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ORIGINAL RESEARCH

Zeros and uniqueness problems related to $F^{(k)} - \alpha(z)$

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Abstract This paper is to establish new results on the zeros of $F^{(k)} - \alpha(z)$, where $F(z)$ is a differential polynomial or difference polynomial of f and $\alpha(z)$ is a small function with respect to f in the sense of Nevanlinna theory. We also obtain that at least one of $F^{(k)} - \alpha(z)$ and $G^{(k)} - \alpha(z)$ has infinitely many zeros, where $F(z)$ and $G(z)$ are crossed differential polynomials or difference polynomials of f and g.

Keywords Zeros · Crossed differential polynomials · Crossed difference polynomials · Meromorphic functions · Value sharing

Mathematics Subject Classification 30D35

1 Introduction

We assume that the reader is familiar with the elementary notations of Nevanlinna theory, see, e.g., [16, 26]. Denote $\mathcal{M}$ for the class of transcendental meromorphic functions and $\mathcal{M}'$ for the class of transcendental meromorphic functions of hyper-order less than one. Denote $\mathcal{E}$ for the class of transcendental entire functions and $\mathcal{E}'$ for the class of transcendental entire functions of hyper-order less than one.

A finite value a is called the generalized Picard exceptional value means that $f(z) - a$ has at most finitely many zeros. The well-known Picard theorem shows that a transcendental meromorphic function $f(z)$ has at most two generalized Picard exceptional values and a transcendental entire function $f(z)$ has at most one generalized Picard exceptional value. Furthermore, Hayman obtained many important results in [15] to consider the zeros of complex differential polynomials, one of the results therein is stated as follows.

Theorem A [15, Theorem 10] *Let $f(z) \in \mathcal{E}$ and $n \geq 2$ be a positive integer. Then $f(z)^n f'(z) - a$ has infinitely many zeros, where a is a non-zero constant.*

Theorem A shows that the possible generalized Picard exceptional value of $f(z)^n f'(z)$ is only zero. Theorem A is also true for transcendental meromorphic functions in $\mathcal{M}$, where the case of $n \geq 3$ is given by Hayman [15, Corollary of Theorem 9], the case of $n = 2$ is given by Mues [22] and $n = 1$ is given by Bergweiler and Eremenko

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[3, Theorem 2], Chen and Fang [5] and Zalcman [28], respectively. By the above results, for any meromorphic function $f(z)$, then $(f(z)^{n+1})' - a$ must have infinitely many zeros by observing that $\left(\frac{f^{n+1}}{n+1}\right)' = f^n f'$, where a is a non-zero constant. Wang and Fang [24, Corollary 1] generalized Theorem A by proving the following theorem.

Theorem B *Let $f \in \mathcal{M}$ and n, k be two positive integers with $n \geq k+1$. Then $(f^n)^{(k)} - a$ has infinitely many zeros, where a is a non-zero constant.*

We recall the recent progress on Theorem B, An and Phuong [1] presented that Theorem B is true for any integer $n \geq 2$, it is an important progress for the reason that there is no any restriction between n and k. Huang and Wang [17, Corollary 2] presented that Theorem B is also valid when the non-zero constant a is replaced by a non-zero polynomial, which is an important application by Lemma 2.3 below given by Fang and Wang [11]. However, as far as we know, it remains an open problem that Theorem B is true or not for the case that a is a non-zero rational function or a small function with respect to $f(z)$ in the sense of Nevanlinna theory. Difference analogue of Theorem A is firstly given by Laine and Yang [19], some further results can be found in [21, Chapter 2]. We state the following result, see [21, Theorem 2.1.8].

Theorem C *Let $f(z) \in \mathcal{M}'$ and $n \geq 4$ be a positive integer. Then $f(z)^n f(z+c) - a(z)$ has infinitely many zeros, where $a(z)$ is a non-zero small function with respect to $f(z)$.*

We also need to know that the condition $n \geq 4$ is sharp, there are counterexamples to show that Theorem C is not true for $n \leq 3$, see [21, Remarks 2.1.4, 2.1.9]. To introduce the main results in the paper, two polynomials will be given firstly. Let

$$P(z) = b(z-\alpha_1)^{m_1} \cdots (z-\alpha_t)^{m_t}$$

and

$$Q(z) = d(z-\beta_1)^{n_1} \cdots (z-\beta_s)^{n_s},$$

where $m_1 + \cdots + m_t = p$, $n_1 + \cdots + n_s = q$, $\alpha_1, \cdots, \alpha_t$ are distinct constants and $\beta_1, \cdots, \beta_s$ are distinct constants. Thus, $P(z)$ is a polynomial of degree p with t different zeros and $Q(z)$ is a polynomial of degree q with s different zeros. Secondly, denote $\triangle_c f = f(z+c) - f(z)$, where c is a non-zero constant. Without loss of generality, we also assume that $q > p$ in our results. The following result can be seen as the difference counterparts to Theorem B given by Gao and Liu [13, Theorems 1.1 and 1.2] and proved by an important result given by Yamanoi [25].

Theorem D *Let k be a positive integer and a be a non-zero constant.*

(i) *Let $f(z) \in \mathcal{M}'$. If $q \geq p+s+t+1$, then $[Q(f(z))P(f(z+c))]^{(k)} - a$ has infinitely many zeros; if $q \geq 2p+s+2t+1$ and $\Delta_c f \not\equiv 0$, then $[Q(f(z))P(\Delta_c f)]^{(k)} - a$ has infinitely many zeros.*
(ii) *Let $f(z) \in \mathcal{E}'$. If $q+p \geq s+t+1$, then $[Q(f(z))P(f(z+c))]^{(k)} - a$ has infinitely many zeros; if $q \geq p+s+t+1$ and $\Delta_c f \not\equiv 0$, then $[Q(f(z))P(\Delta_c f)]^{(k)} - a$ has infinitely many zeros.*

We will improve Theorem D below by a result given by Fang and Wang [11].

Theorem 1.1 *Let k be a positive integer and $p(z)$ be a non-zero polynomial.*

(i) *Let $f(z) \in \mathcal{M}'$. If $q \geq p+s+t+1$, then $[Q(f(z))P(f(z+c))]^{(k)} - p(z)$ has infinitely many zeros; if $q \geq 2p+s+2t+1$ and $\Delta_c f \not\equiv 0$, then $[Q(f(z))P(\Delta_c f)]^{(k)} - p(z)$ has infinitely many zeros.*
(ii) *Let $f(z) \in \mathcal{E}'$. If $q+p \geq s+t+1$, then $[Q(f(z))P(f(z+c))]^{(k)} - p(z)$ has infinitely many zeros; if $q \geq p+s+t+1$ and $\Delta_c f \not\equiv 0$, then $[Q(f(z))P(\Delta_c f)]^{(k)} - p(z)$ has infinitely many zeros.*

Suppose that $Q(f(z))P(f(z+c))$ reduces into $f(z)^n f(z+c)$ or $f(z)^n \Delta_c f$, that is $q = n$, $s = 1$, $p = 1$ and $t = 1$, we have the following corollary.

Corollary 1.1 *Let k be a positive integer and $p(z)$ be a non-zero polynomial.*

(i) *Let $f(z) \in \mathcal{M}'$. If $n \geq 4$, then $[f(z)^n f(z+c)]^{(k)} - p(z)$ has infinitely many zeros; if $n \geq 6$, then $[f(z)^n \Delta_c f]^{(k)} - p(z)$ has infinitely many zeros.*
(ii) *Let $f(z) \in \mathcal{E}'$. If $n \geq 2$, then $[f(z)^n f(z+c)]^{(k)} - p(z)$ has infinitely many zeros; if $n \geq 4$, then $[f(z)^n \Delta_c f]^{(k)} - p(z)$ has infinitely many zeros.*

Remark 1.1 (1) It remains an open problem that whether Theorem 1.1 is true for the case that $p(z)$ is any non-zero small function with respect to $f(z)$ in the sense of Nevanlinna theory.

(2) The condition of finite order can not be removed generally in Corollary 1.1. Since the non-linear difference equations $f(z)^n f(z+c) = s(z)$ and $f(z)^n \Delta_c f = s(z)$ can admit transcendental entire solutions with infinite order, where $s(z)$ is a non-zero polynomial.

Gao and Liu [12, Theorem 1.1] also considered the zero distribution on the crossed complex delay-differential polynomials of different types, namely the zero distribution of $f(z)^n L(g) - a(z)$ and $g(z)^n L(f) - a(z)$, where $L(h)$ takes the derivatives $h^{(k)}(z)(k \geq 1)$ or the shift $h(z+c)$ or the difference $h(z+c) - h(z)(\not\equiv 0)$ or the delay-differential $h^{(k)}(z+c)(k \geq 1)$, where c is a non-zero constant and $a(z)$ is a non-zero small function with respect to $f(z)$ and $g(z)$ in the sense of Nevanlinna theory.

Theorem E *If one of the following conditions is satisfied:*

(i) $L(h) = h^{(k)}(z)$, $n \geq k+4$ *and* $h \in \mathcal{M}$ *or* $n \geq 3$ *and* $h \in \mathcal{E}$;
(ii) $L(h) = h(z+c)$, $n \geq 4$ *and* $h \in \mathcal{M}'$ *or* $n \geq 3$ *and* $h \in \mathcal{E}'$;
(iii) $L(h) = h(z+c) - h(z)$, $n \geq 5$ *and* $h \in \mathcal{M}'$ *or* $n \geq 3$ *and* $h \in \mathcal{E}'$;
(iv) $L(h) = h^{(k)}(z+c)$, $n \geq k+4$ *and* $h \in \mathcal{M}'$ *or* $n \geq 3$ *and* $h \in \mathcal{E}'$;

then at least one of $f(z)^n L(g) - a(z)$ *and* $g(z)^n L(f) - a(z)$ *has infinitely many zeros, where* $a(z)$ *is a non-zero small function with respect to* $f(z)$ *and* $g(z)$.

In this paper, we obtain the following theorem.

Theorem 1.2 *Let* $f, g \in \mathcal{M}'$, k *be a positive integer,* $p_1(z)$ *and* $p_2(z)$ *be two non-zero polynomials. If* $q \geq p+s+t+1$, *then at least one of* $[Q(f)P(g(z+c))]^{(k)} - p_1(z)$ *and* $[Q(g)P(f(z+c))]^{(k)} - p_2(z)$ *has infinitely many zeros; if* $q \geq 2p+s+2t+1$, *then at least one of* $[Q(f(z))P(\Delta_c g)]^{(k)} - p_1(z)$ *and* $[Q(g(z))P(\Delta_c f)]^{(k)} - p_2(z)$ *has infinitely many zeros.*

Corollary 1.2 *Let* $f, g \in \mathcal{M}'$, k *be a positive integer,* $p_1(z)$ *and* $p_2(z)$ *be two non-zero polynomials. If* $n \geq 4$, *then at least one of* $[f(z)^n g(z+c)]^{(k)} - p_1(z)$ *and* $[g(z)^n f(z+c)]^{(k)} - p_2(z)$ *has infinitely many zeros; if* $n \geq 6$, *then at least one of* $[f(z)^n \Delta_c g]^{(k)} - p_1(z)$ *and* $[g(z)^n \Delta_c f]^{(k)} - p_2(z)$ *has infinitely many zeros.*

Remark 1.2 We have not obtained the corresponding result within Theorem 1.1 for $f, g \in \mathcal{E}'$ by the method in the proofs below. The reason is that, related to $[Q(f)P(g(z+c))]^{(k)}$ and $[Q(f(z))P(\Delta_c g)]^{(k)}$, we can not get the corresponding calculation as (2.8) and (2.10).

We proceed to consider the uniqueness problem when $[f(z)^n]^{(k)}$ and $[g(z)^n]^{(k)}$ share a common value. Fang [9, Theorem 1] obtained the following result, the original consideration is devoted to Yang and Hua [27] to consider the uniqueness problem on $f(z)^n f'(z) - a$ and $g(z)^n g'(z) - a$ having the same zeros with the same multiplicities, that is $f(z)^n f'(z)$ and $g(z)^n g'(z)$ share one non-zero constant a CM, CM is the abbreviation of counting multiplicities.

Theorem F *Let* $f, g \in \mathcal{E}$ *and* n, k *be two positive integers with* $n > 2k+4$. *If* $[f(z)^n]^{(k)}$ *and* $[g(z)^n]^{(k)}$ *share the value 1 CM, then either* $f = c_1 e^{cz}$ *and* $g = c_2 e^{-cz}$, *where* c, c_1 *and* c_2 *are constants and* $(-1)^k (c_1 c_2)^n (nc)^{2k} = a^2$ *or* $f = tg$, *where* $t^n = 1$.

There are many improvements of Theorem F which can be found in [4,6,8,10,20,29]. The case of $k = 1$ that is $f^{n-1} f'$ and $g^{n-1} g'$ share the value $\frac{1}{n}$ CM from Theorem F. The cross-sharing problem, considered by Wang and Liu [23], implies to consider f^{n-1} and g' share common values together with g^{n-1} and f' share common values.

Theorem G [12, Corollary 1.6] *Let* $f, g \in \mathcal{E}$ *and* $n \geq 8$. *If* $f(z)^n g'(z)$ *and* $g(z)^n f'(z)$ *share a non-zero constant* a *CM, then* $f(z) = tg(z)$ *and* $t^{n-1} = 1$ *or* $f(z) = c_1 e^{cz}$ *and* $g(z) = c_2 e^{-cz}$ *where* c, c_1 *and* c_2 *are constants and* $(c_1 c_2)^{n+1} c^2 = a^2$.

We will improve Theorem G by giving the following theorem. Before that, we first to consider the multiplicity of zeros of $(f-\beta_1)^{n_1} \cdots (f-\beta_s)^{n_s}$. Let u_1 satisfy the condition that the multiplicity of any zeros of $(f-\beta_{u_1+1})^{n_{u_1+1}}, \cdots, (f-\beta_s)^{n_s}$ are great than or equal to $2+k$ and at least one of multiplicity of zeros of $(f-\beta_1)^{n_1}, \cdots, (f-\beta_{u_1})^{n_{u_1}}$ is less than $2+k$. For the zeros of $(g'-\alpha_1)^{m_1} \cdots (g'-\alpha_t)^{m_t}$, let v_1 satisfy the condition that the multiplicity of any zeros of $(g'-\alpha_{v_1+1})^{n_{v_1+1}}, \cdots, (g'-\alpha_t)^{n_t}$ are great than or equal to $2+k$ and at least one of multiplicity of zeros of $(g'-\alpha_1)^{n_1}, \cdots, (g'-\alpha_{v_1})^{n_{v_1}}$ is less than $2+k$. Similarly as the above, for $(g-\beta_1)^{n_1} \cdots (g-\beta_s)^{n_s}$ and $(f'-\alpha_1)^{m_1} \cdots (f'-\alpha_t)^{m_t}$, we can define u_2 and v_2, where $0 \leq u_1, u_2 \leq s$ and $0 \leq v_1, v_2 \leq t$.

Theorem 1.3 *Let $f, g \in \mathcal{M}$, k be a positive integer, u_1, u_2, v_1, v_2 be defined as the above and*

$$q \geq 2p + \Bigg[2k + 2(k+2)\max((s + 2t - u_1 - 2v_2), (s + 2t - u_2 - 2v_1)) + 2\max\left(\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i\right) + 4\max\left(\sum_{j=1}^{v_1} m_j, \sum_{j=1}^{v_2} m_j\right) + 8\Bigg] + 1.$$

If $[Q(f)P(g')]^{(k)}$ and $[Q(g)P(f')]^{(k)}$ share a non-zero small function $\alpha(z)$ CM, then we have either

$$[Q(f)P(g')]^{(k)} \equiv [Q(g)P(f')]^{(k)}$$

or

$$[Q(f)P(g')]^{(k)} \cdot [Q(g)P(f')]^{(k)} \equiv \alpha^2.$$

For $[Q(f)P(g')]^{(k)} \equiv [Q(g)P(f')]^{(k)}$, if $f = gh$ and h is a constant, then we have either $h^q = h^p$ when $Q(z) = a_q z^q$ and $P(z) = b_p z^p$ or $h^\mu = 1$, where μ is the greatest common divisor of integers from 1 to q satisfying the corresponding coefficients $a_j (1 \leq j \leq q)$ and $b_i (1 \leq i \leq p)$ of $Q(z)$ and $P(z)$ are non-zero constants.

Corollary 1.3 *Let $f, g \in \mathcal{E}$ and $n \geq 2k + 8$. If $[f(z)^n g'(z)]^{(k)}$ and $[g(z)^n f'(z)]^{(k)}$ share a non-zero constant a CM, then either*

$$[f(z)^n g'(z)]^{(k)} \equiv [g(z)^n f'(z)]^{(k)}$$

or

$$[f(z)^n g'(z)]^{(k)} \cdot [g(z)^n f'(z)]^{(k)} \equiv a^2.$$

For $[f(z)^n g'(z)]^{(k)} \equiv [g(z)^n f'(z)]^{(k)}$, if $f = tg$ and t is a constant, then $t^{n-1} = 1$. Furthermore, $[f(z)^n g'(z)]^{(k)} \cdot [g(z)^n f'(z)]^{(k)} \equiv a^2$ implies $f(z) = c_1 e^{Cz}$ and $g(z) = c_2 e^{-Cz}$, where $(-1)^{k+1}(c_1 c_2)^{n+1}[(n-1)C]^{2k} C^2 = a^2$.

The difference analogue of Theorem G is given in [12, Corollary 1.8].

Theorem H *Let $f, g \in \mathcal{E}'$ and $n \geq 8$. If $f(z)^n g(z + c)$ and $g(z)^n f(z + c)$ share a non-zero constant a CM, then $f(z) = c_1 g(z)$ where $c_1^{n-1} = 1$ or $f(z)g(z) = c_2$ where $c_2^{n+1} = a^2$.*

We will establish an improvement of Theorem H below. Furthermore, we consider the multiplicity of zeros of $(g(z + c) - \alpha_1)^{m_1}(g(z + c) - \alpha_t)^{m_t}$, let v_3 satisfy the condition that the multiplicity of any zeros of $(g(z + c) - \alpha_{v_3+1})^{m_{v_3+1}}, \cdots, (g(z + c) - \alpha_t)^{m_t}$ are great than or equal to $2 + k$ and at least one of multiplicity of zeros of $(g(z + c) - \alpha_1)^{m_1}, \cdots, (g(z + c) - \alpha_{v_3})^{m_{v_3}}$ is less than $2 + k$. Similarly as the above, for $(f(z + c) - \alpha_1)^{m_1} \cdots (f(z + c) - \alpha_s)^{m_t}$, we can define v_4, where $0 \leq v_3, v_4 \leq t$.

Theorem 1.4 *Let $f, g \in \mathcal{M}'$, k be a positive integer, u_1, u_2, v_3, v_4 be defined as the above and*

$$q \geq p + \Bigg[2k + 2(k+2)\max((s + t - u_1 - v_4), (s + t - u_2 - v_3)) + 2\max\left(\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i\right) + 2\max\left(\sum_{j=1}^{v_3} m_j, \sum_{j=1}^{v_4} m_j\right) + 8\Bigg] + 1.$$

If $[Q(f)P(g(z + c))]^{(k)}$ and $[Q(g)P(f(z + c))]^{(k)}$ share a non-zero small function $\alpha(z)$ CM, then either

$$[Q(f)P(g(z + c))]^{(k)} \equiv [Q(g)P(f(z + c))]^{(k)}$$

or

$$[Q(f)P(g(z + c))]^{(k)} \cdot [Q(g)P(f(z + c))]^{(k)} \equiv \alpha^2.$$

For $[Q(f)P(g(z + c))]^{(k)} \equiv [Q(g)P(f(z + c))]^{(k)}$, if $f = gh$ and h is a constant, then either $h^q = h^p$ when $Q(z) = a_q z^q$ and $P(z) = b_p z^p$ or $h^\mu = 1$, where μ is the greatest common divisor of integers from 1 to q satisfying the corresponding coefficients $a_j (1 \leq j \leq q)$ and $b_i (1 \leq i \leq p)$ of $Q(z)$ and $P(z)$ are non-zero constants.

Corollary 1.4 *Let f, $g \in \mathcal{M}'$ and n, k be two positive integers with $n \geq 4k+16$. If $[f(z)^n g(z+c)]^{(k)}$ and $[g(z)^n f(z+c)]^{(k)}$ share the value 1 CM, then either*

$$[f(z)^n g(z+c)]^{(k)} \equiv [g(z)^n f(z+c)]^{(k)}$$

or

$$[f(z)^n g(z+c)]^{(k)} \cdot [g(z)^n f(z+c)]^{(k)} \equiv 1.$$

For $[f(z)^n g(z+c)]^{(k)} \equiv [g(z)^n f(z+c)]^{(k)}$, if f=tg and t is a constant, then $t^{n-1} = 1$. Furthermore, $[f(z)^n g(z+c)]^{(k)} \cdot [g(z)^n f(z+c)]^{(k)} \equiv 1$ implies $f(z) = c_1 e^{Cz}$ and $g(z) = c_2 e^{-Cz}$ when f, $g \in \mathcal{E}'$, where $(-1)^k (c_1 c_2)^{n+1} [(n-1)C]^{2k} = 1$.

Remark 1.3 Recall the following two equations in Theorems 1.3 and 1.4,

$$[Q(f)P(g')]^{(k)} \cdot [Q(g)P(f')]^{(k)} \equiv \alpha^2,$$

and

$$[Q(f)P(g(z+c))]^{(k)} \cdot [Q(g)P(f(z+c))]^{(k)} \equiv \alpha^2.$$

It remains an open question to describe the relations between f and g provided that f, $g \in \mathcal{M}$ or $\mathcal{M}'$.

2 Lemmas

In this section, we present some basic facts and results which will be used in the next section.

Lemma 2.1 [26, Theorem 1.7'] *Let $f \in \mathcal{M}$. If the order of f is finite, then*

$$S(r, f) = o(T(r, f)), (r \to \infty).$$

If the order of f is infinite, then

$$S(r, f) = o(T(r, f)), (r \to \infty, r \notin E),$$

where E is a set with finite linear or logarithmic measure.

Lemma 2.2 [11, Theorem 20] *Let $f \in \mathcal{M}$, $k \geq 1$ be an integer and $\varepsilon > 0$. Then*

$$(1-\varepsilon)T(r, f) \leq \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - 1}\right) + S(r, f). \tag{2.1}$$

Lemma 2.3 [11, Proposition 3] *Let $f \in \mathcal{M}$, $k \geq 1$ be an integer, $\varepsilon > 0$ and $p(z) \not\equiv 0$ be a polynomial. Then*

$$(1-\varepsilon)T(r, f) + (k-1)\overline{N}(r, f) \leq \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - p(z)}\right) + S(r, f). \tag{2.2}$$

Remark 2.1 Hayman inequality [15], namely,

$$T(r, f) \leq \left(2 + \frac{1}{k}\right) N\left(r, \frac{1}{f}\right) + \left(2 + \frac{2}{k}\right) N\left(r, \frac{1}{f^{(k)} - 1}\right) + S(r, f), \tag{2.3}$$

where $f(z) \in \mathcal{M}$ and $k \geq 1$ is a positive integer. Lemma 2.3 is an improvement of Hayman inequality on the coefficients and plays an important role in the paper.

Lemma 2.4 [14, Lemma 8.3] *Let $f \in \mathcal{M}'$. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f), \tag{2.4}$$

$$N(r, f(z+c)) = N(r, f) + S(r, f) \tag{2.5}$$

and

$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.6}$$

Lemma 2.5 [13, Lemma 2.4] *Let $f \in \mathcal{M}'$. Then*

$$(q-p)T(r,f)+S(r,f) \le T(r,Q(f(z))P(f(z+c))) \le (p+q)T(r,f)+S(r,f). \tag{2.7}$$

Let $f \in \mathcal{E}'$. Then

$$T(r,Q(f(z))P(f(z+c))) = (p+q)T(r,f)+S(r,f). \tag{2.8}$$

Lemma 2.6 [13, Lemma 2.5] *Let $f \in \mathcal{M}'$ and $\Delta_c f \not\equiv 0$. Then*

$$(q-2p)T(r,f)+S(r,f) \le T(r,Q(f)P(\Delta_c f)) \le (q+2p)T(r,f)+S(r,f). \tag{2.9}$$

Let $f \in \mathcal{E}'$ and $\Delta_c f \not\equiv 0$. Then

$$(q-p)T(r,f)+S(r,f) \le T(r,Q(f)P(\Delta_c f)) \le (q+p)T(r,f)+S(r,f). \tag{2.10}$$

Lemma 2.7 [26, Theorem 1.22] *Let $f \in \mathcal{M}$ and k be a positive integer. Then*

$$T(r,f^{(k)}) \le T(r,f)+k\overline{N}(r,f)+S(r,f). \tag{2.11}$$

Let p be a positive integer and a be a constant. Denote the counting function of the zeros of $f-a$ where an m-fold zero is counted m times if $m \le p$ and p times if $m > p$ by $N_p(r,\frac{1}{f-a})$. The following lemma can be obtained immediately from the proof of [18, Lemma 2.3] which includes the special case $p=2$ and mentioned in [30, Lemma 2.1].

Lemma 2.8 *Let $f \in \mathcal{M}$ and k be a positive integer. Then*

$$N_p\left(r,\frac{1}{f^{(k)}}\right) \le T(r,f^{(k)})-T(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f), \tag{2.12}$$

$$N_p\left(r,\frac{1}{f^{(k)}}\right) \le k\overline{N}(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f). \tag{2.13}$$

Lemma 2.9 [2, Lemma 2] *Let $F, G \in \mathcal{M}$ and $\alpha(z)$ be a non-zero small function with respect to $F(z)$ and $G(z)$. If $F(z)$ and $G(z)$ share $\alpha(z)$ CM, then one of the following three cases holds:*

(i)

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)+S(r,F)+S(r,G)$$

and

$$T(r,G) \le N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)+S(r,F)+S(r,G);$$

(ii) $F \equiv G$;
(iii) $F \cdot G \equiv \alpha^2$.

Lemma 2.10 *Let $f, g \in \mathcal{M}$. If*

$$[Q(f)P(g')]^{(k)} \equiv [Q(g)P(f')]^{(k)} \tag{2.14}$$

and $q \ge 2p+2(s+2t+1)+1$, then $Q(f)P(g') \equiv Q(g)P(f')$. Furthermore, if $f=gh$ and h is a constant, then either $h^q = h^p$ when $Q(z)=a_q z^q$ and $P(z)=b_p z^p$ or $h^\mu = 1$, where μ is the greatest common divisor of integers from 1 to q satisfying the corresponding coefficients $a_j (1 \le j \le q)$ and $b_i (1 \le i \le p)$ of $Q(z)$ and $P(z)$ are non-zero constants.

Proof Integrating (2.14), we obtain

$$Q(f)P(g') \equiv Q(g)P(f') + H(z),$$

where $H(z)$ is a polynomial of $\deg(H(z)) \leq k-1$. If $H(z) \not\equiv 0$, then

$$\frac{Q(f)P(g')}{H(z)} = \frac{Q(g)P(f')}{H(z)} + 1.$$

From the second main theorem of Nevanlinna theory for three small functions [16, Theorem 2.5], we have

$$\begin{aligned}
qT(r, f(z)) - 2pT(r, g(z)) &\leq qT(r, f(z)) - pT(r, g'(z)) + S(r, f) + S(r, g)\\
&\leq T\left(r, \frac{Q(f)P(g')}{H(z)}\right) + S(r, f) + S(r, g)\\
&\leq \overline{N}\left(r, \frac{Q(f)P(g')}{H(z)}\right) + \overline{N}\left(r, \frac{H(z)}{Q(f)P(g')}\right) + \overline{N}\left(r, \frac{H(z)}{Q(g)P(f')}\right)\\
&+ S(r, f) + S(r, g)\\
&\leq \overline{N}(r, f) + \overline{N}(r, g) + s\overline{N}\left(r, \frac{1}{f}\right) + t\overline{N}\left(r, \frac{1}{g'}\right)\\
&+ s\overline{N}\left(r, \frac{1}{g}\right) + t\overline{N}\left(r, \frac{1}{f'}\right) + S(r, f) + S(r, g)\\
&\leq (s + 2t + 1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g).
\end{aligned}$$

Similarly, we can conclude that

$$qT(r, g(z)) - 2pT(r, f(z)) \leq (s + 2t + 1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g).$$

Thus, we get

$$(q - 2p)(T(r, f) + T(r, g)) \leq 2(s + 2t + 1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g),$$

which is a contradiction with $q \geq 2p+2(s+2t+1)+1$. Hence, we get $H(z) \equiv 0$ when $q \geq 2p+2(s+2t+1)+1$. It implies that

$$Q(f)P(g') \equiv Q(g)P(f'). \tag{2.15}$$

Set $h = \frac{f}{g}$. If h is a constant, substituting $f = gh$ into (2.15), we have

$$\frac{Q(gh)}{Q(g)} \equiv \frac{P(g'h)}{P(g')}. \tag{2.16}$$

From (2.16) and Valiron-Mohon'ko Theorem [26, Theorem 1.13], we have

$$qT(r, g) = pT(r, g') + S(r, g) \leq 2pT(r, g) + S(r, g),$$

thus $q \leq 2p$, which is a contradiction with $q \geq 2p+2(s+2t+1)+1$. Thus, both $\frac{Q(gh)}{Q(g)}$ and $\frac{P(g'h)}{P(g')}$ are constants. Assume that $Q(z) = a_q z^q + a_{q-1}z^{q-1} + \cdots + a_0$ and $P(z) = b_p z^p + b_{p-1}z^{p-1} + \cdots + b_0$, where $a_q \neq 0$ and $b_p \neq 0$. From (2.16), we have

$$\frac{[a_q(gh)^q + a_{q-1}(gh)^{q-1} + \cdots + a_0]}{[a_q g^q + a_{q-1}g^{q-1} + \cdots + a_0]} = \frac{[b_p(g'h)^p + b_{p-1}(g'h)^{p-1} + \cdots + b_0]}{[b_p(g')^p + b_{p-1}(g')^{p-1} + \cdots + b_0]}.$$

From the above equation, then either $h^q = h^p$ when all $a_0, \cdots, a_{q-1}, b_0, \cdots, b_{p-1}$ are vanishing or $h^\mu = 1$, μ is the greatest common divisor of index 1 to q such that $a_1, \cdots, a_{q-1}, b_1, \cdots, b_{p-1}$ are not zeros. □

Lemma 2.11 *Let $f, g \in \mathcal{M}'$. If*

$$[Q(f)P(g(z+c))]^{(k)} \equiv [Q(g)P(f(z+c))]^{(k)} \tag{2.17}$$

and $q \geq p + 2(s+t+1) + 1$, then $Q(f)P(g(z+c)) \equiv Q(g)P(f(z+c))$. Furthermore, if $f = gh$ and h is a constant, then either $h^q = h^p$ when $Q(z) = a_q z^q$ and $P(z) = b_p z^p$ or $h^\mu = 1$ where μ is the greatest common divisor of integers from 1 to q satisfying the corresponding coefficients $a_j (1 \leq j \leq q)$ and $b_i (1 \leq i \leq p)$ of $Q(z)$ and $P(z)$ are non-zero constants.

Proof Integrating (2.17), we get

$$Q(f)P(g(z+c)) \equiv Q(g)P(f(z+c)) + H(z),$$

where $H(z)$ is a polynomial of $\deg(H(z)) \leq k-1$. If $H(z) \not\equiv 0$, then

$$\frac{Q(f)P(g(z+c))}{H(z)} = \frac{Q(g)P(f(z+c))}{H(z)} + 1.$$

From the second main theorem of Nevanlinna theory for three small functions [16, Theorem 2.5], we have

$$\begin{aligned}
qT(r, f(z)) - pT(r, g(z)) &\leq T\left(r, \frac{Q(f)P(g(z+c))}{H(z)}\right) + S(r, f) + S(r, g) \\
&\leq \overline{N}\left(r, \frac{Q(f)P(g(z+c))}{H(z)}\right) + \overline{N}\left(r, \frac{H(z)}{Q(f)P(g(z+c))}\right) \\
&+ \overline{N}\left(r, \frac{H(z)}{Q(g)P(f(z+c))}\right) + S(r, f) + S(r, g) \\
&\leq \overline{N}(r, f) + \overline{N}(r, g(z+c)) + s\overline{N}\left(r, \frac{1}{f}\right) + t\overline{N}\left(r, \frac{1}{g(z+c)}\right) \\
&+ s\overline{N}\left(r, \frac{1}{g}\right) + t\overline{N}\left(r, \frac{1}{f(z+c)}\right) + S(r, f) + S(r, g) \\
&\leq (s+t+1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g).
\end{aligned}$$

Similarly, we have

$$qT(r, g(z)) - pT(r, f(z)) \leq (s+t+1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g).$$

Thus, we get

$$(q-p)(T(r, f) + T(r, g)) \leq 2(s+t+1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g),$$

which is a contradiction with $q \geq p + 2(s+t+1) + 1$. Hence, we get $H(z) \equiv 0$, which implies that

$$Q(f)P(g(z+c)) \equiv Q(g)P(f(z+c)). \tag{2.18}$$

If h is a constant, substituting $f = gh$ into (2.18), we have

$$\frac{Q(gh)}{Q(g)} \equiv \frac{P(g(z+c)h)}{P(g(z+c))}. \tag{2.19}$$

From (2.19) and Valiron-Mohon'ko Theorem [26, Theorem 1.13], we have

$$qT(r, g) = pT(r, g(z+c)) + S(r, g) = pT(r, g) + S(r, g).$$

Since $q \geq p + 2(s+t+1) + 1$, we have $\frac{Q(gh)}{Q(g)}$ and $\frac{P(g(z+c)h)}{P(g(z+c))}$ are constants. The conclusion can be proved by the similar discussions as the proof of Lemma 2.10. □

3 Proofs of Theorems

Proof of Theorem 1.1 (i) Let $F = Q(f(z))P(f(z+c))$ and $f(z) \in \mathcal{M}'$. From Lemma 2.3, we have

$$(1-\varepsilon)T(r,F) + (k-1)\overline{N}(r,F) \le \overline{N}\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{F^{(k)}-p(z)}\right) + S(r,F).$$

From (2.7), we have

$$\begin{aligned}(1-\varepsilon)(q-p)T(r,f) &\le \overline{N}\left(r,\frac{1}{Q(f(z))}\right) + \overline{N}\left(r,\frac{1}{P(f(z+c))}\right)\\ &\quad + N\left(r,\frac{1}{F^{(k)}-p(z)}\right) + S(r,f)\\ &\le (s+t)T(r,f) + N\left(r,\frac{1}{F^{(k)}-p(z)}\right) + S(r,f).\end{aligned}$$

Thus, if $q \ge p+s+t+1$, then $[Q(f(z))P(f(z+c))]^{(k)} - p(z)$ has infinitely many zeros. If $f(z) \in \mathcal{E}'$, by (2.8), then $[Q(f(z))P(f(z+c))]^{(k)} - p(z)$ has infinitely many zeros when $q+p \ge s+t+1$.

(ii) Let $G = Q(f(z))P(\Delta_c f)$ and $f(z) \in \mathcal{M}'$. From Lemma 2.3, we have

$$(1-\varepsilon)T(r,G) + (k-1)\overline{N}(r,G) \le \overline{N}\left(r,\frac{1}{G}\right) + N\left(r,\frac{1}{G^{(k)}-p(z)}\right) + S(r,G).$$

From (2.9), we have

$$\begin{aligned}(1-\varepsilon)(q-2p)T(r,f) &\le \overline{N}\left(r,\frac{1}{Q(f(z))}\right) + \overline{N}\left(r,\frac{1}{P(\Delta_c f)}\right)\\ &\quad + N\left(r,\frac{1}{G^{(k)}-p(z)}\right) + S(r,f)\\ &\le (s+2t)T(r,f) + N\left(r,\frac{1}{G^{(k)}-p(z)}\right) + S(r,f).\end{aligned}$$

Thus, if $q \ge 2p+s+2t+1$ and $\Delta_c f \not\equiv 0$, then $[Q(f(z))P(\Delta_c f)]^{(k)} - p(z)$ has infinitely many zeros. If $f(z) \in \mathcal{E}'$ and $\Delta_c f \not\equiv 0$, by (2.10), then $[Q(f(z))P(\Delta_c f)]^{(k)} - p(z)$ has infinitely many zeros when $q \ge p+s+t+1$. □

Proof of Theorem 1.2 (i) Let $F_1 = Q(f)P(g(z+c))$, $G_1 = Q(g)P(f(z+c))$ and $f, g \in \mathcal{M}'$. By Lemma 2.4, we have

$$\begin{aligned}T(r,F_1) &\le qT(r,f) + pT(r,g(z+c)) + O(1)\\ &\le (q+p)(T(r,f)+T(r,g)) + S(r,f) + S(r,g),\end{aligned}$$

so we have $S(r,F_1) = o(T(r,f)+T(r,g))$. We also have $S(r,G_1) = o(T(r,f)+T(r,g))$ from the same reason as above. Furthermore,

$$qT(r,f) = T(r,Q(f)) + O(1) = T\left(r,\frac{F_1}{P(g(z+c))}\right) \le T(r,F_1) + pT(r,g) + S(r,g),$$

that is

$$qT(r,f) - pT(r,g) \le T(r,F_1) + S(r,g).$$

We also have

$$qT(r,g) - pT(r,f) \le T(r,G_1) + S(r,f).$$

From (2.1), we have

$$\begin{aligned}(1-\varepsilon)T(r,F_1) &\le \overline{N}\left(r,\frac{1}{Q(f)}\right)+\overline{N}\left(r,\frac{1}{P(g(z+c))}\right)\\ &\quad+N\left(r,\frac{1}{F_1^{(k)}-p_1(z)}\right)+S(r,F_1)\\ &\le sT(r,f)+tT(r,g)+N\left(r,\frac{1}{F_1^{(k)}-p_1(z)}\right)+S(r,F_1)\end{aligned}$$

and

$$\begin{aligned}(1-\varepsilon)T(r,G_1) &\le \overline{N}\left(r,\frac{1}{Q(g)}\right)+\overline{N}\left(r,\frac{1}{P(f(z+c))}\right)\\ &\quad+N\left(r,\frac{1}{G_1^{(k)}-p_2(z)}\right)+S(r,G_1)\\ &\le sT(r,g)+tT(r,f)+N\left(r,\frac{1}{G_1^{(k)}-p_2(z)}\right)+S(r,G_1).\end{aligned}$$

Using the above two inequalities, we obtain

$$\begin{aligned}(1-\varepsilon)(q-p)(T(r,f)+T(r,g)) &\le (s+t)(T(r,f)+T(r,g))\\ &\quad+N\left(r,\frac{1}{F_1^{(k)}-p_1(z)}\right)+N\left(r,\frac{1}{G_1^{(k)}-p_2(z)}\right)\\ &\quad+S(r,f)+S(r,g).\end{aligned}$$

Thus, if $q \ge p+s+t+1$, then at least one of $[Q(f)P(g(z+c))]^{(k)}-p_1(z)$ and $[Q(g)P(f(z+c))]^{(k)}-p_2(z)$ has infinitely many zeros.

(ii) Let $F_2 = Q(f(z))P(\Delta_c g)$, $G_2 = Q(g(z))P(\Delta_c f)$, $\Delta_c f \not\equiv 0$, $\Delta_c g \not\equiv 0$ and $f, g \in \mathcal{M}'$. By Lemma 2.4, we have

$$\begin{aligned}T(r,F_2) &\le qT(r,f)+p(T(r,\Delta_c g)+T(r,g))\\ &\le (q+2p)(T(r,f)+T(r,g))+S(r,f)+S(r,g),\end{aligned}$$

so we have $S(r,F_2) = o(T(r,f)+T(r,g))$. We also have $S(r,G_2) = o(T(r,f)+T(r,g))$ from the same reason as above. Furthermore,

$$qT(r,f) = T(r,Q(f))+O(1) = T\left(r,\frac{F_2}{P(\Delta_c g)}\right) \le T(r,F_2)+2pT(r,g)+S(r,g),$$

that is

$$qT(r,f)-2pT(r,g) \le T(r,F_2)+S(r,g).$$

We also have

$$qT(r,g)-2pT(r,f) \le T(r,G_2)+S(r,f).$$

From (2.1) again, we have

$$\begin{aligned}(1-\varepsilon)T(r,F_2) &\le \overline{N}\left(r,\frac{1}{Q(f)}\right)+\overline{N}\left(r,\frac{1}{P(\Delta_c g)}\right)+N\left(r,\frac{1}{F_2^{(k)}-p_1(z)}\right)+S(r,F_2)\\ &\le sT(r,f)+2tT(r,g)+N\left(r,\frac{1}{F_2^{(k)}-p_1(z)}\right)+S(r,F_2)\end{aligned}$$

and

$$(1-\varepsilon)T(r,G_2) \le \overline{N}\left(r,\frac{1}{Q(g)}\right)+\overline{N}\left(r,\frac{1}{P(\Delta_c f)}\right)+N\left(r,\frac{1}{G_2^{(k)}-p_2(z)}\right)+S(r,G_2)$$
$$\le sT(r,g)+2tT(r,f)+N\left(r,\frac{1}{G_2^{(k)}-p_2(z)}\right)+S(r,G_2).$$

Combining with the above two inequalities, we obtain

$$(1-\varepsilon)(q-2p)(T(r,f)+T(r,g)) \le (s+2t)(T(r,f)+T(r,g)) + N\left(r,\frac{1}{F_2^{(k)}-p_1(z)}\right)+N\left(r,\frac{1}{G_2^{(k)}-p_2(z)}\right) + S(r,f)+S(r,g).$$

Thus, if $q \ge 2p+s+2t+1$, $\Delta_c f \not\equiv 0$ and $\Delta_c g \not\equiv 0$, then at least one of $[Q(f(z))P(\Delta_c g)]^{(k)}-p_1(z)$ and $[Q(g(z))P(\Delta_c f)]^{(k)}-p_2(z)$ has infinitely many zeros. □

Proof of Theorem 1.3 Let $F_3=[Q(f)P(g')]^{(k)}$, $G_3=[Q(g)P(f')]^{(k)}$ and $f, g \in \mathcal{M}$. From

$$T(r,Q(f)P(g')) \le qT(r,f)+pT(r,g')+S(r,f)+S(r,g)$$
$$\le qT(r,f)+2pT(r,g)+S(r,f)+S(r,g)$$
$$\le (q+2p)(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

we have $S(r,Q(f)P(g'))=o(T(r,f)+T(r,g))$. We also have $S(r,Q(g)P(f'))=o(T(r,f)+T(r,g))$ from the same reason as above. From Lemma 2.8, we have

$$N_2\left(r,\frac{1}{F_3}\right)=N_2\left(r,\frac{1}{[Q(f)P(g')]^{(k)}}\right) \le T(r,F_3)-T(r,Q(f)P(g')) + N_{2+k}\left(r,\frac{1}{Q(f)P(g')}\right)+S(r,f)+S(r,g).$$

Furthermore,

$$qT(r,f)+S(r,f)=T(r,Q(f))=T\left(r,\frac{Q(f)P(g')}{P(g')}\right)$$
$$\le T(r,Q(f)P(g'))+T(r,P(g'))+O(1)$$
$$\le T(r,Q(f)P(g'))+2pT(r,g)+S(r,g),$$

so

$$qT(r,f)-2pT(r,g) \le T(r,Q(f)P(g'))+S(r,f)+S(r,g)$$
$$\le T(r,F_3)-N_2\left(r,\frac{1}{F_3}\right)+N_{2+k}\left(r,\frac{1}{Q(f)P(g')}\right)+S(r,f)+S(r,g). \quad (3.1)$$

Similarly as the above, we obtain

$$qT(r,g)-2pT(r,f) \le T(r,G_3)-N_2\left(r,\frac{1}{G_3}\right)+N_{2+k}\left(r,\frac{1}{Q(g)P(f')}\right) + S(r,f)+S(r,g). \quad (3.2)$$

From Lemma 2.8, we also have

$$N_2\left(r,\frac{1}{F_3}\right) \le k\overline{N}(r,Q(f)P(g'))+N_{2+k}\left(r,\frac{1}{Q(f)P(g')}\right)+S(r,f)+S(r,g)$$
$$\le k\overline{N}(r,f)+k\overline{N}(r,g')+N_{2+k}\left(r,\frac{1}{Q(f)}\right)+N_{2+k}\left(r,\frac{1}{P(g')}\right)+S(r,f)+S(r,g)$$
$$\le kT(r,f)+kT(r,g)+N_{2+k}\left(r,\frac{1}{Q(f)}\right)+N_{2+k}\left(r,\frac{1}{P(g')}\right)+S(r,f)+S(r,g).$$

Now, we estimate

$$N_{2+k}\left(r,\frac{1}{Q(f)}\right)=N_{2+k}\left(r,\frac{1}{d(f-\beta_1)^{n_1}\cdots(f-\beta_s)^{n_s}}\right).$$

Then

$$\begin{aligned}N_{2+k}\left(r,\frac{1}{Q(f)}\right)&=N_{2+k}\left(r,\frac{1}{d(f-\beta_1)^{n_1}\cdots(f-\beta_s)^{n_s}}\right)\\&\le n_1N\left(r,\frac{1}{f-\beta_1}\right)+\cdots+n_{u_1}N\left(r,\frac{1}{f-\beta_{u_1}}\right)\\&\quad+(k+2)\overline{N}\left(r,\frac{1}{f-\beta_{u_1+1}}\right)+\cdots+(k+2)\overline{N}\left(r,\frac{1}{f-\beta_s}\right)\\&\le\left[\sum_{i=1}^{u_1}n_i+(s-u_1)(k+2)\right]T(r,f).\end{aligned}$$

On the other hand, we estimate $N_{2+k}\left(r,\frac{1}{P(g')}\right)=N_{2+k}\left(r,\frac{1}{b(g'-\alpha)^{m_1}\cdots(g'-\alpha_t)^{m_t}}\right)$. Then

$$\begin{aligned}N_{2+k}\left(r,\frac{1}{P(g')}\right)&=N_{2+k}\left(r,\frac{1}{b(g'-\alpha_1)^{m_1}\cdots(g'-\alpha_t)^{m_t}}\right)\\&\le m_1N\left(r,\frac{1}{g'-\alpha_1}\right)+\cdots+m_{v_1}N\left(r,\frac{1}{g'-\alpha_{v_1}}\right)\\&\quad+(k+2)\overline{N}\left(r,\frac{1}{g'-\alpha_{v_1+1}}\right)+\cdots+(k+2)\overline{N}\left(r,\frac{1}{g'-\alpha_t}\right)\\&\le\left[\sum_{j=1}^{v_1}m_j+(t-v_1)(k+2)\right]T(r,g')\\&\le 2\left[\sum_{j=1}^{v_1}m_j+(t-v_1)(k+2)\right]T(r,g)+S(r,g).\end{aligned}$$

Hence, the above three inequalities imply that

$$\begin{aligned}N_2\left(r,\frac{1}{F_3}\right)&\le\left[k+\sum_{i=1}^{u_1}n_i+(s-u_1)(k+2)\right]T(r,f)\\&\quad+\left[k+2\sum_{j=1}^{v_1}m_j+2(t-v_1)(k+2)\right]T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}N_2\left(r,\frac{1}{G_3}\right)&\le\left[k+\sum_{i=1}^{u_2}n_i+(s-u_2)(k+2)\right]T(r,g)\\&\quad+\left[k+2\sum_{j=1}^{v_2}m_j+2(t-v_2)(k+2)\right]T(r,f)+S(r,f)+S(r,g).\end{aligned}$$

In the following, we will prove the (i) of Lemma 2.9 can not hold. Otherwise, by adding $T(r,F_3)$ and $T(r,G_3)$, from (3.1), (3.2) and the following two inequalities

$$\begin{aligned}N_2(r,F_3)&\le 2\overline{N}(r,Q(f)P(g'))\le 2(T(r,f)+T(r,g))+S(r,f)+S(r,g),\\N_2(r,G_3)&\le 2\overline{N}(r,Q(g)P(f'))\le 2(T(r,g)+T(r,f))+S(r,f)+S(r,g),\end{aligned}$$

we conclude that

$$\begin{aligned}(q-2p)(T(r,f)+T(r,g)) &\le N_2\left(r,\frac{1}{F_3}\right)+N_2\left(r,\frac{1}{G_3}\right)+2N_2(r,F_3)+2N_2(r,G_3)\\ &\quad+N_{2+k}\left(r,\frac{1}{Q(f)P(g')}\right)+N_{2+k}\left(r,\frac{1}{Q(g)P(f')}\right)+S(r,f)+S(r,g)\\ &\le\left[2k+(k+2)(s+2t-u_1-2v_2)+\sum_{i=1}^{u_1}n_i+2\sum_{j=1}^{v_2}m_j\right]T(r,f)\\ &\quad+\left[2k+(k+2)(s+2t-u_2-2v_1)+\sum_{i=1}^{u_2}n_i+2\sum_{j=1}^{v_1}m_j\right]T(r,g)\\ &\quad+8(T(r,f)+T(r,g))\\ &\quad+\left[\sum_{i=1}^{u_1}n_i+2\sum_{j=1}^{v_2}m_j+(s+2t-u_1-2v_2)(k+2)\right]T(r,f)\\ &\quad+\left[\sum_{i=1}^{u_2}n_i+2\sum_{j=1}^{v_1}m_j+(s+2t-u_2-2v_1)(k+2)\right]T(r,g)\\ &\quad+S(r,f)+S(r,g)\\ &=\Bigg[2k+2(k+2)\max(s+2t-u_1-2v_2,s+2t-u_2-2v_1)\\ &\quad+2\max\left(\sum_{i=1}^{u_1}n_i,\sum_{i=1}^{u_2}n_i\right)+4\max\left(\sum_{j=1}^{v_1}m_j,\sum_{j=1}^{v_2}m_j\right)+8\Bigg]\\ &\quad(T(r,f)+T(r,g))+S(r,f)+S(r,g),\end{aligned}$$

which is a contradiction with the assumption $q \ge 2p+[2k+2(k+2)\max(s+2t-u_1-2v_2, s+2t-u_2-2v_1)+2\max(\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i)+4\max(\sum_{j=1}^{v_1} m_j, \sum_{j=1}^{v_2} m_j)+8]+1$. Hence, $F_3 \equiv G_3$ or $F_3 \cdot G_3 \equiv \alpha^2$. Finally, for $F_3 \equiv G_3$, from Lemma 2.10, we have if f=gh and h is a constant, then either $h^q = h^p$ when $Q(z) = a_q z^q$ and $P(z) = b_p z^p$ or $h^\mu = 1$ where μ is the greatest common divisor of integers from 1 to q satisfying the corresponding coefficients $a_j (1 \le j \le q)$ and $b_i (1 \le i \le p)$ of $Q(z)$ and $P(z)$ are non-zero constants. For the case of $F_3 \cdot G_3 \equiv \alpha^2$, it implies that

$$[Q(f)P(g')]^{(k)} \cdot [Q(g)P(f')]^{(k)} \equiv \alpha^2.$$

□

Proof of Theorem 1.4 Let $F_4 = [Q(f)P(g(z+c))]^{(k)}$, $G_4 = [Q(g)P(f(z+c))]^{(k)}$ and $f, g \in \mathcal{M}'$. From

$$\begin{aligned}T(r,Q(f)P(g(z+c))) &\le qT(r,f)+pT(r,g(z+c))+S(r,f)+S(r,g)\\ &\le qT(r,f)+pT(r,g)+S(r,f)+S(r,g)\\ &\le (q+p)(T(r,f)+T(r,g))+S(r,f)+S(r,g),\end{aligned}$$

we have $S(r, Q(f)P(g(z+c))) = o(T(r,f)+T(r,g))$. We also have $S(r, Q(g)P(f(z+c))) = o(T(r,f)+T(r,g))$ from the same reason as the above. From Lemma 2.8, we have

$$\begin{aligned}N_2\left(r,\frac{1}{F_4}\right) = N_2\left(r,\frac{1}{[Q(f)P(g(z+c))]^{(k)}}\right) &\le T(r,F_4)-T(r,Q(f)P(g(z+c)))\\ &\quad+N_{2+k}\left(r,\frac{1}{Q(f)P(g(z+c))}\right)+S(r,f)+S(r,g).\end{aligned}$$

Furthermore,

$$\begin{aligned} qT(r,f)+S(r,f) &= T(r,Q(f)) = T\left(r,\frac{Q(f)P(g(z+c))}{P(g(z+c))}\right)\\ &\le T(r,Q(f)P(g(z+c)))+T(r,P(g(z+c)))+O(1)\\ &\le T(r,Q(f)P(g(z+c)))+pT(r,g)+S(r,g), \end{aligned}$$

so

$$\begin{aligned} qT(r,f)-pT(r,g) &\le T(r,Q(f)P(g(z+c)))+S(r,f)+S(r,g)\\ &\le T(r,F_4)-N_2\left(r,\frac{1}{F_4}\right)+N_{2+k}\left(r,\frac{1}{Q(f)P(g(z+c))}\right)+S(r,f)+S(r,g) \end{aligned} \tag{3.3}$$

Similarly as the above, we obtain

$$\begin{aligned} qT(r,g)-pT(r,f) &\le T(r,G_4)-N_2\left(r,\frac{1}{G_4}\right)+N_{2+k}\left(r,\frac{1}{Q(g)P(f(z+c))}\right)\\ &\quad +S(r,f)+S(r,g). \end{aligned} \tag{3.4}$$

From Lemma 2.8, we also have

$$\begin{aligned} N_2\left(r,\frac{1}{F_4}\right) &\le k\overline{N}(r,Q(f)P(g(z+c)))+N_{2+k}\left(r,\frac{1}{Q(f)P(g(z+c))}\right)\\ &\quad +S(r,f)+S(r,g)\\ &\le k\overline{N}(r,f)+k\overline{N}(r,g(z+c))+N_{2+k}\left(r,\frac{1}{Q(f)}\right)\\ &\quad +N_{2+k}\left(r,\frac{1}{P(g(z+c))}\right)+S(r,f)+S(r,g)\\ &\le kT(r,f)+kT(r,g)+N_{2+k}\left(r,\frac{1}{Q(f)}\right)+N_{2+k}\left(r,\frac{1}{P(g(z+c))}\right)\\ &\quad +S(r,f)+S(r,g). \end{aligned}$$

Similarly as Theorem 1.3, we have

$$\begin{aligned} N_{2+k}\left(r,\frac{1}{Q(f)}\right) &= N_{2+k}\left(r,\frac{1}{d(f-\beta_1)^{n_1}\cdots(f-\beta_s)^{n_s}}\right)\\ &\le \left[\sum_{i=1}^{u_1}n_i+(s-u_1)(k+2)\right]T(r,f). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} N_{2+k}\left(r,\frac{1}{P(g(z+c))}\right) &= N_{2+k}\left(r,\frac{1}{b(g(z+c)-\alpha_1)^{m_1}\cdots(g(z+c)-\alpha_t)^{m_t}}\right)\\ &\le m_1N\left(r,\frac{1}{g(z+c)-\alpha_1}\right)+\cdots+m_{v_3}N\left(r,\frac{1}{g(z+c)-\alpha_{v_3}}\right)\\ &\quad +(k+2)\overline{N}\left(r,\frac{1}{g(z+c)-\alpha_{v_3+1}}\right)+\cdots+(k+2)\overline{N}\left(r,\frac{1}{g(z+c)-\alpha_t}\right)\\ &\le \left[\sum_{j=1}^{v_3}m_j+(t-v_3)(k+2)\right]T(r,g). \end{aligned}$$

Hence, the above three inequalities imply that

$$N_2\left(r,\frac{1}{F_4}\right)\le\left[k+\sum_{i=1}^{u_1}n_i+(s-u_1)(k+2)\right]T(r,f)+\left[k+\sum_{j=1}^{v_3}m_j+(t-v_3)(k+2)\right]T(r,g)+S(r,f)+S(r,g).$$

Similarly as the above method, we obtain

$$N_2\left(r,\frac{1}{G_4}\right)\le\left[k+\sum_{i=1}^{u_2}n_i+(s-u_2)(k+2)\right]T(r,g)+\left[k+\sum_{j=1}^{v_4}m_j+(t-v_4)(k+2)\right]T(r,f)+S(r,f)+S(r,g).$$

In the following, we will prove the (i) of Lemma 2.9 can not hold. Otherwise, by adding $T(r,F_4)$ and $T(r,G_4)$, from (3.3), (3.4) and the following two inequalities

$$N_2(r,F)\le 2\overline{N}(r,Q(f)P(g(z+c)))\le 2(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

$$N_2(r,G)\le 2\overline{N}(r,Q(g)P(f(z+c)))\le 2(T(r,g)+T(r,f))+S(r,f)+S(r,g),$$

we conclude that

$$\begin{aligned}(q-p)(T(r,f)+T(r,g))&\le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2N_2(r,F)+2N_2(r,G)\\&\quad+N_{2+k}\left(r,\frac{1}{Q(f)P(g(z+c))}\right)+N_{2+k}\left(r,\frac{1}{Q(g)P(f(z+c))}\right)+S(r,f)+S(r,g)\\&\le\left[2k+(k+2)(s+t-u_1-v_4)+\sum_{i=1}^{u_1}n_i+\sum_{j=1}^{v_4}m_j\right]T(r,f)\\&\quad+\left[2k+(k+2)(s+t-u_2-v_3)+\sum_{i=1}^{u_2}n_i+\sum_{j=1}^{v_3}m_j\right]T(r,g)\\&\quad+8(T(r,f)+T(r,g))\\&\quad+\left[\sum_{i=1}^{u_1}n_i+\sum_{j=1}^{v_4}m_j+(s+t-u_1-v_4)(k+2)\right]T(r,f)\\&\quad+\left[\sum_{i=1}^{u_2}n_i+\sum_{j=1}^{v_3}m_j+(s+t-u_2-v_3)(k+2)\right]T(r,g)\\&\quad+S(r,f)+S(r,g)\\&=\Bigg[2k+2(k+2)\max(s+t-u_1-v_4,s+t-u_2-v_3)\\&\quad+2\max\left(\sum_{i=1}^{u_1}n_i,\sum_{i=1}^{u_2}n_i\right)+2\max\left(\sum_{j=1}^{v_3}m_j,\sum_{j=1}^{v_4}m_j\right)+8\Bigg](T(r,f)+T(r,g))\\&\quad+S(r,f)+S(r,g),\end{aligned}$$

which is a contradiction with $q \geq p + \Big[2k + 2(k+2)\max((s+t-u_1-v_4),(s+t-u_2-v_3)) + 2\max(\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i) + 2\max(\sum_{j=1}^{v_3} m_j, \sum_{j=1}^{v_4} m_j) + 8\Big] + 1$. Hence, $F_4 \equiv G_4$ or $F_4 \cdot G_4 \equiv \alpha^2$. Finally, for $F_4 \equiv G_4$, from Lemma 2.11, we have if f=gh and h is a constant, then either $h^q = h^p$ when $Q(z) = a_q z^q$ and $P(z) = b_p z^p$ or $h^\mu = 1$ where μ is the greatest common divisor of integers from 1 to q satisfying the corresponding coefficients $a_j (1 \leq j \leq q)$ and $b_i (1 \leq i \leq p)$ of $Q(z)$ and $P(z)$ are non-zero constants. For $F_4 \cdot G_4 \equiv \alpha^2$, it implies that

$$[Q(f)P(g(z+c))]^{(k)} \cdot [Q(g)P(f(z+c))]^{(k)} \equiv \alpha^2.$$

□

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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