it follows that

$$h(p) \geq h\left(1 + \frac{ab}{a+b}\right) = \frac{1}{(a+b)^2}\begin{bmatrix} 4(3a^2 + 7a^3 + 3a^4 + 3b^2 + 7b^3 + 3b^4) \\ +ab(24 + 42a + 16a^2 + 42b + 23ab + 16b^2) \\ +3ab(2a + 2b + ab - a^2b - ab^2) \end{bmatrix} > 0.$$

This together with (2.7) gives the desired result. □

Lemma 2.5 *Let $(a, b) \in \mathbb{R}_+^2$ and $u_n(p)$ be defined as in Proposition* 2.1. *Then it holds that*

(i) $u_1(p) \leq 0$ *for* $p \geq ab/(a+b)$;
(ii) $u_2(p) < 0$ *and* $u_3(p) < 0$ *for* $ab/(a+b) \leq p \leq 1 + ab/(a+b)$;
(iii) $u_4(p) < 0$ *for* $ab/(a+b) \leq p \leq 1$.

Proof (i) By Proposition 2.1, it is easy to see that $u_1(p) = ab/(a+b) - p \le 0$ for $p \ge ab/(a+b)$.
(ii) Simplifying with (2.1) gives

$$u_2(p) = \frac{p^2}{2} - \left(\frac{ab}{a+b} + \frac{1}{2}\right)p + \frac{ab(a+1)(b+1)}{2(a+b)(a+b+1)},$$
$$u_3(p) = -\frac{p^3}{6} + \left[\frac{ab}{2(a+b)} + \frac{1}{2}\right]p^2 - \left[\frac{ab}{2(a+b)} + \frac{ab(a+1)(b+1)}{2(a+b)(a+b+1)} + \frac{1}{3}\right]p + \frac{ab(a+1)(b+1)(a+2)(b+2)}{6(a+b)(a+b+1)(a+b+1)}.$$

Moreover, as a quadratic function of p, $u_2(p)$ is strictly convex for $p \in \mathbb{R}$ and also $u_3(p)$ is strictly convex for $p \le 1 + ab/(a+b)$ due to $u_3''(p) = 1 + ab/(a+b) - p \ge 0$.

Hence, $u_2(p) < 0$ and $u_3(p) < 0$ for $ab/(a+b) \le p \le 1 + ab/(a+b)$ follows from

$$u_2\left(\frac{ab}{a+b}\right) = u_2\left(\frac{ab}{a+b} + 1\right) = -\frac{a^2b^2}{2(a+b)^2(a+b+1)} < 0,$$
$$u_3\left(\frac{ab}{a+b}\right) = -\frac{a^2b^2(3a + 2a^2 + 3b + 2ab + 2b^2)}{3(a+b)^3(a+b+1)(a+b+2)} < 0,$$
$$u_3\left(1 + \frac{ab}{a+b}\right) = -\frac{a^2b^2(a-b)^2}{6(a+b)^3(1+a+b)(2+a+b)} < 0.$$

(iii) By (2.1), we write the explicit expression of u_4 as

$$u_4(p) = \frac{p^4}{24} - \left[\frac{ab}{6(a+b)} + \frac{1}{4}\right]p^3 + \left[\frac{ab}{2(a+b)} + \frac{ab(a+1)(b+1)}{4(a+b)(a+b+1)} + \frac{11}{24}\right]p^2 - \left[\frac{ab}{3(a+b)} + \frac{ab(a+1)(b+1)}{4(a+b)(a+b+1)} + \frac{ab(a+1)(b+1)(a+2)(b+2)}{6(a+b)(a+b+1)(a+b+2)} + \frac{1}{4}\right]p + \frac{ab(a+1)(b+1)(a+2)(b+2)(a+3)(b+3)}{24(a+b)(a+b+1)(a+b+2)(a+b+3)}$$

and thereby, by differentiation,

$$u_4'(p) = \frac{p^3}{6} - \left[\frac{ab}{2(a+b)} + \frac{3}{4}\right]p^2 + \left[\frac{ab}{a+b} + \frac{ab(a+1)(b+1)}{2(a+b)(a+b+1)} + \frac{11}{12}\right]p - \left[\frac{ab}{3(a+b)} + \frac{ab(a+1)(b+1)}{4(a+b)(a+b+1)} + \frac{ab(a+1)(b+1)(a+2)(b+2)}{6(a+b)(a+b+1)(a+b+2)} + \frac{1}{4}\right],$$
$$u_4''(p) = \frac{p^2}{2} - \left(\frac{ab}{a+b} + \frac{3}{2}\right)p + \frac{ab}{a+b} + \frac{ab(a+1)(b+1)}{2(a+b)(a+b+1)} + \frac{11}{12}.$$

As a quadratic function of p, $u_4''(p)$ is strictly convex for $p \in \mathbb{R}$. Combining this with $u_4''(0) > 0$ and

$$u_4''\left(1 + \frac{ab}{a+b}\right) = -\frac{(a+b)^3 + (a+b)^2 + 6a^2b^2}{12(a+b)^2(1+a+b)} < 0,$$

it follows that there exists a number $0 < p_0 < 1 + ab/(a+b)$ such that $u_4'(p)$ is strictly increasing for $0 < p < p_0$ and strictly decreasing for $p_0 < p < 1 + ab/(a+b)$. This together with $u_4'(0) < 0$ and the fact that

$$u_4'\left(1 + \frac{ab}{a+b}\right) = \frac{(a+b)^5 + 3(a+b)^4 + 2(a+b)^3 + a^2b^2(6a + 6b + 5a^2 + 2ab + 5b^2)}{12(a+b)^3(1+a+b)(2+a+b)} > 0$$

shows that there exists a number $0 < p_1 < 1 + ab/(a+b)$ such that $u_4(p)$ is strictly decreasing for $0 < p < p_1$ and strictly increasing for $p_1 < p < 1 + ab/(a+b)$.

Therefore, $u_4(p) < 0$ for $ab/(a+b) \le p \le 1$ follows from the piecewise monotonicity of $p \mapsto u_4(p)$ together with the facts that

$$u_4\left(\frac{ab}{a+b}\right) = -\frac{a^2b^2}{8(a+b)^3\prod_{j=0}^{3}(a+b+j)}\left[\begin{array}{c} 8a^3+8b^3+6a^4+6b^4+11a^3b+11ab^3 \\ +16a^2b^2+22(a+b)^2+16(a+b)^3 \\ +ab(a+b)(a+b-ab) \end{array}\right] < 0,$$
$$u_4(1) = -\frac{ab(a+1)(a+2)(b+1)(b+2)(a+b-ab+3)}{24(a+b)(a+b+1)(a+b+2)(a+b+3)} < 0.$$

This completes the proof. □

We now give a key lemma to determine the sign of u_n after a certain term.

Lemma 2.6 *Let $(a,b) \in \mathbb{R}_+^2$ and $u_n = u_n(p)$ be defined as in Proposition* 2.1. *Then the following conclusions hold for the range of (a,b,p) satisfying the conditions in Lemma* 5.4:

(i) *If there exists $m \ge 5$ such that $u_{m-1} \le 0$ and $u_m > 0$, then $u_n > 0$ for all $n \ge m$;*
(ii) *Conversely, if there exists $m \ge 5$ such that $u_{m-1} \ge 0$ and $u_m < 0$, then $u_n < 0$ for all $n \ge m$.*

Proof We only need to prove part (i), and part (ii) can be obtained by changing the sign of all inequalities in the proof of part (i).

First, it follows from Lemma 5.4 that $p \le 1 + ab/(a+b)$ and so $\alpha_n > 0$, $\beta_n > 0$ and $\xi_n > 0$ for $n \ge 3$ by Lemmas 2.2 and 2.3.

Suppose that there exists $m \ge 5$ such that $u_{m-1} \le 0$ and $u_m > 0$, then by (2.1), one has

$$\begin{aligned} u_{m+1} &= \alpha_m u_m + \beta_m \times (-u_{m-1}) \ge \alpha_m u_m > 0, \\ \tilde{u}_{m+1} &= u_{m+1} - \frac{m+1}{m+p+2}u_m = \alpha_m u_m + \beta_m \times (-u_{m-1}) - \frac{m+1}{m+p+2}u_m \\ &= \xi_m u_m + \beta_m \times (-u_{m-1}) > 0, \end{aligned} \tag{2.8}$$

where the last equality follows from the definition of ξ_m.

Now we may assume that $u_k > 0$ for $m+1 \le k \le n$, combining this with $u_m > 0$, then $u_{k-1} > 0$ for $m+1 \le k \le n$.

In the range of (a,b,p) satisfying the conditions in Lemma 5.4, i.e. (a,b,p) satisfies the range considered in Theorem 1.1, then it follows from Lemma 2.3 that $\xi_k > 0$ and $\eta_k \ge 0$ for $k \ge 6$ and so

$$\tilde{u}_{k+1} = \xi_k \tilde{u}_k + \eta_k u_{k-1} \ge \xi_k \tilde{u}_k$$

for $m+1 \le k \le n$. Then by (2.8),

$$\tilde{u}_{n+1} \ge \xi_n \tilde{u}_n \ge \xi_n \xi_{n-1} \tilde{u}_{n-1} \ge \cdots \ge \xi_n \xi_{n-1} \cdots \xi_{m+1} \tilde{u}_{m+1} > 0,$$

which yields

$$\tilde{u}_{n+1} = u_{n+1} - \frac{n+1}{n+p+2}u_n > 0 \Longleftrightarrow u_{n+1} > \frac{n+1}{n+p+2}u_n.$$

Combining this with the assumption that $u_k > 0$ for $m+1 \le k \le n$, we conclude that $u_{n+1} > 0$. This completes the proof by mathematical induction. □

3 Proof of the main theorem

We are now in a position to give the proof of Theorem 1.1 by the recurrence method.

Proof of Theorem 1.1 Differentiating ϕ_p gives

$$\phi_p'(x) = (1-x)^{p-1}\left[\frac{ab}{a+b}F(a,b;a+b+1;x) - pF(a,b;a+b;x)\right]. \tag{3.1}$$

We divide into three cases to complete the proof.

Case 3.1: $(a, b) \in \mathcal{R}_1$. In this case, if $\max\{ab/(a+b), ab-1\} \le p \le 1$, then it follows from Lemma 2.5 that $u_1 \le 0$ and $u_j < 0$ for $2 \le j \le 4$. Suppose that there exists a positive term of the sequence $\{u_n\}_{n\ge 5}$, and let n_0 ($n_0 \ge 5$) denote the item number of the first positive term, that is to say, $u_k \le 0$ for $1 \le k \le n_0 - 1$ and $u_{n_0} > 0$. According to this with Lemma 2.6(i), it can be easily seen that $u_n > 0$ for $n \ge n_0$. This implies

$$\phi_p'(x) = \sum_{n=1}^{\infty} n u_n x^{n-1} = \sum_{n=1}^{n_0-1} n u_n x^{n-1} + \sum_{n=n_0}^{\infty} n u_n x^{n-1} > \sum_{n=1}^{n_0-1} n u_n x^{n-1} \tag{3.2}$$

for $x \in (0, 1)$. On the other hand, for $0 < p \le 1$, it follows from (1.2) and (3.1) that

$$\lim_{x\to 1^-} \phi_p'(x) = -\infty,$$

which is a contradiction with (3.2). Hence, we conclude that $u_n \le 0$ for $n \ge 1$, that is to say, $-\phi_p'(x)$ is absolutely monotonic on $(0, 1)$.

Case 3.2: $(a, b) \in \mathcal{R}_2$. In this case, if $\max\{ab-1, 0\} \le p < ab/(a+b)$, then $u_1 > 0$ by Proposition 2.1 and it follows from Lemma 5.2(i) that $0 \le p < ab/(a+b) < 2/3$ and so $\lim_{x\to 1^-} \phi_p'(x) = -\infty$ by (3.1). This demonstrates that there must be a negative term in the sequence $\{u_n\}_{n\ge 2}$. Now we may assume that u_l ($l \ge 2$) is the first negative term, in other words, $u_n \ge 0$ for $2 \le n \le l-1$ and $u_l < 0$.

In the following we discuss in two cases of l.

- $l = 2$. Then $u_1 > 0$ and $u_2 < 0$. In this case, it follows from Lemma 2.2 and $0 \le p < 1$ that $\alpha_2 > 0$ and $\beta_2 > 0$. This together with (2.1) gives

$$u_3 = \alpha_2 u_2 - \beta_2 u_1 < 0.$$

Moreover, by (2.1) and Lemma 2.4, it follows that

$$u_4 = \alpha_3 u_3 - \beta_3 u_2 = \alpha_3(\alpha_2 u_2 - \beta_2 u_1) - \beta_3 u_2 = (\alpha_3\alpha_2 - \beta_3)u_2 - \alpha_3\beta_2 u_1 < 0. \tag{3.3}$$

- $l = 3$. Then $u_1 > 0$, $u_2 \ge 0$ and $u_3 < 0$. This gives $u_4 = \alpha_3 u_3 - \beta_3 u_2 < 0$ by (2.1) and Lemma 2.2.

In summary, the number l is at least 4, $l \ge 4$. We declare that $u_n \le 0$ for all $n \ge l+1$. Otherwise, there exists a positive term in the sequence $\{u_n\}_{n\ge l+1}$ and we denote by u_m the first positive term, that is, $u_{m-1} \le 0$ and $u_m > 0$. Since $m \ge l+1 \ge 5$, by Lemma 2.6(i), it follows that $u_n > 0$ for all $n \ge m$. So the inequality holds as in (3.2), which is a contradiction with $\lim_{x\to 1^-} \phi_p'(x) = -\infty$.

Therefore, it has been shown that $u_1 > 0$, $u_n \ge 0$ for $2 \le n \le l-1$ and $u_l < 0$, $u_n \le 0$ for $n \ge l+1$. That is to say, $\phi_p'(x)$ is a Positive-Negative type power series on $(0, 1)$.

Case 3.3: $(a, b) \in \mathcal{R}_3$.
We first prove, in this case, that $2ab \le a+b$. Otherwise, $2ab > a+b \ge 2\sqrt{ab}$, that is, $ab > 1$. On the other hand, $a+b+2 \ge 4ab > 2(a+b)$, equivalently, $a+b < 2$ and thereby $ab \le (a+b+2)/4 < 1$, which deduces a contradiction.
If $\max\{ab-1, 1\} \le p \le 1 + ab/(a, b)$ and $p \ne 1$, then $p > 1 > ab/(a+b)$, and thereby $u_1 < 0$ by Proposition 2.1 and $1 < p \le 1 + ab/(a+b) < 2$. According to this with (1.2), twice differentiation of $\phi_p(x)$ yields

$$\begin{aligned}\lim_{x\to 1^-} \Phi_p''(x) &= \lim_{x\to 1^-} \left\{ (1-x)^{p-2} \left[\begin{array}{c} p(p-1)F(a, b; a+b; x) - 2p\frac{ab}{a+b}F(a, b; a+b+1; x) \\ +\frac{ab(a+1)(b+1)}{(a+b)(a+b+1)}F(a, b; a+b+2; x) \end{array} \right] \right\} \\ &= \lim_{x\to 1^-} \left\{ (1-x)^{p-2} \left[p(p-1)\cdot \frac{R(a, b) - \log(1-x)}{B(a, b)} - \frac{2p-1}{B(a, b)} \right] \right\} = +\infty. \end{aligned} \tag{3.4}$$

This shows that there must be a positive term of the sequence $\{u_n\}_{n\ge 2}$. Let u_k be the first positive term, then $u_n \le 0$ for $2 \le n \le k-1$ and $u_k > 0$. Since $ab/(a+b) < p \le 1 + ab/(a+b)$, it follows from Lemma 2.5(ii) that $u_2(p) < 0$ and $u_3(p) < 0$. This deduces $k \ge 4$.

We will prove $u_n \geq 0$ for $n \geq k+1$ by the method of contradiction. As a matter of fact, if there exists a negative term in the sequence $\{u_n\}_{n\geq k+1}$, then let $u_{m'}$ be the first negative term, that is, $u_{m'-1} \geq 0$ and $u_{m'} < 0$ with $m' \geq k+1 \geq 5$. This together with Lemma 2.6(ii) implies that $u_n < 0$ for $n \geq m'$. This shows

$$\Phi_p''(x) = \sum_{n=2}^{m'-1} n(n-1)u_n x^{n-2} + \sum_{n=m'}^{\infty} n(n-1)u_n x^{n-2} < \sum_{n=2}^{m'-1} n(n-1)u_n x^{n-2}$$

for $x \in (0, 1)$, which is a contradiction with (3.4).

To this end, we conclude that $u_n \leq 0$ for $1 \leq n \leq k-1$ and $u_n \geq 0$ for $u \geq k$. In particular, $u_1 < 0$ and $u_k > 0$. That is to say, $\phi_p'(x)$ is a Negative–Positive type power series on $(0, 1)$. □

4 A rational approximation

In this section, we provide a rational approximation for zero-balanced hypergeometric function by applying Theorem 1.1.

Let us denote

$$Q_{p,n}(x) := \phi_p(x) - \sum_{k=0}^{n} u_k x^k = \sum_{k=n+1}^{\infty} u_k x^k$$

for $n \geq 0$. For $(a, b) \in \mathcal{R}_1$, if $\max\{ab/(a+b), ab-1\} \leq p \leq 1$, then by Theorem 1.1$(i)$ we conclude that the function $Q_{p,n}(x)/x^{n+1}$ is strictly decreasing on $(0, 1)$ and

$$\lim_{x\to 0^+} \frac{Q_{p,n}(x)}{x^{n+1}} = u_{n+1} \quad \text{and} \quad \lim_{x\to 1^-} \frac{Q_{p,n}(x)}{x^{n+1}} = -\sum_{k=0}^{n} u_k.$$

This implies

$$\sum_{k=0}^{n} u_k x^k - \left(\sum_{k=0}^{n} u_k\right) x^{n+1} < \phi_p(x) < \sum_{k=0}^{n+1} u_k x^k$$

for $x \in (0, 1)$. We obtain therefore the following proposition.

Proposition 4.1 *Let $u_n = u_n(p)$ for $n \geq 0$ be defined in Proposition* 2.1. *For $(a, b) \in \mathcal{R}_1$, the double inequality*

$$\frac{\sum_{k=0}^{n} u_k x^k - \left(\sum_{k=0}^{n} u_k\right) x^{n+1}}{(1-x)^p} < F(a, b; a+b; x) < \frac{\sum_{k=0}^{n+1} u_k x^k}{(1-x)^p} \tag{4.1}$$

holds for $r \in (0, 1)$ with $n \geq 0$ if $\max\{ab/(a+b), ab-1\} \leq p \leq 1$, *where the lower and upper bounds are sharp. Moreover, for $(a, b) \in \mathcal{R}_2$ and* $\max\{ab-1, 0\} \leq p < ab/(a+b)$, *if $u_N(p) \leq 0$, then the inequality* (4.1) *also holds for $r \in (0, 1)$ with $n \geq N$.*

Similarly, the following proposition can be derived from Theorem 1.1(iii).

Proposition 4.2 *Let $u_n = u_n(p)$ for $n \geq 0$ be defined in Proposition* 2.1. *For $(a, b) \in \mathcal{R}_3$ and* $\max\{ab-1, 1\} \leq p \leq 1 + ab/(a+b)$, *if $u_L(p) \geq 0$, then the inequality*

$$\frac{\sum_{k=0}^{n+1} u_k x^k}{(1-x)^p} < F(a, b; a+b; x) < \frac{\sum_{k=0}^{n} u_k x^k - \left(\sum_{k=0}^{n} u_k\right) x^{n+1}}{(1-x)^p}$$

holds for $r \in (0, 1)$ with $n \geq L$, where the lower and upper bounds are sharp.

Funding This work was supported by the National Natural Science Foundation of China (11971142) and the Natural Science Foundation of Zhejiang Province (LY19A010012).

Data Availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

5 Appendix: Some computational lemmas

Lemma 5.1 *Let* $(a, b) \in \mathbb{R}_+^2$ *and* $\kappa(a, b)$ *be defined by* (1.4). *Then the following statements hold true*:

(i) *If* $ab/(a+b) > ab - 1$ *and* $(1-a)(b-1) \geq 0$, *then* $\kappa(a, b) > 0$.
(ii) *If* $ab/(a+b) \leq ab - 1$ *and* $(1-a)(b-1) < 0$, *then* $\kappa(a, b) < 0$.

Proof For $(a, b) \in \mathbb{R}_+^2$, we can solve the equation $ab = (a+b)(ab-1)$ for $b > 0$, of which a unique positive solution is give by

$$b(a) = \frac{1 + a - a^2 + \sqrt{4a^2 + (1 + a - a^2)^2}}{2a}.$$

Simplifying $\kappa(a, b)$ into a polynomial of b, which gives

$$\kappa(a, b) = 2(a+2)a^3 + a^2 q_1(a) b + a q_2(a) b^2 + q_3(a) b^3 + (2 - 5a) b^4,$$

where $q_1(a) = 12 + 9a - 5a^2$, $q_2(a) = 12 + 14a - 19a^2$ and $q_3(a) = 4 + 9a - 19a^2 - a^3$. Note that $q_j(a)$ $(j = 1, 2, 3)$ is a PN type polynomial, it is easy to find their positive roots, for instance, $a_1 = (9 + \sqrt{321})/10 = 2.6916\cdots > a_2 = (7 + \sqrt{277})/19 = 1.2443\cdots > a_3 = 0.73274\cdots > 2/5$ such that $q_j(a_j) = 0$ $(j = 1, 2, 3)$. From this, we clearly see that when $q_j(a) < 0$, then $q_{j+1}(a) < 0$ and $2 - 5a < 0$. That is to say, $\kappa(a, b)$ is also a PN type polynomial of b.
Let

$$\varrho_1(a) = -\begin{bmatrix} 8 + 8a + 2a^3 + 2a^4 + 42a^9 + 9a^{10} + 6a^{12} + 28a^6 \\ +21a(1 - a^{10}) + (55a^2 + 69a^5)(1 - a^3) + 31a^3(1 - a^4) \end{bmatrix},$$

$$\varrho_2(a) = -\begin{bmatrix} 7 + 21a + 2a^3 + 8a^5 + 9a^9 + 26a^2(1 - a^5) \\ +(10a^5 + 6a^9)(1 - a) + 4a^5(1 - a^3) + (1 - a^7) \end{bmatrix},$$

$$\varrho_3(a) = 1410 + (a - 1)\begin{bmatrix} 1390 + 1288a + 1003a^2 + 612a^3 + 330a^4 + 339a^5 + 336a^6 \\ +3a^6\begin{pmatrix} 62(2 - a) + 37(a - 1)(2 - a) \\ +2(a - 1)(2 - a)^2(7 + 11a + 5a^2) \end{pmatrix} \\ +3a^{11}(2 - a)\big(26 + 9a + 5a(a - 1)^2\big) \end{bmatrix},$$

$$\varrho_4(a) = \big[1 + a^4 + a(2 - a)(1 + 2a)\big]\left[126 + (a - 1)\begin{pmatrix} 106 + 64a + 25a^2 \\ +23a^3 + 23a^4 + 23a^5 \\ 3a^6\big(3(2 - a) + 5a^2(a - 1)\big) \end{pmatrix}\right].$$

Then it is easy to see that $\varrho_1(a) < 0$, $\varrho_2(a) < 0$ for $0 < a \leq 1$ and $\varrho_3(a) > 0$, $\varrho_4(a) > 0$ for $1 < a < 2$. Differentiating $\kappa(a, b(a))$ yields

$$\kappa'(a, b(a)) = \frac{\varrho_1(a) + \varrho_2(a)\sqrt{1 + 2a + 3a^2 - 2a^3 + a^4}}{2a^5\sqrt{1 + 2a + 3a^2 - 2a^3 + a^4}} < 0, \quad 0 < a \leq 1,$$

$$\kappa''(a, b(a)) = \frac{\varrho_3(a) + \varrho_4(a)\sqrt{1 + 2a + 3a^2 - 2a^3 + a^4}}{a^6(1 + 2a + 3a^2 - 2a^3 + a^4)^{3/2}} > 0, \quad 1 < a < 2.$$

Hence, $\kappa(a, b(a))$ is strictly decreasing on $(0, 1]$ and $\kappa(a, b(a))$ is strictly convex on $(1, 2)$.
Without of generality, we may assume that $b \geq a > 0$.
(i) If $ab/(a+b) > ab - 1$ and $(1-a)(b-1) \geq 0$, then it can be reduced as

$$0 < a \leq 1, \quad 1 \leq b < b(a).$$

This together with the monotonicity of $a \mapsto \kappa(a, b(a))$ on $(0, 1)$ implies that $\kappa(a, b(a)) \geq \kappa(1, b(1)) = 0$. Since $\kappa(a, b)$ is a PN type polynomial of b, it follows from Proposition 2.2 and $\kappa(a, b(a)) > 0$ that $\kappa(a, b) \geq 0$ for $0 < a \leq 1$ and $1 \leq b < b(a)$.
(ii) If $ab/(a+b) \leq ab - 1$ and $(1-a)(b-1) < 0$, then it can be solved as

$$1 < a \leq (1 + \sqrt{17})/4, \quad b \geq b(a) \quad \text{or} \quad a > (1 + \sqrt{17})/4, \quad b \geq a.$$

In order to prove $\kappa(a, b) < 0$, as a PN type polynomial of b, it suffices to verify $\kappa(a, b(a)) < 0$ and $\kappa(a, a) < 0$ in each case by Proposition 2.2.

- For $1 < a \le (1+\sqrt{17})/4$ and $b \ge b(a)$, the convexity of $a \mapsto \kappa(a, b(a))$ leads to

$$\kappa(a, b(a)) < \max\left\{\kappa(1, b(1)), \kappa\left((1+\sqrt{17})/4, b((1+\sqrt{17})/4)\right)\right\} = 0.$$

- For $a > (1+\sqrt{17})/4$ and $b \ge a$, it follows that

$$\kappa(a, a) = 32a^3 + 36a^4 - 48a^5 - a^6$$

  is a PN type polynomial of a, then $\kappa(a, a) < 0$ follows from Proposition 2.2 together with

$$\kappa\left((1+\sqrt{17})/4, b((1+\sqrt{17})/4)\right) = -5.739\cdots < 0.$$

This completes the proof. □

Lemma 5.2 *Let $(a, b) \in \mathbb{R}_+^2$. Then it holds that*

(i) *if $ab/(a+b) > ab - 1$, then $2(a+b) > 3ab$;*
(ii) *if $ab \le 2$ and $(1-a)(b-1) \ge 0$, then $2(a+b) \ge 3ab$;*
(iii) *if $a + b + 2 \ge 4ab$, then $\tau(a, b) > 0$, where*

$$\tau(a, b) = 6a^4 + 2a^3(7-2a)b + a^2(16-19a)b^2 + a(14-19a-a^2)b^3 + 2(3-2a)b^4.$$

Proof We may assume that $b \ge a > 0$ without of generality.
(i) If $2(a+b) \le 3ab$, that is, $b \ge 4/3$ and $2b/(3b-2) \le a \le b$. Then

$$\zeta(a) = (a+b)\left[\frac{ab}{a+b} - (ab-1)\right] = b + (1+b-b^2)a - ba^2$$

is a PN type polynomial of a, and thereby, $\zeta(a) < 0$ for $2b/(3b-2) \le a \le b$ by Proposition 2.2 together with the fact that

$$\zeta\left(\frac{2b}{3b-2}\right) = -\frac{b^2}{(3b-2)^2}\left[\frac{2}{3} + 6\left(b - \frac{4}{3}\right)\left(b - \frac{7}{6}\right)\right] < 0.$$

This gives Lemma 5.2(i).
(ii) If $ab \le 2$ and $(1-a)(b-1) \ge 0$, then we have $0 < a \le 1$ and $1 \le b \le 2/a$. Hence

$$2(a+b) - 3ab = 2a + (2-3a)b \ge \min\left\{2a + (2-3a),\ 2a + (2-3a)\cdot\frac{2}{a}\right\} \ge 0.$$

(iii) If $a + b + 2 \ge 4ab$, then combining with $b \ge a > 0$, it can be reduced to

$$0 < a \le \frac{1}{4}, \quad b \ge a \quad \text{or} \quad \frac{1}{4} < a \le 1, \quad a \le b \le \frac{a+2}{4a-1}.$$

We divide the proof into three cases.

- If $0 < a \le 1/4$, then it is easy to see that $\tau(a, b) > 0$.
- If $1/4 < a \le 9/10$ and $b \ge a$, then it is a matter of simple transformation to verify that

$$\tau(a, b) = \begin{bmatrix} a^4(56-46a-a^2) + a^3(112-115a-3a^2)(b-a) \\ +a^2(94-100a-3a^2)(b-a)^2 \\ +\big(a(1-a)(32+a) + 2(3-2a)b\big)(b-a)^3 \end{bmatrix} > 0,$$

  it follows from

$$56 - 46a - a^2 = \frac{1379 + (9-10a)(469+10a)}{100} > 0,$$
$$112 - 115a - 3a^2 = \frac{607 + (9-10a)(1177+30a)}{100} > 0,$$
$$94 - 100a - 3a^2 = \frac{157 + (9-10a)(1027+30a)}{100} > 0,$$

  for $1/4 < a \le 9/10$.

- If $9/10 < a \le 1$ and $a \le b \le (a+2)/(4a-1)$, then it is clear that $1 < b \le (a+2)/(4a-1) < 29/26 < 6/5$, which by simplifying shows

$$\begin{aligned}\tau(a,b) &> 6a^4 + 2a^3(7-2a) + a^2(16-19a)\,(6/5)^2 + a(14-19a-a^2)\,(6/5)^3 + 2(3-2a)\\ &= \frac{2\left[207 + (1-a)\left(168 + 1430a + a^2(818-125a)\right)\right]}{125} > 0.\end{aligned}$$

This completes the proof. □

Lemma 5.3 *Let $f_n(p)$ be defined as in* (2.5). *Then $f_n(p) > 0$ for $n \ge 3$ if $0 \le p \le 1 + ab/(a+b)$.*

Proof We denote by Δf_n the difference of f_n, which is given explicitly by

$$\begin{aligned}\Delta f_n(p) &:= f_{n+1}(p) - f_n(p)\\ &= 3n^2 + (2a+2b+5)n + (a+b)(3+p) + ab - 5p - 2p^2 - 1.\end{aligned}$$

Simplifying leads to

$$\Delta f_1(p) = 7 + 5(a+b) + ab + (a+b-5)p - 2p^2,$$

which is a PN type polynomial of p and so $\Delta f_1(p) > 0$ for $0 \le p \le 1 + ab/(a+b)$ follows from

$$\Delta f_1\left(1 + \frac{ab}{a+b}\right) = \frac{6a^3 + 9a^2b + 2a^3b + 9ab^2 + 2a^2b^2 + 6b^3 + 2ab^3}{(a+b)^2} > 0$$

by Proposition 2.2. This gives $\Delta f_n(p) \ge \Delta f_1(p) > 0$ for $n \ge 1$, and thereby the sequence $\{f_n\}$ is strictly increasing for $n \ge 1$.

Note that

$$f_3(p) = 14(a+b) + 5ab + 27 + (a+b+ab-15)p - (a+b+6)p^2$$

is also a PN type polynomial of p, which is positive for $n \ge 3$ by Proposition 2.2 since

$$f_3\left(1 + \frac{ab}{a+b}\right) = 6 + 5(a+b) + \frac{9a^3 + 5a^3b + 4a^2b^2 + 9b^3 + 5ab^3}{(a+b)^2} > 0.$$

By the monotonicity of $\{f_n\}$, we conclude that $f_n(p) \ge f_3(p) > 0$ for $n \ge 3$. □

Lemma 5.4 *Let $g_n(p)$ be defined as in* (2.6). *Then $g_n(p) \ge 0$ for $n \ge 6$ if (a, b, p) satisfies one of the following conditions*

(i) $(a,b) \in \mathcal{R}_1$ *and* $\max\{ab/(a+b), ab-1\} \le p \le 1$;
(ii) $(a,b) \in \mathcal{R}_2$ *and* $\max\{ab-1, 0\} \le p < ab/(a+b)$;
(iii) $(a,b) \in \mathcal{R}_3$ *and* $\max\{ab-1, 1\} \le p \le 1 + ab/(a+b)$.

Proof We first prove $g_n(p) \ge 0$ for $n \ge 1$ in all cases when $p = ab - 1$. For the time being, it reduces to $g_n(p) = ab(1-a)(b-1)(ab+1)$ for all $n \ge 1$.

- If $(a,b) \in \mathcal{R}_1$ and $\max\{ab/(a+b), ab-1\} \le p \le 1$, then it follows from $p = ab-1$ that $ab/(a+b) \le ab-1$ and so $(a,b) \in \mathcal{R}_{12}$ by Lemma 2.1. This gives $(1-a)(b-1) \ge 0$.
- If $(a,b) \in \mathcal{R}_2$ and $\max\{ab-1, 0\} \le p < ab/(a+b)$, then it is clear that $(1-a)(b-1) \ge 0$.
- If $(a,b) \in \mathcal{R}_3$ and $\max\{ab-1, 1\} \le p \le 1 + ab/(a+b)$, then $p = ab-1 \ge 1$, that is, $ab \ge 2$. This together with $a+b \ge 4ab-2$ implies that $(1-a)(b-1) = a+b-ab-1 \ge 3(ab-1) \ge 3$.

In the following, we focus on $p > ab - 1$. In this case, by (2.6), g_n is increasing for $n \ge 1$. It suffices to prove that $g_6(p) \ge 0$ in all cases.

Simplifying gives

$$g_6(p) = 2(a+b-4ab+2) + (3a+3b-ab+1)p + (a+b-4)p^2 - p^3. \tag{5.1}$$

We divide the proof into three cases.

Case 5.1: $(a,b) \in \mathcal{R}_1$ and $\max\{ab/(a+b), ab-1\} \le p \le 1$. Note that $ab \le 2$ for $(a,b) \in \mathcal{R}_1$, then $ab/(a+b) \le 1$. Otherwise, $ab > a+b \ge 2\sqrt{ab}$ and thereby $ab > 4$, which deduces a contradiction with $ab \le 2$. In other words, it is feasible to take $\max\{ab/(a+b), ab-1\} \le p \le 1$.

In this case, it is easy to see that $g_6'(p) = 1 + 3a + 3b - ab + 2(a+b-4)p - 3p^2$ is a PN type polynomial of p. By Proposition 2.2, we conclude that $g_6(p)$ is increasing on [0, 1] if $g_6'(1) \ge 0$ or there exists $p_2 \in (0,1)$ such that $g_6(p)$ is strictly increasing on $[0, p_2]$ and strictly decreasing on $[p_2, 1]$ if $g_6'(1) < 0$. In either case, in order to prove $g_6(p) \ge 0$, it suffices to show $g_6(\max\{ab/(a+b), ab-1\}) \ge 0$ and $g_6(1) \ge 0$ for $(a,b) \in \mathcal{R}_1$. We divide into two subcases to complete the proof by Lemma 2.1.

Subcase 5.1-1: $(a,b) \in \mathcal{R}_{11}$ and $ab \le 2$. In this case, $\kappa(a,b) \ge 0$ and $ab/(a+b) > ab-1$, and so $ab/(a+b) \le p \le 1$. This in conjunction with Lemma 5.2(i) implies

$$g_6\left(\frac{ab}{a+b}\right) = \frac{\kappa(a,b)}{(a+b)^3} \ge 0 \quad \text{and} \quad g_6(1) = 3\left[2(a+b) - 3ab\right] > 0.$$

Subcase 5.1-2: $(a,b) \in \mathcal{R}_{12}$ and $ab \le 2$.

In this case, $(1-a)(b-1) \ge 0$ and $ab/(a+b) \le ab-1$, and so $ab-1 \le p \le 1$. According to this with Lemma 5.2(ii), it follows that

$$g_6\,(ab-1) = ab(1-a)(b-1)(ab+1) \ge 0 \quad \text{and} \quad g_6(1) = 3\left[2(a+b) - 3ab\right] \ge 0. \tag{5.2}$$

Case 5.2: $(a,b) \in \mathcal{R}_2$ and $\max\{ab-1, 0\} \le p < ab/(a+b)$.

In this case, $(1-a)(b-1) \ge 0$ and $ab-1 < ab/(a+b)$. Then it follows from Lemma 5.2(i) that $2(a+b) > 3ab$ and so $ab-1 \le p < ab/(a+b) < 1$. As in the proof of *Case 5.1*, $g_6(p) \ge 0$ is either strictly increasing on (0, 1) or piecewise monotonic on (0, 1) (precisely, firstly increasing and then decreasing). This together with (5.2) implies $g_6(p) \ge 0$ by Proposition 2.2 for

$$g_6(p) \ge \min\{g_6(ab-1), g_6(1)\} \ge 0.$$

Case 5.3: $(a,b) \in \mathcal{R}_3$ and $\max\{ab-1, 1\} \le p \le 1 + ab/(a+b)$ with $p \ne 1$.

In this case, $a+b+2 \ge 4ab$ and $ab-1 \le 1 + ab/(a+b)$. First, by (5.1), $g_6(p)$ is a PN type polynomial of p. Moreover, according to Lemma 5.2(iii), it follows that

$$g_6\left(1 + \frac{ab}{a+b}\right) = \frac{\tau(a,b)}{(a+b)^3} > 0.$$

This in conjunction with Proposition 2.2 gives $g_6(p) > 0$.

This completes the proof. □

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