Indian J Pure Appl Math (2026) 57:1583–1607
https://doi.org/10.1007/s13226-024-00743-9

ORIGINAL RESEARCH

Some classical and fractal splines for constrained and shape preserving interpolation and applications in solving differential equations

Richa Jain · **Vijender Nallapu**

Received: 17 October 2023 / Accepted: 14 November 2024 / Published online: 30 December 2024

Abstract In this paper, we develop constrained and shape preserving classical splines in one and two variables and fractal splines in two variables. First, we obtain a classical rational spline lying above or below the prescribed piecewise line or contained in a prescribed rectangle. Using a blending technique, we construct a classical bivariate rational spline and study its constrained and shape preserving aspects. The proposed constrained and shape preserving splines are exploited for solving *constraint differential equations*. We construct a bivariate fractal spline [which is the fixed point of the Read-Bajraktarević (RB) operator] as a perturbation of our classical bivariate rational spline proposed in this paper. Further, we investigate suitable conditions on the shape parameters and scaling factors so that the proposed fractal surface lies above or below the given plane, or within a prescribed cuboid. Also, we study the positivity and monotonicity preserving aspects of the proposed bivariate fractal spline. It is observed that our bivariate fractal spline converges to the original function as the mesh norm goes to zero. The proposed theoretical results are verified through numerical examples.

Keywords Constraint differential equations · Constrained rational splines · Bivariate fractal splines · Shape preserving interpolation · Iterated function system · Read-Bajraktarević operator

Mathematics Subject Classification 46N20 · 28A80 · 26C15 · 41A05 · 41A29 · 65D05

1 Introduction

The process of construction of a continuous function using a finite number of data points is called interpolation and it is a rather old problem dating back to ancient Babylon and Greece. Because of its increasing applications in science and engineering (computer graphics, computer-aided design, information sciences, and data visualization), different interpolation schemes are being developed by a huge number of researchers. In computer-aided geometric design [1,2], VLSI, CAM, robotics, data visualization, chemical and physical sciences, and reverse engineering, one needs a representation of a data set having a fundamental shape by a function that interpolates the given data and additionally preserves its fundamental shape. This kind of interpolation is called shape preserving interpolation. Monotonicity, positivity, and convexity (concavity) are the fundamental shapes that are found in data obtained from different environments. In the operation of nuclear and fossil-fired power plants, it is often

Communicated by G.D. Veerappa Gowda.

R. Jain · V. Nallapu (✉)
Department of Mathematics, Visvesvaraya National Institute of Technology Nagpur, South Ambazari Road, Nagpur, Maharashtra 440006, India
E-mail: vijendernallapu@gmail.com

R. Jain
E-mail: richajain8797@gmail.com

desirable to have the values of thermodynamic properties such as enthalpy and specific volume of steam and water. Furthermore, enthalpy and specific volume can be best considered as positive surfaces constructed over a domain in the temperature-pressure plane. Thus, we require shape preserving interpolants for scientific data to produce physically reasonable curves and surfaces. On the other hand, there are practical problems wherein one requires (i) interpolating curves (surfaces) that lie completely above or below a prefixed curve (plane), (ii) interpolating curves (surfaces) that lie within a given rectangle (cuboid), and this kind of interpolation is called constrained interpolation. Considering the importance of constrained interpolants, we develop sufficient conditions on the shape parameters so that the rational spline given in [3] is a constrained interpolant. Next, we construct a constrained bivariate rational spline by using the blending technique [4].

There exist many differential equations whose solutions (i) have a shape, or (ii) lie above or below a prescribed piecewise line (plane), or lie between the prescribed piecewise lines (planes), or lie in a prescribed rectangle (cuboid), and we refer such differential equations as *constraint differential equations*, for instance, the governing equation of the LR-circuit is given by

$$L\frac{di}{dt} + iR = V, i(0) = 0,$$

where i is current, L is inductance, R is resistance, and V is voltage. The analytic solution of the above equation is given by

$$i(t) = \frac{V}{R}\left(1 - e^{\frac{-R}{L}t}\right).$$

From the aforementioned expression, one can easily verify that the current i is positive (non-negative) and monotonically increasing, and hence the above differential equation is a *constraint differential equation.* The existing numerical methods may not provide an approximate solution for a constraint differential equation whose analytic solution exists, but is difficult to get. In this manuscript, we develop a methodology to acquire the approximate solution for a constraint differential equation.

However, in times where all efforts are directed towards the construction of smooth interpolants, the fact that many experimental and natural signals are "rough" having a dense set of nondifferentiable points or even nowhere differentiable was overlooked. Barnsley [5] evolved a fractal interpolation technique targeting data/samples that need continuous and nowhere differentiable representation. Some interesting results in the theory of fractal interpolation are studied in [6–10]. The construction of fractal splines with different boundary conditions is reported in [11,12]. Very recently, using the theory of univariate fractal splines, different types of shape preserving and constrained univariate fractal splines [13–16] are developed. Further, these shape preserving and constrained univariate fractal splines are the fractal analogue of some standard classical existing shape preserving univariate splines.

Considering the theory of shape preserving and constrained univariate fractal splines, one may seek shape preserving and constrained bivariate fractal splines. To the best of our knowledge, there is no research reported in the direction of shape preserving and constrained bivariate fractal splines which are fixed points of the Read-Bajraktarević (RB) operator defined on a suitable function space. Of course, shape preserving and constrained bivariate fractal splines [17–21] are developed in the literature, but they are not fixed points of RB-operator, and hence they need not be self-referential. In the current paper, we study the shape preserving and constrained bivariate fractal splines and their convergence properties. In the literature, there exist many approaches [22–27] for the construction of bivariate fractal interpolation functions and each approach has its specific advantages and disadvantages. Very recently, a general construction of fractal interpolation surfaces which are fixed points of the RB-operator was developed in [28]. Taking this methodology on one hand and rational functions on the other, we develop positive, monotonic, and constrained bivariate fractal splines which are the fixed points of the RB-operator. As a result, the proposed constrained and shape preserving bivariate fractal splines are self-referential.

By modifying estimated first order partial derivatives suitably, Brodlie et al. [30] developed a positive surface scheme using a bicubic interpolant. However, classical shape preserving interpolants have derivatives of all orders except perhaps at a finite number of points in the rectangular domain. Consequently, these interpolants may not be suitable for preserving the shape of given Hermite data wherein the variables representing the derivatives are modelled using functions of varying irregularity/fractality (smooth to nowhere differentiable). Such data arise naturally and abundantly in nonlinear control systems (e.g., a pendulum-cart system) and in some fluid dynamics problems (e.g., the motion of a falling sphere in a non-Newtonian fluid). The proposed constrained and shape preserving bivariate fractal spline is $\mathcal{C}^1$-smooth, but its first order partial derivatives are functions of varying irregularity/fractality.

2 Rational spline and its constrained aspects

In this section, first, we recall the rational spline given in [3] and then discuss its constrained aspects.

Let Σ_s be the set of all first s natural numbers and M be a natural number bigger than one. Let the given data be $\{(x_i, z_i, d_i) : i \in \Sigma_M\}$, where $x_1 < x_2 \cdots < x_{M-1} < x_M$, d_i is the derivative value at x_i. Denote $I_i = [x_i, x_{i+1}]$, and $h_i = x_{i+1} - x_i$, $i \in \Sigma_{M-1}$. The $\mathcal{C}^1$- rational spline S constructed in [3] is given by

$$S(x) = \frac{P_i(x)}{Q_i(x)}, x \in I_i, i \in \Sigma_{M-1}, \tag{1}$$

where

$$P_i(x) = z_i(1-\theta)^3 + \big(r_i z_i + h_i d_i\big)\theta(1-\theta)^2 + \big(r_i z_{i+1} - h_i d_{i+1}\big)\theta^2(1-\theta) + z_{i+1}\theta^3, \tag{2}$$

$$Q_i(x) = 1 + (r_i - 3)\theta(1-\theta), \theta = \frac{x - x_i}{x_{i+1} - x_i}, \tag{3}$$

the free parameter r_i is selected to be strictly bigger than -1 to ensure that $Q_i(x) > 0$ for all $x \in I$, which in turn avoids a singularity of the rational expression occurring in (1). The parameter r_i can be varied to alter the shape of the interpolant S and hence is called the shape parameter.

The following proposition provides the convergence analysis of the rational spline S.

Proposition 1 *[3] Let the interpolation data $\{(x_i, z_i, d_i) : i \in \Sigma_M)\}$ be obtained from $f \in \mathcal{C}^4[x_1, x_M]$. Then for $x \in [x_i, x_{i+1}]$,*

$$\begin{aligned} |f(x) - S(x)| \le & \frac{h_i}{4\nu_i} \max\{|f^{(1)}(x_i) - d_i|, |f^{(1)}(x_{i+1}) - d_{i+1}|\} \\ & + \frac{h_i^2}{384\nu_i}\Big\{h_i^2\|f^{(4)}\|_\infty\big(1 + |r_i - 3|/4\big) + 4|r_i - 3|\big(h_i\|f^{(3)}\|_\infty) + 3\|f^{(2)}\|_\infty\big)\Big\}, \end{aligned} \tag{4}$$

where $\nu_i = \begin{cases} (1+r_i)/4, & \text{if } -1 < r_i < 3, \\ 1, & \text{if } r_i \ge 3, \end{cases}$ *and $\|.\|_\infty$ denotes the supremum norm on $[x_i, x_{i+1}]$.*

It is plain to verify that $\nu_i \to 0$ as $r_i \to -1$. Subsequently, $\dfrac{1}{\nu_i} \to \infty$ as $r_i \to -1$. As a result, the error bound in (4) goes to infinity as $r_i \to -1$. To make $\dfrac{1}{\nu_i}$ bounded, we modify ν_i as follows.

$$\nu_i = \begin{cases} (1+r_i)/4, & \text{if } -1 < \kappa < r_i < 3, \\ 1, & \text{if } r_i \ge 3, \end{cases}$$

where κ is a fixed number.

Proposition 2 *[3] Let $\{(x_i, z_i, d_i) : i \in \Sigma_M\}$ be a monotonic data set. Then the sufficient conditions for the rational spline S to be monotonic is that $r_i \ge \dfrac{d_i + d_{i+1}}{\Delta_i}$.*

2.1 Constrained aspect of univariate rational spline

In this section, we discuss constrained aspects of classical rational spline S. The following theorem provides sufficient conditions on the shape parameters r_i, $i \in \Sigma_{M-1}$ for the rational spline lying above the given piecewise line whenever the given interpolating data lies above the given piecewise line.

Theorem 1 *Let* $\{(x_i, z_i, d_i) : i \in \Sigma_M\}$ *be a given data set which lies above the piecewise line* $l(x) = m_i x + t_i, m_i, t_i \in \mathbb{R}, x \in I_i, i \in \Sigma_{M-1}$. *Then the rational spline S lies above the piecewise line l if the shape parameters* $r_i, i \in \Sigma_{M-1}$ *are chosen as*

$$r_i > \max\left\{-1, \frac{-h_i(d_i - m_i)}{z_i - l_i}, \frac{h_i(d_{i+1} - m_i)}{z_{i+1} - l_{i+1}}\right\}, l_i = l(x_i).$$

Proof Since the data set $\{(x_i, z_i, d_i) : i \in \Sigma_M\}$ lies above the piecewise line $l(x) = m_i x + t_i$, it follows that $z_i \geq l(x_i), i \in \Sigma_M$.
Since $r_i > -1$, we can verify that $Q_i(x) > 0$ for all $x \in I_i, i \in \Sigma_{M-1}$.
Therefore, $S(x) = \frac{P_i(x)}{Q_i(x)} \geq l(x)$ for all $x \in I_i, i \in \Sigma_{M-1}$ is identical to

$$P_i(x) - Q_i(x)l(x) \geq 0 \;\forall x \in I_i,\; i \in \Sigma_{M-1}. \tag{5}$$

Since $l_i = l(x_i), i \in \Sigma_{M-1}$, we get

$$l_i = l(x_i) = m_i x_i + t_i \text{ and } l_{i+1} = l(x_{i+1}) = m_i x_{i+1} + t_i, i \in \Sigma_{M-1}.$$

Solving the above equations, we get

$$m_i = \frac{l_{i+1} - l_i}{x_{i+1} - x_i}, t_i = \frac{x_{i+1}l_i - x_i l_{i+1}}{x_{i+1} - x_i}.$$

Substituting m_i and t_i in $l(x) = m_i x + t_i, x \in I_i, i \in \Sigma_M$, we obtain

$$l(x) = (1-\theta)l_i + \theta l_{i+1}, \theta = \frac{x - x_i}{x_{i+1} - x_i}, x \in I_i, i \in \Sigma_{M-1}. \tag{6}$$

One can see that

$$\begin{aligned}
Q_i(x)l(x) =& Q_i(x)[(1-\theta)l_i + \theta l_{i+1}],\\
=&[1 + (r_i - 3)\theta(1-\theta)][(1-\theta)l_i + \theta l_{i+1}],\\
=&(1-\theta)l_i + \theta l_{i+1} + (r_i - 3)\theta(1-\theta)^2 l_i + (r_i - 3)\theta^2(1-\theta)l_{i+1},\\
=&(1-\theta)^3 l_i + \{(r_i - 1)l_i + l_{i+1}\}\theta(1-\theta)^2 + \{(r_i - 1)l_{i+1} + l_i\}\theta^2(1-\theta) + \theta^3 l_{i+1}.
\end{aligned} \tag{7}$$

Using (2) and (7), we can get

$$P_i(x) - Q_i(x)l(x) = (z_i - l_i)(1-\theta)^3 + A_i\theta(1-\theta)^2 + B_i\theta^2(1-\theta) + (z_{i+1} - l_{i+1})\theta^3, \tag{8}$$

where $A_i = r_i(z_i - l_i) - m_i h_i + h_i d_i$, $B_i = r_i(z_{i+1} - l_{i+1}) + m_i h_i - h_i d_{i+1}$.
From (8), it is clear that $P_i(x) - Q_i(x)l(x) \geq 0$ for all $x \in I_i, i \in \Sigma_{M-1}$ if $A_i \geq 0$ and $B_i \geq 0$.
Next, the inequalities $A_i \geq 0$ and $B_i \geq 0$ yield the required condition on the shape parameters. □

Remark 1 By taking $l(x) = 0$ in Theorem 1, one can see that the rational spline S is positive whenever the given data is positive, provided $r_i > \max\left\{-1, \frac{-h_i d_i}{z_i}, \frac{h_i d_{i+1}}{z_{i+1}}\right\}, i \in \Sigma_{M-1}$.

The following theorem provides sufficient conditions on the shape parameters $r_i, i \in \Sigma_{M-1}$ for the rational spline S, lies below the given piecewise line whenever the given data lies below the given piecewise line.

Theorem 2 *Let* $\{(x_i, z_i, d_i) : i \in \Sigma_M\}$ *be a given data set which lies below the piecewise line* $l^*(x) = m_i^* x + t_i^*, m_i^*, t_i^* \in \mathbb{R}, x \in I_i, i \in \Sigma_{M-1}$. *Then the rational spline S lies below the piecewise line* l^* *if the corresponding shape parameters* $r_i, i \in \Sigma_{M-1}$ *are chosen as*

$$r_i > \max\left\{-1, \frac{h_i(d_i - m_i^*)}{l_i^* - z_i}, \frac{-h_i(d_{i+1} - m_i^*)}{l_{i+1}^* - z_{i+1}}\right\}, l_i^* = l_i^*(x_i).$$

The proof is very similar to that of the previous theorem and hence it is omitted.
The following theorem discusses the sufficient condition for the rational spline S lying between two straight lines.

Theorem 3 *Let $\{(x_i, z_i, d_i) : i \in \Sigma_M\}$ be a given data set which lies between the piecewise lines $l(x) = m_i x + t_i$, $i \in \Sigma_{M-1}$ and $l^*(x) = m_i^* x + t_i^*$, $x \in I_i$, $i \in \Sigma_{M-1}$. Then the rational spline S lies between piecewise lines l and l^* if the corresponding shape parameters r_i, $i \in \Sigma_{M-1}$ are chosen as*

$$r_i > \max\left\{-1, \frac{-h_i(d_i - m_i)}{z_i - l_i}, \frac{h_i(d_{i+1} - m_i)}{z_{i+1} - l_{i+1}}, \frac{h_i(d_i - m_i^*)}{l_i^* - z_i}, \frac{-h_i(d_{i+1} - m_i^*)}{l_{i+1}^* - z_{i+1}}\right\}, l_i = l_i(x_i), l_i^* = l_i^*(x_i).$$

Proof From the Theorem 1, it follows that the sufficient condition for the rational spline S lying above the piecewise line l is that the shape parameters r_i, $i \in \Sigma_{M-1}$ satisfy the inequality

$$r_i > \max\left\{-1, \frac{-h_i(d_i - m_i)}{z_i - l_i}, \frac{h_i(d_{i+1} - m_i)}{z_{i+1} - l_{i+1}}\right\}. \tag{9}$$

Next, from the Theorem 2, it can be seen that the sufficient condition for the rational spline S lying below the piecewise line l^* is that the shape parameters r_i, $i \in \Sigma_{M-1}$ satisfy the inequality

$$r_i > \max\left\{-1, \frac{h_i(d_i - m_i^*)}{l_i^* - z_i}, \frac{-h_i(d_{i+1} - m_i^*)}{l_{i+1}^* - z_{i+1}}\right\}. \tag{10}$$

Combining (9) and (10), we complete the proof. □

2.2 Containment of rational spline in a rectangle

The following theorem provides sufficient condition on shape parameter r_i, $i \in \Sigma_{M-1}$ for the rational spline lies within the rectangle whenever the given interpolating data lies within the rectangle.

Theorem 4 *Let $\{(x_i, z_i, d_i) : i \in \Sigma_M\}$ be a given data set. Then the classical rational spline S lies within the rectangle $[x_1, x_M] \times [\Lambda_1, \Lambda_2]$ if the corresponding shape parameters r_i, $i \in \Sigma_{M-1}$ satisfy the following conditions:*

$$r_i > \max\left\{-1, \frac{-h_i d_i}{z_i - \Lambda_1}, \frac{h_i d_{i+1}}{z_{i+1} - \Lambda_1}, \frac{h_i d_i}{\Lambda_2 - z_i}, \frac{-h_i d_{i+1}}{\Lambda_2 - z_{i+1}}\right\}, \tag{11}$$

where $\Lambda_1 < \min\{z_i : i \in \Sigma_M\}$ and $\Lambda_2 > \max\{z_i : i \in \Sigma_M\}$.

Proof We can see that the rational spline S lies within the rectangle $[x_1, x_M] \times [\Lambda_1, \Lambda_2]$ if $\Lambda_1 < S(x) < \Lambda_2$ for all $x \in [x_1, x_M]$. It is equivalent to

$$U_i(x) = P_i(x) - \Lambda_1 Q_i(x) \geq 0\ \forall\, x \in I_i, i \in \Sigma_{M-1} \text{ and}$$
$$V_i(x) = \Lambda_2 Q_i(x) - P_i(x) \geq 0\ \forall\, x \in I_i, i \in \Sigma_{M-1}.$$

Since $r_i > -1$, one can see that $Q_i(x) > 0$ for all $x \in I_i$, $i \in \Sigma_{M-1}$.

By using a degree of elevation, one can rewrite Q_i as follows:

$$Q_i(x) = (1-\theta)^3 + r_i\theta(1-\theta)^2 + r_i\theta^2(1-\theta) + \theta^3\ \forall x \in I_i,\ i \in \Sigma_{M-1}. \tag{12}$$

Using (2) and (12), we can notice that the sufficient conditions for $U_i(x) \geq 0$ for all $x \in I_i$, $i \in \Sigma_{M-1}$, are

$$r_i(z_i - \Lambda_1) + h_i d_i \geq 0,\ \text{and}\ r_i(z_{i+1} - \Lambda_1) - h_i d_{i+1} \geq 0,$$

which lead to $r_i > \max\left\{\frac{-h_i d_i}{z_i - \Lambda_1}, \frac{h_i d_{i+1}}{z_{i+1} - \Lambda_1}\right\}$.

Similarly, sufficient conditions for $V_i(x) \geq 0$ for all $x \in I_i$, $i \in \Sigma_{M-1}$, are

$$r_i(\Lambda_2 - z_i) - h_i d_i \geq 0,\ \text{and}\ r_i(\Lambda_2 - z_{i+1}) + h_i d_{i+1} \geq 0.$$

The aforementioned inequalities lead to $r_i > \max\left\{\frac{h_i d_i}{\Lambda_2 - z_i}, \frac{h_i d_{i+1}}{\Lambda_2 - z_{i+1}}\right\}$.

Combining all the restriction on shape parameters r_i, we get the desired condition on the shape parameters r_i, $i \in \Sigma_{M-1}$. □

3 Solving constraint ordinary differential equations by rational spline

Consider the boundary value problem (BVP)

$$F(x, z, z', z'') = 0,\ x \in [\beta_1, \beta_2],\ \Omega_1 z(\beta_1) + \Omega_2 z'(\beta_1) = \tau_1,\ \Omega_3 z(\beta_2) + \Omega_4 z'(\beta_2) = \tau_2, \tag{13}$$

or the initial value problem (IVP)

$$F(x, z, z', z'') = 0,\ x \in [\beta_1, \beta_2],\ z(\beta_1) = \tau_1^*,\ z'(\beta_1) = \tau_2^*, \tag{14}$$

subject to the constraint(s): $z(x) > l(x)$ for all $x \in [\beta_1, \beta_2]$, $z(x) < l^*(x)$ for all $x \in [\beta_1, \beta_2]$, $l(x) < z(x) < l^*(x)$ for all $x \in [\beta_1, \beta_2]$, or $\Lambda_1 < z(x) < \Lambda_2$ for all $x \in [\beta_1, \beta_2]$, where l and l^* are piecewise lines, and $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \tau_1, \tau_2, \tau_1^*, \tau_2^*, \Lambda_1, \Lambda_2 \in \mathbb{R}$.

Now, our main aim is obtaining the solution for the given BVP (13) or IVP (14) which obeys the given constraint(s). For this purpose, we proceed as follows. Let $\beta_1 = x_1 < x_2 < \ldots < x_{M-1} < x_M = \beta_2$ be a partition of $[\beta_1, \beta_2]$. First, we solve the given BVP or IVP numerically, to get $z(x_i), i \in \Sigma_M$ approximately. Next, using the numerical methods such as arithmetic mean and geometric mean [31,32], we calculate derivative values $z'(x_i), i \in \Sigma_M$ approximately. Thus, we obtain the data set $\{(x_1, z_1, d_1), (x_2, z_2, d_2), \ldots (x_M, z_M, d_M)\}$, where $z_i = z(x_i)$ and $d_i = z'(x_i)$. One should eliminate the points from the above data set which do not satisfy the given constraint. We construct the rational spline S with respect to the data set $\{(x_i, z_i, d_i), i \in \Sigma_M\}$ and the shape parameters chosen according to Theorem 1 if the constraint is $z(x) > l(x)$ for all $x \in [\beta_1, \beta_2]$, or Theorem 2 if the constraint is $z(x) < l^*(x)$ for all $x \in [\beta_1, \beta_2]$, or Theorem 3 if the constraint is $l(x) < z(x) < l^*(x)$ for all $x \in [\beta_1, \beta_2]$, or Theorem 4 if the constraint is $\Lambda_1 < z(x) < \Lambda_2$ for all $x \in [\beta_1, \beta_2]$. Then the rational spline S will be an approximate solution for the given BVP or IVP with prescribed constraint(s).

From the Proposition 1, it follows that the rational spline S converges to the exact solution of the constraint IVP (13) [constraint BVP (14)] as $M \to \infty$, where M is the number of data points pertaining to IVP (13) [constraint BVP (14)].

3.1 Numerical examples

Let us consider the IVP: $8z'' + 2z' + z = x^3 - x + 4$, $x \in [0, 10]$, $z(0) = 150$ and $z'(0) = -25$, subject to any one of the constraints: $z(x) > l(x)$ for all $x \in [0, 10]$, or $z(x) < l^*(x)$ for all $x \in [0, 10]$, or $l(x) < z(x) < l^*(x)$ for all $x \in [0, 10]$, or $10 < z(x) < 300$ for all $x \in [0, 10]$, where

$$l(x) = \begin{cases} -11x + 46, & \text{if } 0 \le x \le 4, \\ 4x - 14, & \text{if } 4 \le x \le 8, \\ 65x - 502, & \text{if } 8 \le x \le 10, \end{cases} \text{ and } l^*(x) = \begin{cases} -34x + 164, & \text{if } 0 \le x \le 4, \\ 6x + 4, & \text{if } 4 \le x \le 8, \\ 124x - 940, & \text{if } 8 \le x \le 10. \end{cases}$$

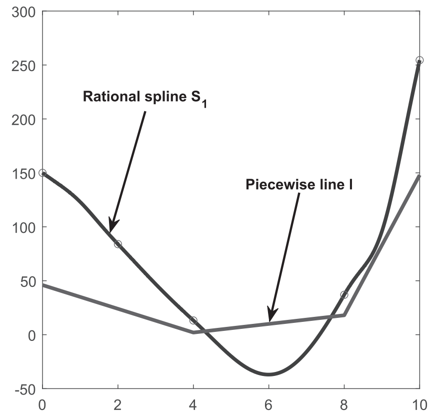
By solving the given IVP using the Runge-Kutta method, we get the data set $\{(0, 150, -25), (2, 83.963, -37.055), (4, 13.173, -32.093), (8, 36.99, 61.01), (10, 254.491, 160.583)\}$.

Let us consider the given IVP with the constraint $z(x) > l(x)$ for all $x \in [0, 10]$. The rational spline S_1 is constructed in Figure 1(a) by using the arbitrarily chosen shape parameters $r_1 = 1, r_2 = 2, r_3 = 2$, and $r_4 = 1$. We can easily notice that the rational spline S_1 is not satisfying the constraint $S_1(x) > l(x)$ for all $x \in [0, 10]$. Therefore, by using the shape parameters chosen according to Theorem 1, we have generated the rational spline S_2 in Figure 1(b), and one can easily verify that $S_2(x) > l(x)$ for all $x \in [0, 10]$. Thus, the rational spline S_2 is an approximate solution for the given IVP with the constraint $z(x) > l(x)$ for all $x \in [0, 10]$.

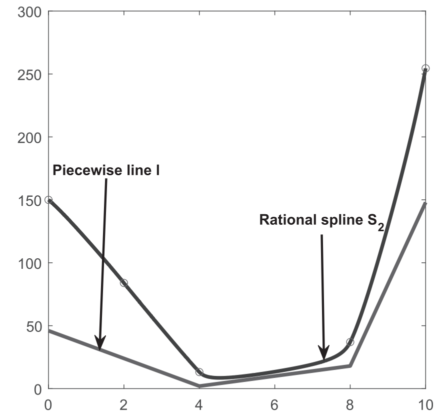
Using the shape parameters chosen according to Theorem 2, we have generated the rational spline S_3 in Figure 1(c), and one can easily verify that $S_3(x) < l^*(x)$ for all $x \in [0, 10]$. Thus, the rational spline S_3 is an approximate solution for the given IVP with the constraint $z(x) < l^*(x)$ for all $x \in [0, 10]$.

Using the shape parameters chosen according to Theorem 3, we have generated the rational spline S_4 in Figure 1(d), and one can easily verify that $l(x) < S_4(x) < l^*(x)$ for all $x \in [0, 10]$. Thus, the rational spline S_4 is an approximate solution for the given IVP with the constraint $l(x) < z(x) < l^*(x)$ for all $x \in [0, 10]$.

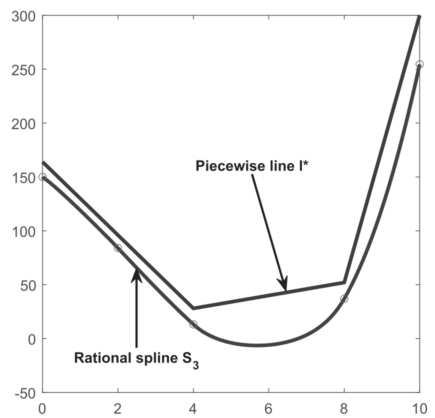
Let us consider the given IVP with the constraint $10 < z(x) < 300$ for all $x \in [0, 10]$. The rational spline S_5 is constructed in Figure 1(e) by using the arbitrarily chosen shape parameters $r_1 = 1, r_2 = 1, r_3 = 5$, and $r_4 = 1$. It is clear that S_5 does not fulfil the constraint $10 < S_5(x) < 300$ for all $x \in [0, 10]$. Therefore, by using the shape parameters chosen according to Theorem 4, we have generated the rational spline S_6 in Figure 1(f), and we can verify that $10 < S_6(x) < 300$ for all $x \in [0, 10]$. Thus, the rational spline S_6 is an approximate solution for the given IVP with the constraint $10 < z(x) < 300$ for all $x \in [0, 10]$.



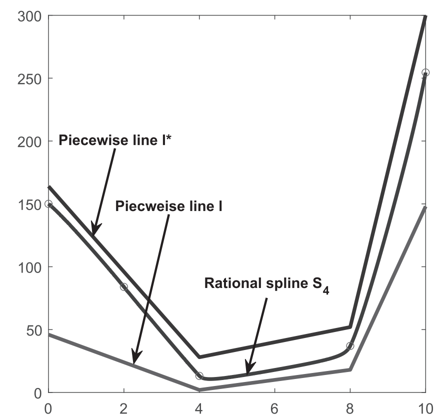
(a) Unconstrained rational spline S_1 with $r_1 = 1$, $r_2 = 2$, $r_3 = 2$, $r_4 = 1$.



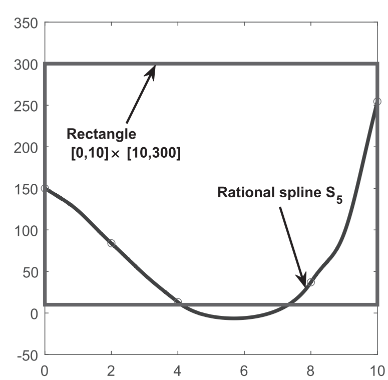
(b) Constrained rational spline S_2 with $r_1 = 5$, $r_2 = 5$, $r_3 = 15$, and $r_4 = 5$.



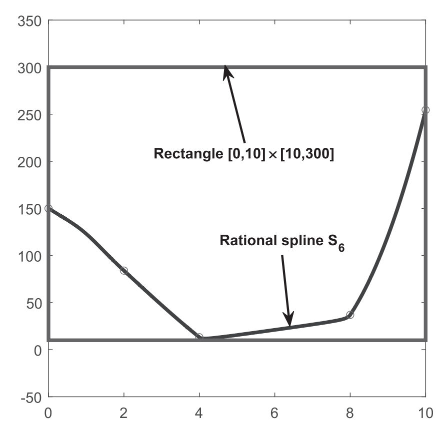
(c) Constrained rational spline S_3 with $r_1 = 5$, $r_2 = 3$, $r_3 = 5$, and $r_4 = 3$.



(d) Constrained rational spline S_4 with $r_1 = 5$, $r_2 = 7$, $r_3 = 25$, and $r_4 = 6$.



(e) Unconstrained rational spline S_5 with $r_1 = 1$, $r_2 = 1$, $r_3 = 5$, and $r_4 = 1$.



(f) Constrained rational spline S_6 with $r_1 = 1, r_2 = 3, r_3 = 45$, and $r_4 = 3$.

Fig. 1 Solutions for IVP with different constraints

4 Rational surface and its constrained aspects

In this section, we construct a bivariate rational spline and discuss its convergence analysis, constrained and shape preserving aspects.

4.1 Construction of bivariate rational spline

Let M and N be natural numbers bigger than one. Let $\Delta^* = \{x_1, x_2 \ldots, x_{M-1}, x_M, y_1, y_2 \ldots, y_{N-1}, y_N\}$, where $x_1 < x_2 \cdots < x_{M-1} < x_M$; $y_1 < y_2 \cdots < y_{N-1} < y_N$. Let the given surface interpolation data be $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j});\ (i, j) \in \Sigma_M \times \Sigma_N\}$, where $z_{x,i,j}$ and $z_{y,i,j}$ are partial derivative values with respect to x and y respectively at point (x_i, y_j). Let $\Sigma_M = \{1, 2, \ldots, M\}$, $\Sigma_N = \{1, 2, \ldots, N\}$, $\Sigma_{M-1} = \{1, 2, \ldots, M-1\}$, and $\Sigma_{N-1} = \{1, 2, \ldots, N-1\}$. Denote $I = [x_1, x_M]$, $J = [y_1, y_N]$, $\mathcal{D} = I \times J$, $I_i = [x_i, x_{i+1}]$, $J_j = [y_j, y_{j+1}]$, $\mathcal{D}_{i,j} = I_i \times J_j$, $h_i = x_{i+1} - x_i$, and $h^*_j = y_{j+1} - y_j$. Suppose that Δ^* is a partition of $\mathcal{D}$. Using the rational spline (1), we define univariate $\mathcal{C}^1$-rational spline $S(x, y_j)$, $j \in \Sigma_N$ that interpolates the univariate data $\{(x_i, z_{i,j}, z_{x,i,j});\ i \in \Sigma_M\}$ as follows:

$$S(x, y_j) = \frac{P_{i,j}(x)}{Q_{i,j}(x)},\ x \in I_i,\ i \in \Sigma_{M-1}, \tag{15}$$

where

$$P_{i,j}(x) = z_{i,j}(1-\theta)^3 + \big(r_{i,j} z_{i,j} + h_i z_{x,i,j}\big)\theta(1-\theta)^2 + \big(r_{i,j} z_{i+1,j} - h_i z_{x,i+1,j}\big)\theta^2(1-\theta) + z_{i+1,j}\theta^3,$$

$Q_{i,j}(x) = 1 + (r_{i,j} - 3)\theta(1-\theta)$, $\theta = \dfrac{x - x_i}{x_{i+1} - x_i}$, and the free parameter $r_{i,j}$ is selected to be strictly bigger than -1 to ensure that $Q_{i,j}(x) > 0$ for all $x \in I$, which in turn avoids a singularity of the rational expression occurring in (15). The parameter $r_{i,j}$ can be varied to alter the shape of the interpolant $S(., y_j)$ and hence is called the shape parameter.

Again, using the rational spline (1), we define univariate $\mathcal{C}^1$-rational spline $S^*(x_i, y)$, $i \in \Sigma_M$ that interpolates the univariate data $\{(y_j, z_{i,j}, z_{y,i,j});\ j \in \Sigma_N\}$ as follows:

$$S^*(x_i, y) = \frac{P^*_{i,j}(y)}{Q^*_{i,j}(y)},\ y \in J_j,\ j \in \Sigma_{N-1}, \tag{16}$$

where

$$P^*_{i,j}(y) = z_{i,j}(1-\phi)^3 + \big(r^*_{i,j} z_{i,j} + h^*_j z_{y,i,j}\big)\phi(1-\phi)^2 + \big(r^*_{i,j} z_{i,j+1} - h^*_j z_{y,i,j+1}\big)\phi^2(1-\phi) + z_{i,j+1}\phi^3,$$

$Q^*_{i,j}(y) = 1 + (r^*_{i,j} - 3)\phi(1-\phi)$, $\phi = \dfrac{y - y_j}{y_{j+1} - y_j}$, and the free parameter $r^*_{i,j}$ is selected to be strictly bigger than -1 to ensure that $Q^*_{i,j}(y) > 0$ for all $y \in J$, which in turn avoids a singularity of the rational expression occurring in (16). The parameter $r^*_{i,j}$ can be varied to alter the shape of the interpolant $S^*(x_i, .)$ and hence is called the shape parameter.

One can easily notice that (i) for $j \in \Sigma_N$, rational spline $S(x, y_j)$, is defined along the grid line $y = y_j$ (ii) for $i \in \Sigma_M$, rational spline $S^*(x_i, y)$, is defined along the grid line $x = x_i$. In what follows, we shall blend these univariate rational splines using well-known partially blending technique [4] to obtain the desired surface. Exploiting the network of univariate rational splines $S(x, y_j)$, $j \in \Sigma_N$, $S^*(x_i, y)$, $i \in \Sigma_M$, and blending technique [4], the bivariate rational spline Φ is defined over $\mathcal{D}_{i,j}$, $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$ as follows

$$\begin{aligned}\Phi(x, y) =& b_{y,0}(\phi)S(x, y_j) + b_{y,1}(\phi)S(x, y_{j+1}) + a_{x,0}(\theta)S^*(x_i, y)\\ &+ a_{x,1}(\theta)S^*(x_{i+1}, y) - a_{x,0}(\theta)b_{y,0}(\phi)z_{i,j} - a_{x,0}(\theta)b_{y,1}(\phi)z_{i,j+1}\\ &- a_{x,1}(\theta)b_{y,0}(\phi)z_{i+1,j} - a_{x,1}(\theta)b_{y,1}(\phi)z_{i+1,j+1},\end{aligned} \tag{17}$$

where $a_{x,0}(\theta) = (1-\theta)^2(1+2\theta)$, $a_{x,1}(\theta) = \theta^2(3-2\theta)$, $b_{y,0}(\phi) = (1-\phi)^2(1+2\phi)$, and $b_{y,1}(\phi) = \phi^2(3-2\phi)$, are called the blending functions as they blend four separate boundary curves $S(x, y_j)$, $S(x, y_{j+1})$, $S^*(x_i, y)$, and $S^*(x_{i+1}, y)$ to give a single well-defined surface.
Further, one can verify that

(i) the function Φ is $\mathcal{C}^1$-continuous over $\mathcal{D}$.
(ii) Φ interpolates Δ.

4.2 Convergence analysis of bivariate rational spline

The following theorem provides the convergence analysis of the bivariate rational spline.

Theorem 5 *Let Δ^* be a partition of $\mathcal{D}$. Let the interpolation data $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j}) : (i, j) \in \Sigma_M \times \Sigma_N)\}$ be obtained from $f \in \mathcal{C}^4(\mathcal{D})$. Then*

$$\begin{aligned}\|f-\Phi\|_\infty \leq & \|\Delta^*\|\Big(\Big\|\frac{\partial f}{\partial x}\Big\|_\infty + \Big\|\frac{\partial \Phi}{\partial x}\Big\|_\infty\Big) + \frac{\|\Delta^*\|\nu^*}{4}Z^* \\ & + \frac{\|\Delta^*\|^2\nu^*}{384}\Big\{\|\Delta^*\|^2\Big\|\frac{\partial^4 f}{\partial y^4}\Big\|_\infty \max_{(i,j)\in\Sigma_{M-1}\times\Sigma_{N-1}}\Big(1+|r^*_{i,j}-3|/4\Big) \\ & + \max_{(i,j)\in\Sigma_{M-1}\times\Sigma_{N-1}} 4|r^*_{i,j}-3|\Big(\|\Delta^*\|\Big\|\frac{\partial^3 f}{\partial y^3}\Big\|_\infty + 3\Big\|\frac{\partial^2 f}{\partial y^2}\Big\|_\infty\Big)\Big\},\end{aligned} \tag{18}$$

where $\|\Delta^*\| = \max_{(i,j)\in\Sigma_{M-1}\times\Sigma_{N-1}}\{|x_{i+1}-x_i|, |y_{j+1}-y_j|\}$, $\nu^* = \max_{(i,j)\in\Sigma_{M-1}\times\Sigma_{N-1}} \frac{1}{\nu_{i,j}}$,

$$\nu_{i,j} = \begin{cases}(1+r^*_{i,j})/4, & \text{if } -1<\kappa<r^*_{i,j}<3,\\ 1, & \text{if } r^*_{i,j}\geq 3,\end{cases}$$

$$Z^* = \max_{(i,j)\in\Sigma_{M-1}\times\Sigma_{N-1}}\Big\{\Big|\frac{\partial f}{\partial y}|_{(x_i,y_j)} - z_{y,i,j}\Big|, \Big|\frac{\partial f}{\partial y}|_{(x_i,y_j)} - z_{y,i,j+1}\Big|\Big\}.$$

Proof We find an upper bound for $\|f-\Phi\|_\infty$. Expanding $f(x, y)$, $(x, y) \in \mathcal{D}_{i,j}$ at the point $(x_i, y) \in \mathcal{D}_{i,j}$ by using Taylor theorem, we get

$$f(x,y) = f(x_i,y) + (x-x_i)\frac{\partial f(\xi,y)}{\partial x},\ \ (\xi,y)\in\mathcal{D}_{i,j}.$$

This implies

$$|f(x,y)-f(x_i,y)| \leq \|\Delta^*\|\ \Big\|\frac{\partial f}{\partial x}\Big\|_\infty. \tag{19}$$

Similarly, by applying above procedure to the classical interpolant Φ, we obtain

$$|\Phi(x,y)-\Phi(x_i,y)| \leq \|\Delta^*\|\Big\|\frac{\partial \Phi}{\partial x}\Big\|_\infty. \tag{20}$$

Now using (19)-(20), it is easy to see that

$$\left.\begin{aligned}|f(x,y)-\Phi(x,y)| \leq & |f(x,y)-f(x_i,y)| + |f(x_i,y)-\Phi(x_i,y)| \\ & + |\Phi(x_i,y)-\Phi(x,y)|, \\ \leq & \|\Delta^*\|\Big(\Big\|\frac{\partial f}{\partial x}\Big\|_\infty + \Big\|\frac{\partial \Phi}{\partial x}\Big\|_\infty\Big) + |f(x_i,y)-\Phi(x_i,y)|.\end{aligned}\right\} \tag{21}$$

From Proposition 1, it is known that

$$\begin{aligned}|f(x_i,y)-\Phi(x_i,y)| \leq & \frac{\|\Delta^*\|}{4\nu_{i,j}}\max\Big\{\Big|\frac{\partial f}{\partial y}|_{(x_i,y_j)} - z_{y,i,j}\Big|, \Big|\frac{\partial f}{\partial y}|_{(x_i,y_j)} - z_{y,i,j+1}\Big|\Big\} \\ & + \frac{\|\Delta^*\|^2}{384\nu_{i,j}}\Big\{h_j^{*2}\Big\|\frac{\partial^4 f}{\partial y^4}\Big\|_\infty\Big(1+|r^*_{i,j}-3|/4\Big) \\ & + 4|r^*_{i,j}-3|\Big(h^*_j\Big\|\frac{\partial^3 f}{\partial y^3}\Big\|_\infty + 3\Big\|\frac{\partial^2 f}{\partial y^2}\Big\|_\infty\Big)\Big\}.\end{aligned} \tag{22}$$

Substituting (22) in (21), we have

$$\begin{aligned}|f(x,y)-\Phi(x,y)| \leq &\|\Delta^*\|\Big(\Big\|\frac{\partial f}{\partial x}\Big\|_\infty + \Big\|\frac{\partial \Phi}{\partial x}\Big\|_\infty\Big) + \frac{\|\Delta^*\|}{4\nu_{i,j}}\max\Big\{\Big|\frac{\partial f}{\partial y}\Big|_{(x_i,y_j)} - z_{y,i,j}\Big|,\\ &\Big|\frac{\partial f}{\partial y}\Big|_{(x_i,y_j)} - z_{y,i,j+1}\Big|\Big\} + \frac{\|\Delta^*\|^2}{384\nu_{i,j}}\Big\{h_j^{*2}\Big\|\frac{\partial^4 f}{\partial y^4}\Big\|_\infty\Big(1+|r_{i,j}^*-3|/4\Big)\\ &+4|r_{i,j}^*-3|\Big(h_j^*\Big\|\frac{\partial^3 f}{\partial y^3}\Big\|_\infty + 3\Big\|\frac{\partial^2 f}{\partial y^2}\Big\|_\infty\Big)\Big\}.\end{aligned} \tag{23}$$

Since (23) is true for every $(x, y) \in \mathcal{D}_{i,j}$, $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$, we obtain the desired result. □

Convergence results: Theorem 5 reveals that the bivariate rational spline Φ converges uniformly to the original function $f \in \mathcal{C}^4(\mathcal{D})$ as $\|\Delta^*\| \to 0$.

4.3 Constrained aspect of bivariate rational spline

Proposition 3 *[4] A surface which is composition of blended surface patches preserves all properties of the network of the boundary curves.*

The following theorem provides the sufficient conditions on the shape parameters $r_{i,j}$ and $r^*_{i,j}$ for the bivariate rational spline lies above the given piecewise plane whenever the given surface data lying above the given piecewise plane.

Theorem 6 *Let $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j}) : (i, j) \in \Sigma_M \times \Sigma_N\}$ be a surface interpolation data set that lies above the piecewise plane $g(x, y) = \sigma_{i,j}\Big[1 - \frac{x}{\gamma_{i,j}} - \frac{y}{\delta_{i,j}}\Big]$, $(x, y) \in \mathcal{D}_{i,j}$, $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$, $\sigma_{i,j} \in \mathbb{R}$, $\gamma_{i,j}, \delta_{i,j} \in \mathbb{R} - \{0\}$, that is $z_{i,j} > g_{i,j} = g(x_i, y_j)$ for all $(i, j) \in \Sigma_M \times \Sigma_N$. Then the classical bivariate rational spline $\Phi(x, y) > g(x, y)$ for all $(x, y) \in \mathcal{D}$ if the corresponding shape parameters are chosen as*

$$r_{i,j} > \max\left\{-1, \frac{-h_i z_{x,i,j} - \frac{\sigma_{i,j}}{\gamma_{i,j}}h_i}{z_{i,j}-g_{i,j}}, \frac{h_i z_{x,i+1,j} + \frac{\sigma_{i,j}}{\gamma_{i,j}}h_i}{z_{i,j}-g_{i+1,j}}\right\}, (i,j) \in \Sigma_{M-1}\times\Sigma_N,$$

$$r^*_{i,j} > \max\left\{-1, \frac{-h^*_j z_{y,i,j} - \frac{\sigma_{i,j}}{\delta_{i,j}}h^*_j}{z_{i,j}-g_{i,j}}, \frac{h^*_j z_{y,i,j+1} + \frac{\sigma_{i,j}}{\delta_{i,j}}h^*_j}{z_{i,j}-g_{i,j+1}}\right\}, (i,j) \in \Sigma_{M}\times\Sigma_{N-1}.$$

Proof Using Proposition 3, we get that the surface $\Phi(x, y) > g(x, y)$ for all $(x, y) \in \mathcal{D}$ if $S(x, y_j) > g(x, y_j)$ for all $x \in I$, $j \in \Sigma_N$, and $S^*(x_i, y) > g(x_i, y)$ for all $y \in J$, $i \in \Sigma_M$. From Theorem 1, it follows that the univariate rational spline $S(x, y_j)$, $j \in \Sigma_N$ lies above the piecewise line $g(x, y_j)$, $j \in \Sigma_N$, if

$$r_{i,j} > \max\left\{-1, \frac{-h_i z_{x,i,j} - \frac{\sigma_{i,j}}{\gamma_{i,j}}h_i}{z_{i,j}-g_{i,j}}, \frac{h_i z_{x,i+1,j} + \frac{\sigma_{i,j}}{\gamma_{i,j}}h_i}{z_{i,j}-g_{i+1,j}}\right\}, i \in \Sigma_{M-1}. \tag{24}$$

Again, from the Theorem 1, it follows that the univariate rational spline $S^*(x_i, y)$ lies above the piecewise line $g(x_i, y)$, $i \in \Sigma_{M-1}$, if

$$r^*_{i,j} > \max\left\{-1, \frac{-h^*_j z_{y,i,j} - \frac{\sigma_{i,j}}{\delta_{i,j}}h^*_j}{z_{i,j}-g_{i,j}}, \frac{h^*_j z_{y,i,j+1} + \frac{\sigma_{i,j}}{\delta_{i,j}}h^*_j}{z_{i,j}-g_{i,j+1}}\right\}, j \in \Sigma_{N-1}. \tag{25}$$

Combining (24) and (25), we complete the proof. □

Remark 2 From the above theorem, one can observe that when Δ is a positive data set, then bivariate rational spline Φ preserves the positivity if scaling factors are chosen as

$$r_{i,j} > \max\left\{-1, \frac{-h_i z_{x,i,j}}{z_{i,j}}, \frac{h_i z_{x,i+1,j}}{z_{i,j}}\right\}, (i,j) \in \Sigma_{M-1}\times\Sigma_N,$$

$$r^*_{i,j} > \max\left\{-1, \frac{-h^*_j z_{y,i,j}}{z_{i,j}}, \frac{h^*_j z_{y,i,j+1}}{z_{i,j}}\right\}, (i,j) \in \Sigma_{M}\times\Sigma_{N-1}.$$

The following theorem provides the sufficient conditions on the shape parameters $r_{i,j}$ and $r^*_{i,j}$ for the bivariate rational spline lies below the given piecewise plane whenever the given surface data lying below the given piecewise plane.

Theorem 7 *Let $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j}) : (i, j) \in \Sigma_M \times \Sigma_N\}$ be a surface interpolation data set that lies below the piecewise plane $g^*(x, y) = \sigma^*_{i,j}\Big[1 - \dfrac{x}{\gamma^*_{i,j}} - \dfrac{y}{\delta^*_{i,j}}\Big]$, $(x, y) \in \mathcal{D}_{i,j}$, $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$, $\sigma^*_{i,j} \in \mathbb{R}$, $\gamma^*_{i,j}, \delta^*_{i,j} \in \mathbb{R} - \{0\}$, that is $z_{i,j} < g^*_{i,j} = g^*(x_i, y_j)$ for all $(i, j) \in \Sigma_M \times \Sigma_N$. Then the classical rational interpolating surface $\Phi(x, y) < g^*(x, y)$ for all $(x, y) \in \mathcal{D}$ if the corresponding shape parameters are chosen as*

$$r_{i,j} > \max\left\{-1, \frac{h_i z_{x,i,j} + \frac{\sigma^*i,j}{\gamma^*_{i,j}} h_i}{g^*_{i,j} - z_{i,j}}, \frac{-h_i z_{x,i+1,j} - \frac{\sigma^*i,j}{\gamma^*_{i,j}} h_i}{g^*_{i+1,j} - z_{i+1,j}}\right\}, (i, j) \in \Sigma_{M-1} \times \Sigma_N,$$

$$r^*_{i,j} > \max\left\{-1, \frac{h^*_j z_{y,i,j} + \frac{\sigma^*i,j}{\delta^*_{i,j}} h^*_j}{g^*_{i,j} - z_{i,j}}, \frac{-h^*_j z_{y,i,j+1} - \frac{\sigma^*i,j}{\delta^*_{i,j}} h^*_j}{g^*_{i,j+1} - z_{i,j+1}}\right\}, (i, j) \in \Sigma_M \times \Sigma_{N-1}.$$

Proof The proof is very similar to that previous theorem and hence it is omitted. □

The following theorem provides sufficient conditions on the shape parameters $r_{i,j}$ and $r^*_{i,j}$ for the bivariate rational spline that lies within a cuboid whenever the given surface data lies within a cuboid.

Theorem 8 *Let $\Delta = \{(x_i, y_i, z_{i,j}, z_{x,i,j}, z_{y,i,j}) : (i, j) \in \Sigma_M \times \Sigma_N\}$ be a surface interpolation data which lies in the cuboid $\mathbb{D} = [x_1, x_M] \times [y_1, y_N] \times [\Lambda_{11}, \Lambda_{22}]$. Then the classical rational interpolating surface Φ lies inside the cuboid $\mathbb{D}$ if the associated shape parameters are chosen as*

$$r_{i,j} > \max\left\{-1, \frac{-h_i z_{x,i,j}}{z_{i,j} - \Lambda_{11}}, \frac{h_i z_{x,i+1,j}}{z_{i+1,j} - \Lambda_{11}}, \frac{h_i z_{x,i,j}}{\Lambda_{22} - z_{i,j}}, \frac{-h_i z_{x,i+1,j}}{\Lambda_{22} - z_{i+1,j}}\right\}, (i, j) \in \Sigma_{M-1} \times \Sigma_N,$$

and

$$r^*_{i,j} > \max\left\{-1, \frac{-h^*_j z_{y,i,j}}{z_{i,j} - \Lambda_{11}}, \frac{h^*_j z_{y,i,j+1}}{z_{i,j+1} - \Lambda_{11}}, \frac{h^*_j z_{y,i,j}}{\Lambda_{22} - z_{i,j}}, \frac{-h^*_j z_{y,i,j+1}}{\Lambda_{22} - z_{i,j+1}}\right\}, (i, j) \in \Sigma_M \times \Sigma_{N-1},$$

where $\Lambda_{11} < \min\{z_{i,j} : (i, j) \in \Sigma_M \times \Sigma_N\}$ and $\Lambda_{22} > \max\{z_{i,j} : (i, j) \in \Sigma_M \times \Sigma_N\}$.

Proof Using Proposition 3, we get that the surface $\Phi(x, y)$ lies within the cuboid $[x_1, x_M]\times[y_1, y_N]\times[\Lambda_{11}, \Lambda_{22}]$ i.e; $\Lambda_{11} < \Phi(x, y) < \Lambda_{22}$ for all $(x, y) \in \mathcal{D}$ if $\Lambda_{11} < S(x, y_j) < \Lambda_{22}$ for all $x \in I$, $j \in \Sigma_N$ and $\Lambda_{11} < S(x_i, y) < \Lambda_{22}$ for all $y \in J$, $i \in \Sigma_M$.

From Theorem 4, the univariate rational spline $S(x, y_j)$, $j \in \Sigma_N$ lies within the rectangle $[x_1, x_M] \times [\Lambda_{11}, \Lambda_{22}]$ if

$$r_{i,j} > \max\left\{-1, \frac{-h_i z_{x,i,j}}{z_{i,j} - \Lambda_{11}}, \frac{h_i z_{x,i+1,j}}{z_{i+1,j} - \Lambda_{11}}, \frac{h_i z_{x,i,j}}{\Lambda_{22} - z_{i,j}}, \frac{-h_i z_{x,i+1,j}}{\Lambda_{22} - z_{i+1,j}}\right\}, i \in \Sigma_{M-1}. \tag{26}$$

Again, from Theorem 4, the univariate rational spline $S^*(x_i, y)$, $i \in \Sigma_M$ lies within the rectangle $[y_1, y_N] \times [\Lambda_{11}, \Lambda_{22}]$ if

$$r^*_{i,j} > \max\left\{-1, \frac{-h^*_j z_{y,i,j}}{z_{i,j} - \Lambda_{11}}, \frac{h^*_j z_{y,i,j+1}}{z_{i,j+1} - \Lambda_{11}}, \frac{h^*_j z_{y,i,j}}{\Lambda_{22} - z_{i,j}}, \frac{-h^*_j z_{y,i,j+1}}{\Lambda_{22} - z_{i,j+1}}\right\}, j \in \Sigma_{N-1}. \tag{27}$$

Combining (26) and (27), we complete the proof. □

4.4 Monotonicity preserving rational surface

Definition 1 A surface data set $\{(x_i, y_j, z_{i,j}); i \in \Sigma_{M-1}, j \in \Sigma_{N-1}\}$ is said to monotonically increasing if $z_{i+1,j} \geq z_{i,j}, i \in \Sigma_{M-1}$, for each fixed $j \in \Sigma_N$; and $z_{i,j+1} \geq z_{i,j}, j \in \Sigma_{N-1}$, for each fixed $i \in \Sigma_M$.

Definition 2 A surface data set $\{(x_i, y_j, z_{i,j}); i \in \Sigma_{M-1}, j \in \Sigma_{N-1}\}$ is said to monotonically decreasing if $z_{i+1,j} \leq z_{i,j}, i \in \Sigma_{M-1}$, for each fixed $j \in \Sigma_N$; and $z_{i,j+1} \leq z_{i,j}, j \in \Sigma_{N-1}$, for each fixed $i \in \Sigma_M$.

Definition 3 A surface data set $\{(x_i, y_j, z_{i,j}); i \in \Sigma_{M-1}, j \in \Sigma_{N-1}\}$ is said to be monotonic if it is either monotonically increasing or decreasing.

The following theorem provides the sufficient conditions on the shape parameters $r_{i,j}$ and $r^*_{i,j}$ for the monotonicity preserving rational surface whenever the given surface data is monotonic.

Theorem 9 *Let* $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j}) : (i, j) \in \Sigma_M \times \Sigma_N\}$ *be a monotonic surface data set. Then the classical rational interpolating surface preserves the monotonicity if the corresponding shape parameters are chosen as*

$$\begin{aligned} r_{i,j} &> \max\left\{-1, \frac{z_{x,i,j} + z_{x,i+1,j}}{z_{i+1,j} - z_{i,j}}\right\}, (i, j) \in \Sigma_{M-1} \times \Sigma_N, \\ \textit{and } r^*_{i,j} &> \max\left\{-1, \frac{z_{y,i,j} + z_{y,i,j+1}}{z_{i,j+1} - z_{i,j}}\right\}, (i, j) \in \Sigma_M \times \Sigma_{N-1}. \end{aligned} \tag{28}$$

Proof Using Proposition 3, we get that the surface Φ is monotonic if $S(x, y_j)$ is monotonic for all $x \in I, j \in \Sigma_N$, and $S^*(x_i, y)$ for all $y \in J, i \in \Sigma_M$. By using Proposition 2, it follows that the univariate rational spline $S(x, y_j), j \in \Sigma_N$ is monotonic if

$$r_{i,j} > \max\left\{-1, \frac{z_{x,i,j} + z_{x,i+1,j}}{z_{i+1,j} - z_{i,j}}\right\}, (i, j) \in \Sigma_{M-1} \times \Sigma_N. \tag{29}$$

Again, by using Proposition 2, it follows that the univariate rational spline $S^*(x_i, y)$ is monotonic if

$$r^*_{i,j} > \max\left\{-1, \frac{z_{y,i,j} + z_{y,i,j+1}}{z_{i,j+1} - z_{i,j}}\right\}, (i, j) \in \Sigma_M \times \Sigma_{N-1}. \tag{30}$$

We complete the proof by combining (29) and (30). □

5 Solving constraint partial differential equations by bivariate rational spline

Let us consider the second-order partial differential equation (PDE):

$$F(x, y, z, z_x, z_y, z_{xx}, z_{yy}, z_{xy}) = 0, (x, y) \in [\beta_1, \beta_2] \times [\lambda_1, \lambda_2] \tag{31}$$

subject to the given initial and boundary conditions, and constraints $z(x) > g(x)$ for all $x \in [\beta_1, \beta_2] \times [\lambda_1, \lambda_2]$ or $z(x) < g^*(x)$ for all $x \in [\beta_1, \beta_2] \times [\lambda_1, \lambda_2]$, or $\Lambda_{11} < z(x) < \Lambda_{22}$ for all $x \in [\beta_1, \beta_2] \times [\lambda_1, \lambda_2]$, where g and g^* are piecewise planes and $\Lambda_{11}, \Lambda_{22} \in \mathbb{R}$. We seek an approximate solution for the PDE (31) which obeys the given constraint. To achieve this, we proceed as follows.

Let $\beta_1 = x_1 < x_2 < \ldots < x_{M-1} < x_M = \beta_2, \lambda_1 = y_1 < y_2 \ldots < y_N = \lambda_2$ be a partition of $[\beta_1, \beta_2] \times [\lambda_1, \lambda_2]$. By exploiting a suitable numerical method [33,34], we solve the PDE (31) to get $z(x_i, y_j), (i, j) \in \Sigma_M \times \Sigma_N$ approximately. Next, using the numerical methods such as arithmetic mean and geometric mean [31,32], we obtain the partial derivative values $z_x(x_i, y_j), z_y(x_i, y_j) \in \Sigma_M \times \Sigma_N$ approximately. One should eliminate the points from the above data set which do not satisfy the given constraint. Then we obtain data set $\{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j}) : (i, j) \in \Sigma_M \times \Sigma_N\}$, where $z_{i,j} = z(x_i, y_j)$, $z_{x,i,j} = z_x(x_i, y_j)$, and $z_{y,i,j} = z_y(x_i, y_j)$. We construct the bivariate rational spline Φ with respect to the above surface data and the shape parameters $r_{i,j}$ and $r^*_{i,j}$ chosen according to Theorem 6 if the constraint is $z(x) > g(x)$ for all $x \in [\beta_1, \beta_2] \times [\lambda_1, \lambda_2]$ or Theorem 7 if the constraint is $z(x) < g^*(x)$ for all $x \in [\beta_1, \beta_2] \times [\lambda_1 \lambda_2]$, or Theorem 8 if the constraint is $\Lambda_{11} < z(x) < \Lambda_{22}$ for all $x \in [\beta_1, \beta_2] \times [\lambda_1, \lambda_2]$.

From the Theorem 5, it follows that the bivariate rational spline S converges to the exact solution of constraint PDE (31) as $M \to \infty$ and $N \to \infty$.

Table 1 Surface data pertaining to the PDE (32)

↓ y/x →	0	0.1	0.2	0.3
0	(0, 3.142, 0)	(0.309, 2.988, −3.050)	(0.588, 2.542, −5.801)	(0.809, 1.847, −7.985)
0.01	(0, 2.846, 0)	(0.282, 2.707, −2.763)	(0.535, 2.303, −5.256)	(0.737, 1.673, −7.234)
0.02	(0, 2.579, 0)	(0.256, 2.453, −2.504)	(0.488, 2.086, −4.762)	(0.671, 1.516, −6.554)
0.03	(0, 2.336, 0)	(0.234, 2.222, −2.268)	(0.444, 1.890, −4.314)	(0.611, 1.373, −5.938)
0.04	(0, 2.117, 0)	(0.213, 2.013, −2.055)	(0.405, 1.713, −3.909)	(0.557, 1.244, −5.380)
0.05	(0, 1.918, 0)	(0.194, 1.824, −1.862)	(0.368, 1.552, −3.542)	(0.507, 1.127, −4.875)
0.06	(0, 1.738, 0)	(0.176, 1.653, −1.687)	(0.336, 1.406, −3.209)	(0.462, 1.021, −4.416)

Table 2 Surface data pertaining to the PDE (32)

↓ y/x →	0.4	0.5	0.6	0.7
0	(0.951, 0.971, −9.387)	(1, 0, −9.870)	(0.951, −0.971, −9.387)	(0.809, −1.847, −7.985)
0.01	(0.866, 0.880, −8.504)	(0.911, 0, −8.942)	(0.866, −0.880, −8.504)	(0.737, −1.673, −7.234)
0.02	(0.789, 0.797, −7.705)	(0.830, 0, −8.102)	(0.789, −0.797, −7.705)	(0.671, −1.516, −6.554)
0.03	(0.719, 0.722, −6.981)	(0.756, 0, −7.340)	(0.719, −0.722, −6.981)	(0.611, −1.373, −5.938)
0.04	(0.655, 0.654, −6.325)	(0.688, 0, −6.650)	(0.655, −0.654, −6.325)	(0.557, −1.244, −5.380)
0.05	(0.596, 0.593, −5.731)	(0.627, 0, −6.025)	(0.596, −0.593, −5.731)	(0.507, −1.127, −4.875)
0.06	(0.543, 0.537, −5.192)	(0.571, 0, −5.459)	(0.543, −0.537, −5.192)	(0.462, −1.021, −4.416)

Table 3 Surface data pertaining to the PDE (32)

↓ y/x →	0.8	0.9	1
0	(0.588, −2.542, −5.801)	(0.309, −2.988, −3.05)	(0, −3.142, 0)
0.01	(0.535, −2.303, −5.256)	(0.282, −2.707, −2.763)	(0, −2.846, 0)
0.02	(0.488, −2.086, −4.762)	(0.256, −2.453, −2.504)	(0, −2.579, 0)
0.03	(0.444, −1.890, −4.314)	(0.234, −2.222, −2.268)	(0, −2.336, 0)
0.04	(0.405, −1.713, −3.909)	(0.213, −2.013, −2.055)	(0, −2.117, 0)
0.05	(0.368, −1.552, −3.542)	(0.194, −1.824, −1.862)	(0, −1.918, 0)
0.06	(0.336, −1.406, −3.209)	(0.176, −1.653, −1.687)	(0, −1.738, 0)

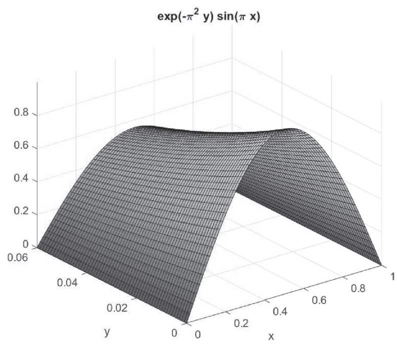
5.1 Numerical examples

Let us consider the PDE

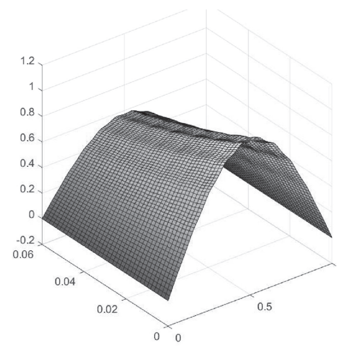
$$z_y = z_{xx}, 0 \leq x \leq 1, t \geq 0, z(0, y) = z(1, y) = 0 \text{ and } z(x, 0) = \sin(\pi x) \tag{32}$$

subject to the constraint $0 \leq z(x, y) \leq 1$ for all $(x, y) \in [0, 1] \times [0, 0.06]$. The exact solution of the given PDE is plotted in Figure 2(a). The triplets shown in the body of the Table 1-3 indicate the function values and the partial derivative values, i.e., $(z_{i,j}, z_{x,i,j}, z_{y,i,j})$. Solving the PDE (32) by the Lassonen (BTCS) scheme [34], we acquire $z_{i,j}$ given in Table 1-3. Next, employing the arithmetic mean and geometric mean [31,32], we obtain the partial derivative values $z_{x,i,j}$ and $z_{y,i,j}$. Thus, we get the surface data in Table 1-3 pertaining to the PDE (32).

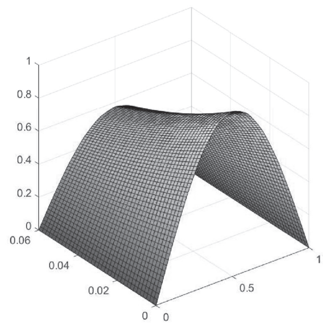
It can be seen that the above data lies within the cuboid $[0, 1] \times [0, 0.06] \times [0, 1]$, i.e., $0 \leq z_{i,j} \leq 1$ for all (i, j). The bivariate rational spline Φ_1 is constructed in Figure 2(b) by using the surface data given in Table 1-3 and the arbitrarily chosen shape parameters. We can easily notice that the bivariate rational spline Φ_1 is not satisfying the condition $0 \leq \Phi_1(x, y) \leq 1$ for all $(x, y) \in [0, 1] \times [0, 0.06]$. Hence, Φ_1 is not a solution for the PDE (32). Next, the bivariate rational spline Φ_2 is constructed in Figure 2(c) by using the surface data given Table 1-3 and the shape parameters chosen according to Theorem 8. Further, it is clear that $0 \leq \Phi_2(x, y) \leq 1$ for all $(x, y) \in [0, 1] \times [0, 0.06]$, and hence Φ_2 is an approximate solution for the given PDE. The xz view of Figure 2(c) is given in Figure 2(d).



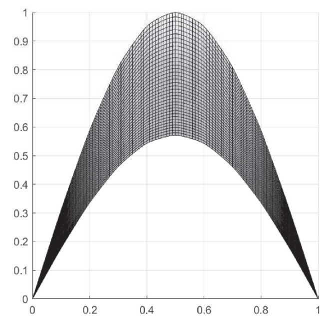
(a) Exact solution of the PDE



(b) Unconstrained bivariate rational spline Φ_1 with $r_{i,j} = 0.5, r^*_{i,j} = -0.5$.



(c) Constrained bivariate rational spline with $r_{i,j} = 5, r^*_{i,j} = 10$.



(d) xz view of Figure 2(c)

Fig. 2 Exact and approximate solutions of PDE (32)

6 Bivariate $\mathcal{C}^1$-fractal rational spline

Let $\Delta^* = \{x_1, x_2 \ldots, x_{M-1}, x_M, y_1, y_2 \ldots, y_{N-1}, y_N\}$, where $x_1 < x_2 \cdots < x_{M-1} < x_M; y_1 < y_2 \cdots < y_{N-1} < y_N$. Let the given surface interpolation data be $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j});\ (i, j) \in \Sigma_M \times \Sigma_N\}$, where $z_{x,i,j}$ and $z_{y,i,j}$ are derivatives value with respect to x and y respectively at point (x_i, y_j). Let $\Sigma_M = \{1, 2, \ldots, M\}$, $\Sigma_N = \{1, 2, \ldots, N\}$, $\Sigma_{M-1} = \{1, 2, \ldots, M-1\}$, $\Sigma_{N-1} = \{1, 2, \ldots, N-1\}$. Denote $I = [x_1, x_M]$, $J = [y_1, y_N]$, $\mathcal{D} = I \times J$, $I_i = [x_i, x_{i+1}]$, $J_j = [y_j, y_{j+1}]$, $\mathcal{D}_{i,j} = I_i \times J_j$, $h_i = x_{i+1} - x_i$, $h^*_j = y_{j+1} - y_j$. and $\mathcal{B}(\mathcal{D})$ denotes the boundary of $\mathcal{D}$. Suppose that Δ^* is a partition of $\mathcal{D}$. For any $i \in \Sigma_{M-1}$, let $u_i : I \to I_i$ be a contractive homeomorphism satisfying

$$\begin{cases} u_i(x_1) = x_i,\ \ u_i(x_M) = x_{i+1}, \text{and} \\ |u_i(x^*) - u_i(x^{**})| \le \zeta_i |x^* - x^{**}|\ \forall\, x^*, x^{**} \in I, \end{cases} \tag{33}$$

where $0 < \zeta_i < 1$ is a given constant. Similarly, for any $j \in \Sigma_{N-1}$, let $v_j : J \to J_j$ be a contractive homeomorphism satisfying

$$\begin{cases} v_j(y_1) = y_j,\ \ v_j(y_N) = y_{j+1}, \text{and} \\ |v_j(y^*) - v_j(y^{**})| \le \varrho_j |y^* - y^{**}|\ \ \forall\, y^*, y^{**} \in J, \end{cases} \tag{34}$$

where $0 < \varrho_i < 1$ is a given constant. We take $u_i(x) = a_i x + b_i$. and $v_j(y) = c_j y + d_j$, where a_i, b_i, c_j,and d_j are constants to be determined by (33)-(34).

Let $K = I \times J \times \mathbb{R}$. For each $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$, let $F_{i,j} : K \to \mathbb{R}$ be a continuous function defined by

$$F_{i,j}(x, y, z) = \alpha_{i,j} z + \Phi(u_i(x), v_j(y)) - \alpha_{i,j} b(x, y),$$

where $\alpha_{i,j} \in \mathbb{R}$ such that $|\alpha_{i,j}| < a_i c_j$ and Φ is rational surface defined in (17), $b \in \mathcal{C}^1(\mathcal{D})$, $b \neq \Phi$, such that

$$b(x, y) = \Phi(x, y), \left.\frac{\partial b}{\partial x}\right|_{(x,y)} = \left.\frac{\partial \Phi}{\partial x}\right|_{(x,y)}, \text{ and } \left.\frac{\partial b}{\partial y}\right|_{(x,y)} = \left.\frac{\partial \Phi}{\partial y}\right|_{(x,y)} \forall (x, y) \in \mathcal{B}(\mathcal{D}).$$

Now, for each $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$, we define $W_{i,j} : K \to I_i \times J_j \times \mathbb{R}$ by

$$W_{i,j}(x, y, z) = \big(u_i(x), v_j(y), F_{i,j}(x, y, z)\big). \tag{35}$$

Then $\{K; W_{i,j} : (i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}\}$ is an IFS.

Using [28], we can show that the above IFS has a unique attractor G. In the following theorem, we construct a bivariate FIF whose graph is G.

Theorem 10 *Let Δ^* be a partition of $\mathcal{D}$. Let the interpolation data be $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j});$ $(i, j) \in \Sigma_M \times \Sigma_N\}$. Let Φ be the bivariate function defined in* (17) *Consider the IFS $\{K; \big(u_i(x), v_j(y),$ $F_{i,j}(x, y, z)), (i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}\}$. Suppose that $|\alpha_{i,j}| < a_i c_j$, $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$, $b \in \mathcal{C}^1(\mathcal{D})$ such that $b \neq \Phi$,*

$$b(x, y) = \Phi(x, y), \left.\frac{\partial b}{\partial x}\right|_{(x,y)} = \left.\frac{\partial \Phi}{\partial x}\right|_{(x,y)}, \text{ and } \left.\frac{\partial b}{\partial y}\right|_{(x,y)} = \left.\frac{\partial \Phi}{\partial y}\right|_{(x,y)} \forall (x, y) \in \mathcal{B}(\mathcal{D}).$$

Then the above IFS determines a bivariate fractal interpolation function $\Phi^\alpha \in \mathcal{C}^1(\mathcal{D})$ that interpolates Δ.

Proof Let $\mathcal{C}^1_\Phi(\mathcal{D}) = \Big\{h \in \mathcal{C}^1(\mathcal{D}) \; : \; h(x, y) = \Phi(x, y), \left.\frac{\partial h}{\partial x}\right|_{(x,y)} = \left.\frac{\partial \Phi}{\partial x}\right|_{(x,y)}, \left.\frac{\partial h}{\partial y}\right|_{(x,y)} = \left.\frac{\partial \Phi}{\partial y}\right|_{(x,y)}$ for all $(x, y) \in \mathcal{B}(\mathcal{D})\Big\}$. Then $\mathcal{C}^1_\Phi(\mathcal{D})$ is a complete metric space with respect to the metric induced by the $\mathcal{C}^1$-norm, namely,

$$\|h\|_{\mathcal{C}^1} = \|h\|_\infty + \left\|\frac{\partial h}{\partial x}\right\|_\infty + \left\|\frac{\partial h}{\partial y}\right\|_\infty.$$

Now define Read-Bajraktarević operator $T : \mathcal{C}^1_\Phi(\mathcal{D}) \to \mathcal{C}^1_\Phi(\mathcal{D})$ as

$$(Th)(x, y) = \Phi(x, y) + \alpha_{i,j}[(h - b)\big(u_i^{-1}(x), v_j^{-1}(y)\big)], (x, y) \in \mathcal{D}_{i,j}, (i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}. \tag{36}$$

Since $\Phi \in \mathcal{C}^1(\mathcal{D})$, $b \in \mathcal{C}^1(\mathcal{D})$ and $h \in \mathcal{C}^1(\mathcal{D})$, it follow that operator Th is continuous over each sub rectangle $\mathcal{D}_{i,j}$. Now for continuity of Th over $\mathcal{D}$ one need to show that Th is continuous along the each grid line $x = x_{i+1}$, $i \in \Sigma_{M-2}$ and $y = y_{j+1}$, $j \in \Sigma_{N-2}$. To check continuity along the grid line $x = x_{i+1}$, we can take $y = (1 - \mu)y_j + \mu y_{j+1}$, $0 \le \mu \le 1$,

$$\lim_{\substack{(x,y)\to(x_{i+1},y) \\ (x,y)\in\mathcal{D}_{i,j}}} Th(x, y) = \Phi\big(x_{i+1}, (1 - \mu)y_j + \mu y_{j+1}\big) + \alpha_{i,j}[(h - b)\big(x_M, (1 - \mu)y_1 + \mu y_N\big)],$$

$$\text{and} \lim_{\substack{(x,y)\to(x_{i+1},y) \\ (x,y)\in\mathcal{D}_{i+1,j}}} Th(x, y) = \Phi\big(x_{i+1}, (1 - \mu)y_j + \mu y_{j+1}\big) + \alpha_{i+1,j}[(h - b)\big(x_1, (1 - \mu)y_1 + \mu y_N\big)].$$

Since $b(x, y) = \Phi(x, y)$ for all $(x, y) \in \mathcal{B}(\mathcal{D})$, it follows that

$$\lim_{\substack{(x,y)\to(x_{i+1},y) \\ (x,y)\in\mathcal{D}_{i,j}}} Th(x, y) = \lim_{\substack{(x,y)\to(x_{i+1},y) \\ (x,y)\in\mathcal{D}_{i+1,j}}} Th(x, y).$$

Similarly, one can show that Th is continuous along the each grid line $y = y_{j+1}$ $j \in \Sigma_{N-1}$.

For all $(x, y) \in \mathcal{D}_{i,j}$, $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$, partial differentiation of Th with respect to x and y respectively, leads to

$$\frac{\partial (Th)(x, y)}{\partial x} = \frac{1}{a_i}\left\{\frac{\partial \Phi(x, y)}{\partial x} + \alpha_{i,j}\frac{\partial (h-b)(u_i^{-1}(x), v_j^{-1}(y))}{\partial x}\right\}, \tag{37}$$

$$\frac{\partial (Th)(x, y)}{\partial y} = \frac{1}{c_j}\left\{\frac{\partial \Phi(x, y)}{\partial y} + \alpha_{i,j}\frac{\partial (h-b)(u_i^{-1}(x), v_j^{-1}(y))}{\partial y}\right\}. \tag{38}$$

Using the conditions on the function b and the arguments used for establishing continuity of Th over $\mathcal{D}$, we can show that $\frac{\partial (Th)}{\partial x}$ and $\frac{\partial (Th)}{\partial y}$ are continuous over $\mathcal{D}$.

It is easy to verify that $Th(x, y) = T\Phi(x, y)$, $\frac{\partial (Th)}{\partial x}\Big|_{(x,y)} = \frac{\partial (T\Phi)}{\partial x}\Big|_{(x,y)}$, and $\frac{\partial (Th)}{\partial y}\Big|_{(x,y)} = \frac{\partial (T\Phi)}{\partial y}\Big|_{(x,y)}$ for all $(x, y) \in \mathcal{B}(\mathcal{D})$.

Therefore, $Th \in \mathcal{C}^1_\Phi(\mathcal{D})$. Next, for $h, \psi \in \mathcal{C}^1_\Phi(\mathcal{D})$, from equation (36)- (38), we obtain

$$\|T\psi - Th\|_\infty \le \max\{|\alpha_{i,j}| : i \in \Sigma_{M-1}, j \in \Sigma_{N-1}\}\|\psi - h\|_\infty,$$
$$\left\|\frac{\partial (T\psi)}{\partial x} - \frac{\partial (Th)}{\partial x}\right\|_\infty \le \max\{|\alpha_{i,j}|a_i^{-1} : i \in \Sigma_{M-1}, j \in \Sigma_{N-1}\}\left\|\frac{\partial \psi}{\partial x} - \frac{\partial h}{\partial x}\right\|_\infty,$$
$$\left\|\frac{\partial (T\psi)}{\partial y} - \frac{\partial (Th)}{\partial y}\right\|_\infty \le \max\{|\alpha_{i,j}|c_j^{-1} : i \in \Sigma_{M-1}, j \in \Sigma_{N-1}\}\left\|\frac{\partial \psi}{\partial y} - \frac{\partial h}{\partial y}\right\|_\infty$$

respectively. By using the above inequalities, we can write

$$\begin{aligned}\|T\psi - Th\|_{\mathcal{C}^1} &= \|T\psi - Th\|_\infty + \left\|\frac{\partial (T\psi)}{\partial x} - \frac{\partial (Th)}{\partial x}\right\|_\infty + \left\|\frac{\partial (T\psi)}{\partial y} - \frac{\partial (Th)}{\partial y}\right\|_\infty,\\ &\le \max\{|\alpha_{i,j}|a_i^{-1}c_j^{-1} : i \in \Sigma_{M-1}, j \in \Sigma_{N-1}\}\|\psi - h\|_{\mathcal{C}^1}.\end{aligned} \tag{39}$$

Since $|\alpha_{i,j}| < a_i c_j$, it follows that T is a contraction map on $\mathcal{C}^1_\Phi(\mathcal{D})$, and hence by the Banach fixed point theorem, T has a unique fixed point, say $\Phi^\alpha \in \mathcal{C}^1(\mathcal{D})$. Therefore,

$$\Phi^\alpha\big(u_i(x), v_j(y)\big) = \alpha_{i,j}\Phi(x, y) + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j}b(x, y), (x, y) \in \mathcal{D}. \tag{40}$$

Further, we can verify that Φ^α interpolates the data Δ. Thus, we complete the proof. □

6.1 Convergence analysis of bivariate fractal rational spline

The objective of this section is to obtain an upper bound for the interpolation error $\Phi^\alpha - f$, where the data generating function $f \in \mathcal{C}^2(\mathcal{D})$. The implicit and recursive nature of bivariate fractal function Φ^α make it difficult to apply the traditional approaches for error analysis available in the classical interpolation theory for obtaining its convergence results. Owing to this reason, we establish the upper bound for $\Phi^\alpha - f$, using the triangle inequality

$$\|\Phi^\alpha - f\|_\infty \le \|\Phi^\alpha - \Phi\|_\infty + \|\Phi - f\|_\infty. \tag{41}$$

Theorem 11 *Let Δ^* be a partition of $\mathcal{D}$. Let the interpolation data $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j}) : (i, j) \in \Sigma_M \times \Sigma_N)\}$ be obtained from $f \in \mathcal{C}^2(\mathcal{D})$. Then*

$$\begin{aligned}\big\|f - \Phi^\alpha\big\|_\infty \le &\frac{\|\Delta^*\|}{(1-|\alpha|_\infty)|I|\,|J|}\|\Phi - b\|_\infty\\ &+ \|\Delta^*\|\Big(\Big\|\frac{\partial f}{\partial x}\Big\|_\infty + \Big\|\frac{\partial \Phi}{\partial x}\Big\|_\infty\Big) + \frac{\|\Delta^*\|\nu^*}{4}Z^*\\ &+ \frac{\|\Delta^*\|^2\nu^*}{384}\Big\{\|\Delta^*\|^2\Big\|\frac{\partial^4 f}{\partial y^4}\Big\|_\infty \max_{(i,j)\in\Sigma_{M-1}\times\Sigma_{N-1}}\Big(1 + |r^*_{i,j} - 3|/4\Big)\\ &+ \max_{(i,j)\in\Sigma_{M-1}\times\Sigma_{N-1}} 4|r^*_{i,j} - 3|\Big(\|\Delta^*\|\Big\|\frac{\partial^3 f}{\partial y^3}\Big\|_\infty + 3\Big\|\frac{\partial^2 f}{\partial y^2}\Big\|_\infty\Big)\Big\},\end{aligned} \tag{42}$$

where $\|\Delta^*\| = \max_{(i,j)\in\Sigma_{M-1}\times\Sigma_{N-1}} \{|x_{i+1}-x_i|, |y_{j+1}-y_j|\},\ |I| = x_M - x_1,\ |J| = y_N - y_1,$

$$|\alpha|_\infty = \max_{(i,j)\in\Sigma_{M-1}\times\Sigma_{N-1}} |\alpha_{i,j}|,\ \nu^* = \max_{(i,j)\in\Sigma_{M-1}\times\Sigma_{N-1}} \frac{1}{\nu_{i,j}},$$

$$\nu_{i,j} = \begin{cases} (1+r^*_{i,j})/4, & \text{if } -1 < \kappa < r^*_{i,j} < 3, \\ 1, & \text{if } r^*_{i,j} \geq 3, \end{cases}$$

$$Z^* = \max_{(i,j)\in\Sigma_{M-1}\times\Sigma_{N-1}} \left\{ \left| \frac{\partial f}{\partial y}\Big|_{(x_i,y_j)} - z_{y,i,j} \right|, \left| \frac{\partial f}{\partial y}\Big|_{(x_i,y_j)} - z_{y,i,j+1} \right| \right\}.$$

Proof The bivariate fractal interpolation function Φ^α satisfy the following self-referential equation: for all $(x, y) \in \mathcal{D},\ (i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$,

$$\Phi^\alpha\big(u_i(x), v_j(y)\big) = \alpha_{i,j}\Phi^\alpha(x, y) + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j}b(x, y). \tag{43}$$

Subsequently, from (43), we get

$$\begin{aligned} |\Phi^\alpha\big(u_i(x), v_j(x)\big) - \Phi\big(u_i(x), v_j(x)\big)| &\leq |\alpha_{i,j}|\big(|\Phi^\alpha(x, y) - \Phi(x, y)| + |\Phi(x, y) - b(x, y)|\big) \\ &\leq |\alpha|_\infty\big(\|\Phi^\alpha - \Phi\|_\infty + \|\Phi - b\|_\infty\big). \end{aligned} \tag{44}$$

The above inequality yields

$$\|\Phi^\alpha - \Phi\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|\Phi - b\|_\infty. \tag{45}$$

Since the scaling factors of $\mathcal{C}^1$-FRIF Φ^α obey the condition $|\alpha_{i,j}| < a_i c_j = \big(\frac{x_{i+1}-x_i}{x_M-x_1}\big)\big(\frac{y_{j+1}-y_j}{y_N-y_1}\big)$ and using (45), we get that

$$\|\Phi^\alpha - \Phi\|_\infty \leq \frac{\|\Delta^*\|}{(1 - |\alpha|_\infty)|I|\ |J|}\|\Phi - b\|_\infty. \tag{46}$$

From (18), we get an upper bound for $\|f - \Phi\|_\infty$. Substituting (46) and (18) in (41), we obtain the desired result. □

Convergence results: Theorem 11 reveals that the $\mathcal{C}^1$-fractal rational spline Φ^α converges uniformly to the original function $f \in \mathcal{C}^4(\mathcal{D})$ as $\|\Delta^*\| \to 0$.

7 Constrained aspect of fractal rational surface

In this section, we discuss the constrained aspect of fractal rational surface Φ^α. The following theorem provides the sufficient condition on the scaling factors $\alpha_{i,j}$, $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$ so that fractal rational surface Φ^α is lying above the function φ whenever the bivariate rational spline Φ lies above φ.

Theorem 12 *Let the given surface data be* $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j}); (i, j) \in \Sigma_M \times \Sigma_N\}$. *Let* Φ *be the bivariate function defined in* (17). *Let* $\varphi \in \mathcal{C}^1(\mathcal{D})$ *such that* $\Phi(x, y) > \varphi(x, y)$ *for all* $(x, y) \in \mathcal{D}$. *Let* $b \in \mathcal{C}^1(D)$ *such that* $b \neq \Phi$, *satisfy the conditions :* $b(x, y) = \Phi(x, y)$, $\frac{\partial b}{\partial x}\Big|_{(x,y)} = \frac{\partial \Phi}{\partial x}\Big|_{(x,y)}$, *and* $\frac{\partial b}{\partial y}\Big|_{(x,y)} = \frac{\partial \Phi}{\partial y}\Big|_{(x,y)}$ *for all* $(x, y) \in \mathcal{B}(\mathcal{D})$. *Then the bivariate fractal function* Φ^α *corresponding to IFS* (35) *satisfy* $\Phi^\alpha(x, y) > \varphi(x, y)$ *for all* $(x, y) \in \mathcal{D}$ *if the scaling factors* $\alpha_{i,j}$, $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$ *are chosen as follows:*

$$0 \leq \alpha_{i,j} \leq \min\left\{a_i c_j, \frac{m(\Phi - \varphi, \mathcal{D}_{i,j})}{M^* - m_*(\varphi)}\right\},$$

where $M^* = \max\{b(x, y) : (x, y) \in \mathcal{D}\}$, $m(\Phi - \varphi, \mathcal{D}_{i,j}) = \min\{(\Phi - \varphi)(x, y) : (x, y) \in \mathcal{D}_{i,j}\}$, *and* $m_*(\varphi) = \min\{\varphi(x, y) : (x, y) \in \mathcal{D}\}$.

Proof Since $\Phi(x,y) > \varphi(x,y)$ for all $(x,y) \in \mathcal{D}$ and $b(x,y)$ matches with $\Phi(x,y)$ along the boundary of $\mathcal{D}$, it is easy to verify that $M^* - m_*(\varphi) > 0$ and $m(\Phi - \varphi, \mathcal{D}_{i,j}) > 0$ for all $(i,j) \in \Sigma_{M-1} \times \Sigma_{N-1}$. From (40), it follows that the bivariate fractal function Φ^α satisfy the following functional equation

$$\Phi^\alpha\big(u_i(x), v_j(y)\big) = \alpha_{i,j}\Phi^\alpha(x,y) + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j}b(x,y),\ (x,y) \in \mathcal{D},\ (i,j) \in \Sigma_{M-1} \times \Sigma_{N-1}.$$

At $(n+1)$-th iteration, Φ^α is computed at $\big((M-1)^{n+2}+1\big)\big((N-1)^{n+2}+1\big)$ distinct points in $\mathcal{D}$ by using the values of Φ^α at n-th iteration at $\big((M-1)^{n+1}+1\big)\big((N-1)^{n+1}+1\big)$ number of distinct points in $\mathcal{D}$.

We observe that

$$\Phi^\alpha(x_i, y_j) = \Phi(x_i, y_j) > \varphi(x_i, y_j),\ (i,j) \in \Sigma_M \times \Sigma_N.$$

Therefore, to prove that $\Phi^\alpha(z,t) > \varphi(z,t)$ for all $(z,t) \in \mathcal{D}$, it is sufficient to show that if $\Phi^\alpha(x,y) > \varphi(x,y)$ for all $(x,y) \in \mathcal{D}$, then $\Phi^\alpha\big(u_i(x), v_j(y)\big) > \varphi\big(u_i(x), v_j(y)\big)$, $(x,y) \in \mathcal{D}$ for all $(i,j) \in \Sigma_{M-1} \times \Sigma_{N-1}$.

Suppose that $\Phi^\alpha(x,y) > \varphi(x,y)$, $(x,y) \in \mathcal{D}$. We have to establish that

$$\alpha_{i,j}\Phi^\alpha(x,y) + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j}b(x,y) - \varphi\big(u_i(x), v_j(y)\big) > 0,\ (x,y) \in \mathcal{D}. \tag{47}$$

As per our assumption $\Phi^\alpha(x,y) > \varphi(x,y)$, $(x,y) \in \mathcal{D}$ and $\alpha_{i,j} \geq 0$ for all $(i,j) \in \Sigma_{M-1} \times \Sigma_{N-1}$ and by the definition of M^*, $m_*(\varphi)$ and $m(\Phi - \varphi, \mathcal{D}_{i,j})$, we get

$$\alpha_{i,j}\Phi^\alpha(x,y) + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j}b(x,y) - \varphi\big(u_i(x), v_j(y)\big) \geq \alpha_{i,j}\varphi(x,y) + m(\Phi - \varphi, \mathcal{D}_{i,j}) - \alpha_{i,j}M^*.$$

we can conclude that $\alpha_{i,j} \leq \dfrac{m(\Phi - \varphi, \mathcal{D}_{i,j})}{M^* - m_*(\varphi)}$ assures (47). Thus, we complete the proof. □

Similarly, we can prove the following theorem.

Theorem 13 *Let the given surface data be* $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j});\ (i,j) \in \Sigma_M \times \Sigma_N\}$. *Let* Φ *be the bivariate function defined in* (17). *Let* $\varphi \in \mathcal{C}^1(\mathcal{D})$ *such that* $\Phi(x,y) < \varphi(x,y)$ *for all* $(x,y) \in \mathcal{D}$. *Let* $b \in \mathcal{C}^1(D)$ *such that* $b \neq \Phi$, *satisfy the conditions* $b(x,y) = \Phi(x,y)$, $\dfrac{\partial b}{\partial x}\Big|_{(x,y)} = \dfrac{\partial \Phi}{\partial x}\Big|_{(x,y)}$, *and* $\dfrac{\partial b}{\partial y}\Big|_{(x,y)} = \dfrac{\partial \Phi}{\partial y}\Big|_{(x,y)}$ *for all* $(x,y) \in \mathcal{B}(\mathcal{D})$. *Then the bivariate fractal function* Φ^α *corresponding to IFS* (35) *satisfy* $\Phi^\alpha(x,y) < \varphi(x,y)$ *for all* $(x,y) \in \mathcal{D}$ *if the scaling factors* $\alpha_{i,j}$, $(i,j) \in \Sigma_{M-1} \times \Sigma_{N-1}$ *are chosen as follows:*

$$0 \leq \alpha_{i,j} \leq \min\left\{a_i c_j, \frac{m(\varphi - \Phi, \mathcal{D}_{i,j})}{M^*(\varphi) - m_*}\right\},$$

where $m_* = \min\{b(x,y) : (x,y) \in \mathcal{D}\}$, $m(\varphi - \Phi, \mathcal{D}_{i,j}) = \min\{(\varphi - \Phi)(x,y) : (x,y) \in \mathcal{D}_{i,j}\}$, *and* $M^*(\varphi) = \max\{\varphi(x,y) : (x,y) \in \mathcal{D}\}$.

Theorem 14 *Let* $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j}) : (i,j) \in \Sigma_M \times \Sigma_N\}$ *be a surface data set that lies above the piecewise plane* $g(x,y) = \sigma_{i,j}\Big[1 - \dfrac{x}{\gamma_{i,j}} - \dfrac{y}{\delta_{i,j}}\Big]$, $(x,y) \in \mathcal{D}_{i,j}$, $(i,j) \in \Sigma_{M-1} \times \Sigma_{N-1}$, $\sigma_{i,j} \in \mathbb{R}$, $\gamma_{i,j}, \delta_{i,j} \in \mathbb{R} - \{0\}$ *that is* $z_{i,j} > g_{i,j} = g(x_i, y_j)$ *for all* $(i,j) \in \Sigma_M \times \Sigma_N$. *Then the fractal interpolating surface* $\Phi^\alpha(x,y) > g(x,y)$ *for all* $(x,y) \in \mathcal{D}$ *if the corresponding shape parameters and scaling factors are chosen as*

$$r_{i,j} > \max\left\{-1, \frac{-h_i z_{x,i,j} - \frac{\sigma i,j}{\gamma_{i,j}}h_i}{z_{i,j} - g_{i,j}}, \frac{h_i z_{x,i+1,j} + \frac{\sigma i,j}{\gamma_{i,j}}h_i}{z_{i,j} - g_{i+1,j}}\right\},\ (i,j) \in \Sigma_{M-1} \times \Sigma_N,$$

$$r^*_{i,j} > \max\left\{-1, \frac{-h^*_j z_{y,i,j} - \frac{\sigma i,j}{\delta_{i,j}}h^*_j}{z_{i,j} - g_{i,j}}, \frac{h^*_j z_{y,i,j+1} + \frac{\sigma i,j}{\delta_{i,j}}h^*_j}{z_{i,j} - g_{i,j+1}}\right\},\ (i,j) \in \Sigma_M \times \Sigma_{N-1},$$

$$\textit{and } 0 \leq \alpha_{i,j} \leq \min\left\{a_i c_j, \frac{m(\Phi - g, \mathcal{D}_{i,j})}{M^* - m_*(g)}\right\},\ (i,j) \in \Sigma_{M-1} \times \Sigma_{N-1},$$

where $M^* = \max\{b(x,y) : (x,y) \in \mathcal{D}\}$, $m(\Phi - g, \mathcal{D}_{i,j}) = \min\{(\Phi - g)(x,y) : (x,y) \in \mathcal{D}_{i,j}\}$, *and* $m_*(g) = \min\{g(x,y) : (x,y) \in \mathcal{D}\}$.

Proof The proof follows from the proofs of Theorem 6 and Theorem 12. □

Remark 3 The following remark is immediate by taking $g(x, y) = 0$ in the previous theorem (Theorem 14). Let $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j}); (i, j) \in \Sigma_M \times \Sigma_N\}$ be a positive surface interpolation data. Let Φ be the bivariate function defined in (17) and $b \in \mathcal{C}^1(D)$ such that $b \neq \Phi$, $b(x, y) = \Phi(x, y)$, $\frac{\partial b}{\partial y}\Big|_{(x,y)} = \frac{\partial \Phi}{\partial y}\Big|_{(x,y)}$, and $\frac{\partial b}{\partial x}\Big|_{(x,y)} = \frac{\partial \Phi}{\partial x}\Big|_{(x,y)}$ for all $(x, y) \in \mathcal{B}(\mathcal{D})$. Then the bivariate fractal function $\Phi^\alpha(x, y) > 0$ for all $(x, y) \in \mathcal{D}$ if the associated shape parameters and scaling factors are chosen as

$$r_{i,j} \geq \max\left\{-1, \frac{-h_i z_{x,i,j}}{z_{i,j}}, \frac{h_i z_{x,i+1,j}}{z_{i+1,j}}\right\}, (i, j) \in \Sigma_M \times \Sigma_{N-1},$$

$$r^*_{i,j} \geq \max\left\{-1, \frac{-h^*_j z_{y,i,j}}{z_{i,j}}, \frac{h^*_j z_{y,i+1,j}}{z_{i,j+1}}\right\}, (i, j) \in \Sigma_{M-1} \times \Sigma_N,$$

$$\text{and } 0 \leq \alpha_{i,j} \leq \min\left\{a_i c_j, \frac{m(\Phi, \mathcal{D}_{i,j})}{M^*}\right\}, (i, j) \in \Sigma_{M-1} \times \Sigma_{N-1},$$

where $M^* = \max\{b(x, y) : (x, y) \in \mathcal{D}\}$, $m(\Phi, \mathcal{D}_{i,j}) = \min\{\Phi(x, y) : (x, y) \in \mathcal{D}_{i,j}\}$.

Theorem 15 *Let $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j}) : (i, j) \in \Sigma_M \times \Sigma_N\}$ be a surface data set that lies below the piecewise plane $g^*(x, y) = \sigma^*_{i,j}\Big[1 - \frac{x}{\gamma^*_{i,j}} - \frac{y}{\delta^*_{i,j}}\Big]$, $(x, y) \in \mathcal{D}_{i,j}$, $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$, $\sigma^*_{i,j} \in \mathbb{R}$, $\gamma^*_{i,j}, \delta^*_{i,j} \in \mathbb{R} - \{0\}$, that is $z_{i,j} < g^*_{i,j} = g^*(x_i, y_j)$ for all $(i, j) \in \Sigma_M \times \Sigma_N$. Then the fractal interpolating surface $\Phi^\alpha(x, y) < g^*(x, y)$ for all $(x, y) \in \mathcal{D}$ if the corresponding shape parameters and scaling factors are chosen as*

$$r_{i,j} > \max\left\{-1, \frac{h_i z_{x,i,j} + \frac{\sigma^*{i,j}}{\gamma^*_{i,j}} h_i}{g^*_{i,j} - z_{i,j}}, \frac{-h_i z_{x,i+1,j} - \frac{\sigma^*{i,j}}{\gamma^*_{i,j}} h_i}{g^*_{i+1,j} - z_{i+1,j}}\right\}, (i, j) \in \Sigma_{M-1} \times \Sigma_N,$$

$$r^*_{i,j} > \max\left\{-1, \frac{h^*_j z_{y,i,j} + \frac{\sigma^*{i,j}}{\delta^*_{i,j}} h^*_j}{g^*_{i,j} - z_{i,j}}, \frac{-h^*_j z_{y,i,j+1} - \frac{\sigma^*{i,j}}{\delta^*_{i,j}} h^*_j}{g^*_{i,j+1} - z_{i,j+1}}\right\}, (i, j) \in \Sigma_M \times \Sigma_{N-1},$$

$$\textit{and } 0 \leq \alpha_{i,j} \leq \min\left\{a_i c_j, \frac{m(g^* - \Phi, \mathcal{D}_{i,j})}{M^*(g^*) - m_*}\right\}, i \in \Sigma_{M-1}, j \in \Sigma_{N-1},$$

where $m_* = \min\{b(x, y) : (x, y) \in \mathcal{D}\}$, $m(g^* - \Phi, \mathcal{D}_{i,j}) = \min\{(g^* - \Phi)(x, y) : (x, y) \in \mathcal{D}_{i,j}\}$, and $M^*(g^*) = \max\{g^*(x, y) : (x, y) \in \mathcal{D}\}$.

Proof The proof follows from the proofs of Theorem 7 and Theorem 13. □

Theorem 16 *Let the given surface data be $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j}); (i, j) \in \Sigma_M \times \Sigma_N\}$. Suppose that $\Lambda_{11} < \min\{z_{i,j} : (i, j) \in \Sigma_M \times \Sigma_N\}$ and $\Lambda_{22} > \max\{z_{i,j} : (i, j) \in \Sigma_M \times \Sigma_N\}$. Let Φ be the bivariate function defined in (17) and $b \in \mathcal{C}^1(D)$ such that $b \neq \Phi$, $b(x, y) = \Phi(x, y)$, $\frac{\partial b}{\partial x}\Big|_{(x,y)} = \frac{\partial \Phi}{\partial x}\Big|_{(x,y)}$, and $\frac{\partial b}{\partial y}\Big|_{(x,y)} = \frac{\partial \Phi}{\partial y}\Big|_{(x,y)}$ for all $(x, y) \in \mathcal{B}(\mathcal{D})$. Then the bivariate fractal function Φ^α corresponding to IFS (35) contained in the cuboid $\mathcal{D} \times [\Lambda_{11}, \Lambda_{22}]$ if the associated shape parameters and scaling factor are chosen as*

$$r_{i,j} > \max\left\{-1, \frac{-h_i z_{x,i,j}}{z_{i,j} - \Lambda_{11}}, \frac{h_i z_{x,i+1,j}}{z_{i+1,j} - \Lambda_{11}}, \frac{h_i z_{x,i,j}}{\Lambda_{22} - z_{i,j}}, \frac{-h_i z_{x,i+1,j}}{\Lambda_{22} - z_{i+1,j}}\right\}, (i, j) \in \Sigma_{M-1} \times \Sigma_N,$$

$$r^*_{i,j} > \max\left\{-1, \frac{-h^*_j z_{y,i,j}}{z_{i,j} - \Lambda_{11}}, \frac{h^*_j z_{y,i,j+1}}{z_{i,j+1} - \Lambda_{11}}, \frac{h^*_j z_{y,i,j}}{\Lambda_{22} - z_{i,j}}, \frac{-h^*_j z_{y,i,j+1}}{\Lambda_{22} - z_{i,j+1}}\right\}, (i, j) \in \Sigma_M \times \Sigma_{N-1},$$

and for $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$,

$$\max\left\{-a_i c_j, \frac{-(m_{i,j} - \Lambda_{11})}{\Lambda_{22} - m_*}, \frac{-(\Lambda_{22} - M_{i,j})}{M^* - \Lambda_{11}}\right\} \leq \alpha_{i,j} \leq \min\left\{a_i c_j, \frac{m_{i,j} - \Lambda_{11}}{M^* - \Lambda_{11}}, \frac{\Lambda_{22} - M_{i,j}}{\Lambda_{22} - m_*}\right\},$$

where $m_* = \min\{b(x, y) : (x, y) \in \mathcal{D}\}$, $M^* = \max\{b(x, y) : (x, y) \in \mathcal{D}\}$, $m_{i,j} = \min\{\Phi(x, y) : (x, y) \in \mathcal{D}_{i,j}\}$, $M_{i,j} = \max\{\Phi(x, y) : (x, y) \in \mathcal{D}_{i,j}\}$.

Proof The conditions on the shape parameters $r_{i,j}$ and $r^*_{i,j}$ which are mentioned in the hypothesis and Theorem 8 ensure that $\Lambda_{11} \leq \Phi(x, y) \leq \Lambda_{22}$ for all $(x, y) \in \mathcal{D}$. As a result, we obtain that $m_{i,j} \geq \Lambda_{11}$ and $\Lambda_{22} \geq M_{i,j}$. Since the function b coincide with Φ along the boundary of $\mathcal{D}$, we get $M^* \geq \Lambda_{11}$ and $\Lambda_{22} \geq m_*$. Therefore, $\dfrac{m_{i,j} - \Lambda_{11}}{M^* - \Lambda_{11}} > 0$ and $\dfrac{\Lambda_{22} - M_{i,j}}{\Lambda_{22} - m_*} > 0$. It is easy to verify that the graph of the bivariate fractal function Φ^α is the attractor of the IFS $\mathcal{I} = \big\{K; \big(u_i(x), v_j(y), F_{i,j}(x, y, z)\big) : (i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}\big\}$, where $u_i(x) = a_i x + b_i$, $v_j(y) = c_j y + d_j$, $F_{i,j}(x, y, z) = \alpha_{i,j} z + \Phi(u_i(x), v_j(y)) - \alpha_{i,j} b(x, y)$, $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$. Also I and J are attractors of the IFSs $\{I; u_i(x) : i \in \Sigma_{M-1}\}$ and $\{J; v_j(y) : j \in \Sigma_{N-1}\}$ respectively. Therefore, it is sufficient to settle

$$\Lambda_{11} \leq F_{i,j}(x, y, z) \leq \Lambda_{22}, \forall (i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}, \text{ and } (x, y, z) \in \mathcal{D} \times [\Lambda_{11}, \Lambda_{22}] \tag{48}$$

to guarantee that the graph $G(\Phi^\alpha) = \big\{\big(x, y, \Phi^\alpha(x, y)\big) : (x, y) \in \mathcal{D}\big\}$ of Φ^α is containing in the cuboid $\mathcal{D} \times [\Lambda_{11}, \Lambda_{22}]$.

Suppose $(x, y, z) \in \mathcal{D} \times [\Lambda_{11}, \Lambda_{22}]$.

Case I. Let $0 \leq \alpha_{i,j} < a_i c_j$. Then, $\Lambda_{11} \leq z \leq \Lambda_{22}$ implies

$$\begin{aligned}\alpha_{i,j}\Lambda_{11} + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y) &\leq \alpha_{i,j} z + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y), \\ &\leq \alpha_{i,j}\Lambda_{22} + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y).\end{aligned}$$

Hence, the following conditions are sufficient for establishing (48)

$$\begin{aligned}\alpha_{i,j}\Lambda_{11} + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y) - \Lambda_{11} \geq 0 \\ \text{and } \alpha_{i,j}\Lambda_{22} + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y) - \Lambda_{22} \leq 0.\end{aligned} \tag{49}$$

But $\alpha_{i,j}\Lambda_{11} + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y) - \Lambda_{11} \geq \alpha_{i,j}\Lambda_{11} + m_{i,j} - \alpha_{i,j} M^* - \Lambda_{11}$,
$\alpha_{i,j}\Lambda_{22} + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y) - \Lambda_{22} \leq \alpha_{i,j}\Lambda_{22} + M_{i,j} - \alpha_{i,j} m_* - \Lambda_{22}$.

It is easy to verify that if $\alpha_{i,j} \leq \dfrac{m_{i,j} - \Lambda_{11}}{M^* - \Lambda_{11}}$ and $\alpha_{i,j} \leq \dfrac{\Lambda_{22} - M_{i,j}}{\Lambda_{22} - m_*}$, then the inequalities in (49) are valid. Thus, the sufficient condition for (48) is

$$\alpha_{i,j} \leq \min\left\{\frac{m_{i,j} - \Lambda_{11}}{M^* - \Lambda_{11}}, \frac{\Lambda_{22} - M_{i,j}}{\Lambda_{22} - m_*}\right\}. \tag{50}$$

Case II. Let $-a_i c_j < \alpha_{i,j} \leq 0$. Then, $\Lambda_{11} \leq z \leq \Lambda_{22}$ implies

$$\begin{aligned}\alpha_{i,j}\Lambda_{22} + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y) &\leq \alpha_{i,j} z + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y), \\ &\leq \alpha_{i,j}\Lambda_{11} + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y).\end{aligned}$$

Hence, the following conditions are sufficient for establishing the (48)

$$\begin{aligned}\alpha_{i,j}\Lambda_{22} + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y) - \Lambda_{11} \geq 0 \\ \text{and } \alpha_{i,j}\Lambda_{11} + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y) - \Lambda_{22} \leq 0.\end{aligned} \tag{51}$$

But $\alpha_{i,j}\Lambda_{22} + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y) - \Lambda_{11} \geq \alpha_{i,j}\Lambda_{22} + m_{i,j} - \alpha_{i,j} m_* - \Lambda_{11}$,
$\alpha_{i,j}\Lambda_{11} + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y) - \Lambda_{22} \leq \alpha_{i,j}\Lambda_{11} + M_{i,j} - \alpha_{i,j} M^* - \Lambda_{22}$.

It is easy to verify that if $\alpha_{i,j} \geq \dfrac{-(m_{i,j} - \Lambda_{11})}{\Lambda_{22} - m_*}$ and $\alpha_{i,j} \geq \dfrac{-(\Lambda_{22} - M_{i,j})}{M^* - \Lambda_{11}}$, then the inequalities in (51) are valid. Thus, the adequate condition for (48) is

$$\alpha_{i,j} \geq \max\left\{\frac{-(m_{i,j} - \Lambda_{11})}{\Lambda_{22} - m_*}, \frac{-(\Lambda_{22} - M_{i,j})}{M^* - \Lambda_{11}}\right\}. \tag{52}$$

From the inequalities (50) and (52), we conclude the theorem. □

Remark 4 If we choose scaling factor $\alpha_{i,j} = 0$ for all $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$, then fractal function Φ^α coincides with bivariate rational spline Φ.

Table 4 Surface interpolation data

$\downarrow y/x \rightarrow$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
0	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
$\frac{\pi}{2}$	(0, 0, 0)	(1.5, 0.5, 1.5)	(3, 1, 0)	(1.5, 0.5, −1.5)	(0, 0, 0)
π	(0, 0, 0)	(1, 0, 1)	(2, 0, 0)	(1, 0, −1)	(0, 0, 0)
$\frac{3\pi}{2}$	(0, 0, 0)	(1.5, −0.5, 1.5)	(3, −1, 0)	(1.5, −0.5, −1.5)	(0, 0, 0)
2π	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)

7.1 Numerical examples

In this section, we shall verify our findings with some numerical experiments. For this purpose, let us take the data set given in Table 4 wherein the triplets shown in the body of the table indicate the function values and partial derivative values.

It can be seen that the above data lies above the plane $z = -0.5$ and is contained in the cuboid $[0, 2\pi] \times [0, 2\pi] \times [0, 3.5]$. We choose the base function b as Hermite interpolant [29] of Φ. The bivariate fractal interpolating function Φ_1^α is constructed in Figure 3(a) by using the arbitrarily chosen shape parameters and scaling factors. We can easily notice that the bivariate fractal function Φ_1^α is neither lying above the plane $z = -0.5$ and nor lying within the cuboid $[0, 2\pi] \times [0, 2\pi] \times [0, 3.5]$. Thus, one can notice that arbitrarily selection of the shape parameters and scaling factors may not provide the constrained bivariate fractal interpolation surfaces. Therefore, by using the shape parameters and scaling factors chosen according to Theorem 14, we have generated the bivariate fractal interpolation surface Φ_2^α in Figure 3(b) which lies above the plane $z = -0.5$. Similarly, by using the shape parameters and scaling factors chosen according to Theorem 16, we have constructed the bivariate fractal interpolating surface Φ_3^α in Figure 3(c) which lies within cuboid $[0, 2\pi] \times [0, 2\pi] \times [0, 3.5]$. Figure 3(d) represents the partial derivative $\frac{\partial \Phi_3^\alpha}{\partial x}$. It is very clear that the partial derivative $\frac{\partial \Phi_3^\alpha}{\partial x}$ is nowhere differentiable, and one can't experience this kind of phenomenon with classical bivariate splines.

Remark 5 One can solve the constraint partial differential equations by using our constraint bivariate fractal spline as we have done in Section 5.

8 Monotonicity preserving fractal rational surface

In this section, we discuss the sufficient conditions on shape parameters and scaling factor so that Φ^α is monotonic whenever the given data is monotonic.

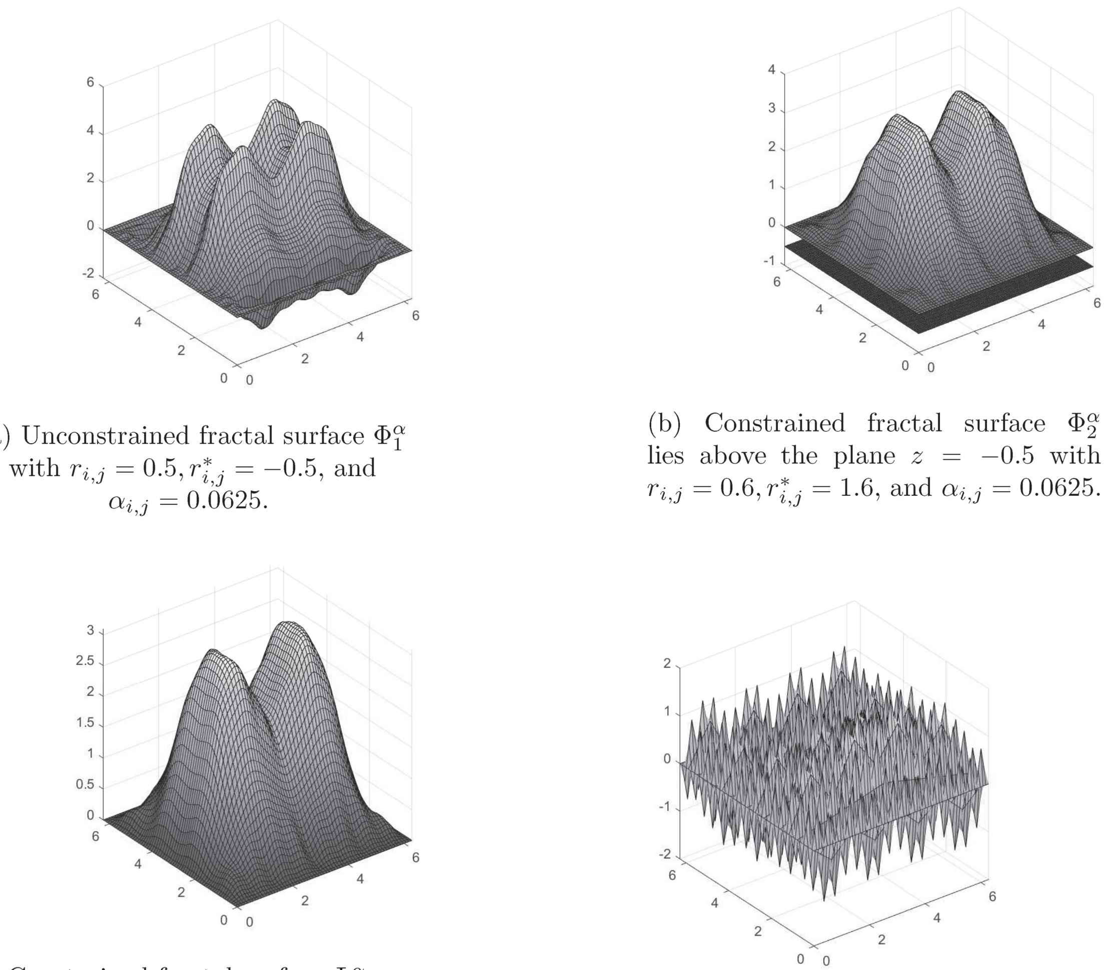
Theorem 17 *Let $\Delta = \{(x_i, y_j, z_{i,j}, z_{x,i,j}, z_{y,i,j}); i \in \Sigma_M, j \in \Sigma_N\}$ be a monotonically increasing data and Φ be the bivariate spline given in (17). Assume $F_{i,j}(x, y, z) = \alpha_{i,j} z + \Phi(u_i(x), v_j(y)) - \alpha_{i,j} b(x, y)$, where $|\alpha_{i,j}| < a_i c_j$, $i \in \Sigma_{M-1}$, $j \in \Sigma_{N-1}$, $b \in \mathcal{C}^1(\mathcal{D})$ satisfying $b \neq \Phi$,*

$$b(x, y) = \Phi(x, y), \ \frac{\partial b}{\partial x}\Big|_{(x,y)} = \frac{\partial \Phi}{\partial x}\Big|_{(x,y)}, \ \textit{and} \ \frac{\partial b}{\partial y}\Big|_{(x,y)} = \frac{\partial \Phi}{\partial y}\Big|_{(x,y)}$$

for all $(x, y) \in \mathcal{B}(\mathcal{D})$. Then Φ^α is monotonically increasing over $\mathcal{D}$ if the corresponding shape parameters and scaling factors are chosen as

$$\begin{aligned} r_{i,j} &> \max\left\{-1, \frac{z_{x,i,j} + z_{x,i+1,j}}{z_{i+1,j} - z_{i,j}}\right\}, (i, j) \in \Sigma_{M-1} \times \Sigma_N, \\ \textit{and } r^*_{i,j} &> \max\left\{-1, \frac{z_{y,i,j} + z_{y,i,j+1}}{z_{i,j+1} - z_{i,j}}\right\}, (i, j) \in \Sigma_M \times \Sigma_{N-1}. \end{aligned} \tag{53}$$

$$0 \le \alpha_{i,j} \le \min\left\{a_i c_j, \frac{a_i m_x(\frac{\partial \Phi}{\partial x}, \mathcal{D}_{i,j})}{M_x^*}, \frac{c_j m_y(\frac{\partial \Phi}{\partial y}, \mathcal{D}_{i,j})}{M_y^*}\right\}, (i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}, \tag{54}$$



(a) Unconstrained fractal surface Φ_1^α with $r_{i,j} = 0.5, r^*_{i,j} = -0.5$, and $\alpha_{i,j} = 0.0625$.

(b) Constrained fractal surface Φ_2^α lies above the plane $z = -0.5$ with $r_{i,j} = 0.6, r^*_{i,j} = 1.6$, and $\alpha_{i,j} = 0.0625$.

(c) Constrained fractal surface Φ_3^α with $r_{i,j} = 4, r^*_{i,j} = 2$, and $\alpha_{i,j} = 0.0625$ lies within the cuboid $[0, 2\pi] \times [0, 2\pi] \times [0, 3.5]$.

(d) Partial derivative $\dfrac{\partial \Phi_3^\alpha}{\partial x}$.

Fig. 3 Constrained fractal surfaces

where $M_x^* = \max\left\{\frac{\partial b}{\partial x}\Big|_{(x,y)} : (x, y) \in \mathcal{D}\right\}$, $m_x\left(\frac{\partial \Phi}{\partial x}, \mathcal{D}_{i,j}\right) = \min\left\{\frac{\partial \Phi}{\partial x}\Big|_{(x,y)} : (x, y) \in \mathcal{D}_{i,j}\right\}$, $M_y^* = \max\left\{\frac{\partial b}{\partial y}\Big|_{(x,y)} : (x, y) \in \mathcal{D}\right\}$, $m_y\left(\frac{\partial \Phi}{\partial y}, \mathcal{D}_{i,j}\right) = \min\left\{\frac{\partial \Phi}{\partial y}\Big|_{(x,y)} : (x, y) \in \mathcal{D}_{i,j}\right\}$.

Proof In view of the conditions on the shape parameters $r_{i,j}$ and $r^*_{i,j}$ which are mentioned in the hypothesis, Theorem 9 ensures that $\frac{\partial \Phi}{\partial x}\Big|_{(x,y)} > 0$, and $\frac{\partial \Phi}{\partial y}\Big|_{(x,y)} > 0$ for all $(x, y) \in \mathcal{D}$. Since $\frac{\partial b}{\partial x}\Big|_{(x,y)} = \frac{\partial \Phi}{\partial x}\Big|_{(x,y)}$ and $\frac{\partial b}{\partial y}\Big|_{(x,y)} = \frac{\partial \Phi}{\partial y}\Big|_{(x,y)}$ for all $(x, y) \in \mathcal{B}(\mathcal{D})$ it is easy to see that $M_x^* > 0$, $M_y^* > 0$ and $m_x\left(\frac{\partial \Phi}{\partial x}, \mathcal{D}_{i,j}\right) > 0$, $m_y\left(\frac{\partial \Phi}{\partial y}, \mathcal{D}_{i,j}\right) > 0$ for all $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$. Also, our bivariate fractal function Φ^α satisfies the following functional equation :

$$\Phi^\alpha\big(u_i(x), v_j(y)\big) = \alpha_{i,j}\Phi^\alpha(x, y) + \Phi\big(u_i(x), v_j(y)\big) - \alpha_{i,j} b(x, y), (x, y) \in \mathcal{D}, (i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}.$$

This implies that

$$\frac{\partial \Phi^\alpha}{\partial x}\Big|_{\left(u_i(x),v_j(y)\right)} = \frac{\alpha_{i,j}}{a_i}\frac{\partial \Phi^\alpha}{\partial x}\Big|_{(x,y)} + \frac{\partial \Phi}{\partial x}\Big|_{\left(u_i(x),v_j(y)\right)} - \frac{\alpha_{i,j}}{a_i}\frac{\partial b}{\partial x}\Big|_{(x,y)}, (x, y) \in \mathcal{D}, (i, j) \in \Sigma_{M-1} \times \Sigma_{N-1},$$

and

$$\frac{\partial \Phi^\alpha}{\partial y}\Big|_{\left(u_i(x),v_j(y)\right)} = \frac{\alpha_{i,j}}{c_j}\frac{\partial \Phi^\alpha}{\partial y}\Big|_{(x,y)} + \frac{\partial \Phi}{\partial y}\Big|_{\left(u_i(x),v_j(y)\right)} - \frac{\alpha_{i,j}}{c_j}\frac{\partial b}{\partial y}\Big|_{(x,y)}, (x, y) \in \mathcal{D}, (i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}.$$

At $(n+1)$-th iteration $\frac{\partial \Phi^\alpha}{\partial x}$ and $\frac{\partial \Phi^\alpha}{\partial y}$ are computed at $\big((M-1)^{n+2}+1\big)\big((N-1)^{n+2}+1\big)$ distinct points in $\mathcal{D}$ by using the values of $\frac{\partial \Phi^\alpha}{\partial x}$ and $\frac{\partial \Phi^\alpha}{\partial y}$ at n-th iteration at $\big((M-1)^{n+1}+1\big)\big((N-1)^{n+1}+1\big)$ number of distinct points respectively in $\mathcal{D}$.

We observe that

$$\frac{\partial \Phi^\alpha}{\partial x}\Big|_{(x_i,y_j)} = \frac{\partial \Phi}{\partial x}\Big|_{(x_i,y_j)} > 0, (i, j) \in \Sigma_M \times \Sigma_N,$$
$$\frac{\partial \Phi^\alpha}{\partial y}\Big|_{(x_i,y_j)} = \frac{\partial \Phi}{\partial y}\Big|_{(x_i,y_j)} > 0, (i, j) \in \Sigma_M \times \Sigma_N.$$

Therefore, to prove that $\frac{\partial \Phi^\alpha}{\partial x}\Big|_{(z,t)} > 0$ and $\frac{\partial \Phi^\alpha}{\partial y}\Big|_{(z,t)} > 0$ for all $(z, t) \in \mathcal{D}$, it is sufficient to show that if $\frac{\partial \Phi^\alpha}{\partial x}\Big|_{(x,y)} > 0$ and $\frac{\partial \Phi^\alpha}{\partial y}\Big|_{(x,y)} > 0$ for $(x, y) \in \mathcal{D}$ then $\frac{\partial \Phi^\alpha}{\partial x}\Big|_{\left(u_i(x),v_j(y)\right)} > 0$ and $\frac{\partial \Phi^\alpha}{\partial y}\Big|_{\left(u_i(x),v_j(y)\right)} > 0$, $(x, y) \in \mathcal{D}$, for all $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$.

Suppose $\frac{\partial \Phi^\alpha}{\partial x}\Big|_{(x,y)} > 0$ and $\frac{\partial \Phi^\alpha}{\partial y}\Big|_{(x,y)} > 0$, for $(x, y) \in \mathcal{D}$. Then, we have to establish that

$$\begin{aligned} &\frac{\alpha_{i,j}}{a_i}\frac{\partial \Phi^\alpha}{\partial x}\Big|_{(x,y)} + \frac{\partial \Phi}{\partial x}\Big|_{\left(u_i(x),v_j(y)\right)} - \frac{\alpha_{i,j}}{a_i}\frac{\partial b}{\partial x}\Big|_{(x,y)} > 0,\\ \text{and } &\frac{\alpha_{i,j}}{c_j}\frac{\partial \Phi^\alpha}{\partial y}\Big|_{(x,y)} + \frac{\partial \Phi}{\partial y}\Big|_{\left(u_i(x),v_j(y)\right)} - \frac{\alpha_{i,j}}{c_j}\frac{\partial b}{\partial y}\Big|_{(x,y)} > 0. \end{aligned} \tag{55}$$

Since $\frac{\partial \Phi^\alpha}{\partial x}\Big|_{(x,y)} > 0$, $\frac{\partial \Phi^\alpha}{\partial y}\Big|_{(x,y)} > 0$, for $(x, y) \in \mathcal{D}$ and $\alpha_{i,j} \geq 0$ for all $(i, j) \in \Sigma_{M-1} \times \Sigma_{N-1}$, and by the definition of M_x^*, M_y^*, $m_x\Big(\frac{\partial \Phi}{\partial x}, \mathcal{D}_{i,j}\Big)$, and $m_y\Big(\frac{\partial \Phi}{\partial y}, \mathcal{D}_{i,j}\Big)$, we get

$$\frac{\alpha_{i,j}}{a_i}\frac{\partial \Phi^\alpha}{\partial x}\Big|_{(x,y)} + \frac{\partial \Phi}{\partial x}\Big|_{\left(u_i(x),v_j(y)\right)} - \frac{\alpha_{i,j}}{a_i}\frac{\partial b}{\partial x}\Big|_{(x,y)} \geq m_x\Big(\frac{\partial \Phi}{\partial x}, \mathcal{D}_{i,j}\Big) - \frac{\alpha_{i,j}}{a_i}M_x^*,$$
$$\frac{\alpha_{i,j}}{c_j}\frac{\partial \Phi^\alpha}{\partial y}\Big|_{(x,y)} + \frac{\partial \Phi}{\partial y}\Big|_{\left(u_i(x),v_j(y)\right)} - \frac{\alpha_{i,j}}{c_j}\frac{\partial b}{\partial y}\Big|_{(x,y)} \geq m_y\Big(\frac{\partial \Phi}{\partial y}, \mathcal{D}_{i,j}\Big) - \frac{\alpha_{i,j}}{c_j}M_y^*.$$

We can conclude that $\alpha_{i,j} \leq \min\left\{\frac{a_i m_x\left(\frac{\partial \Phi}{\partial x}, \mathcal{D}_{i,j}\right)}{M_x^*}, \frac{c_j m_y\left(\frac{\partial \Phi}{\partial y}, \mathcal{D}_{i,j}\right)}{M_y^*}\right\}$ assures (55). □

9 Conclusion

In this manuscript, first we have studied constrained aspects of the univariate rational spline [3] and developed a methodology for solving constrained ordinary differential equations by our constrained univariate rational spline. Subsequently, we have constructed a blending surface using the network of univariate rational splines and discussed its constrained aspects. Solving constrained partial differential equations by our constrained blending surface technique is established. Using [28], we have constructed a self-similar bivariate fractal spline as a perturbation of the blending surface proposed in this paper. Suitable conditions on the shape parameters and scaling factors so that the proposed fractal surface lies above or below the given plane, or within a prescribed cuboid are developed. Also, we studied the shape preserving aspects and convergence analysis of the proposed bivariate fractal spline.

It may be noted that the present article confines to a particular shape property of the given surface data set. It would be interesting to study other shape properties such as the comonotonicity, coconvexity, and copositivity of the proposed bivariate fractal spline. We have derived a set of sufficient conditions for a constrained or shape preserving interpolation on the basis of parameters involved in the bivariate fractal spline. The parameters are restricted to satisfy inequalities that provide suitable lower or upper bounds for them, but left completely free otherwise. An optimal choice for the parameters within the permissible ranges is a challenging problem which still remains an open problem.

Funding The first author acknowledges the financial support from DST, India under the INSPIRE fellowship scheme (IF200483). The second author is thankful for the sponsored research project: MTR/2021/000905-MATRICS from the Science and Engineering Research Board (SERB), Government of India.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Data repositories The authors ensure that datasets on which the conclusions of the paper rely are presented in the main manuscript.

Declarations

Competing interest The authors have no relevant financial or non-financial interests to disclose.

References

1. G. Farin, Curves and Surfaces for Computer Aided Geometric Design, Academic Press, 2002.
2. M. P. Groover and E. W. Zimmers Jr., CAD/CAM: Computer-Aided Design and Manufacturing, Pearson Education, Upper Saddle River, NJ, 1997.
3. R. Delbourgo and J. A. Gregory, Shape Preserving Piecewise Rational Interpolation, Siam Journal on Scientific and Statistical Computing 6(1), 967-976, 1985.
4. G. Casciola and L. Romani, Rational interpolants with tension parameters, J. Curves and Surfaces, 41-50, 2003.
5. M. F. Barnsley, Fractal functions and interpolation, Constr. Approx. 2, 303-329, 1986.
6. M. A. Navascués, Fractal polynomial interpolation, Z. Anal. Anwend. 24(2), 401-418, 2005.
7. M. A. Navascués, Non-smoothpolynomials, Int. J. Math. Anal. 1, 159-174, 2007.
8. M. A. Navascués, Fractal approximation, Complex Anal. Oper. Theory 4, 953-974, 2010.
9. M. A. Navascués, Fractal bases of L_p spaces, Fractals 20, 141-148, 2012.
10. M. A. Navascués and A. K. B. Chand, Fundamental sets of fractal functions, Acta Appl. Math. 100, 247-261, 2008.
11. M. F. Barnsley and A. N. Harrington, The calculus of fractal interpolation functions, J. Approx. Theory 57, 14-34, 1989.
12. A. K. B. Chand and G. P. Kapoor, Generalized cubic spline fractal interpolation functions, SIAM J. Num. Anal. 44(2), 655-676, 2006.
13. A. K. B. Chand, N. Vijender and M. A. Navascués, Shape preservation of scientific data through rational fractal splines, Calcolo 51, 329-362, 2014.
14. N. Vijender, Fractal perturbation of shaped functions: convergence independent of scaling, Mediterranean Journal of Mathematics 15(211), 2018.
15. A. K. B. Chand, N. Vijender, and R. P. Agarwal, Rational iterated function system for positive/monotonic shape preservation, Advances in Difference Equations, 1-19, 2014.
16. P. Viswanathan and A. K. B. Chand, α-fractal rational splines for constrained interpolation, Electronic Transactions on Numerical Analysis 41, 420-442, 2014.
17. A. K. B. Chand, N. Vijender, Positive blending Hermite rational cubic spline fractal interpolation surfaces, Calcolo 52,1-24, 2015 .
18. A. K. B. Chand and N. Vijender, Monotonicity/Symmetricity Preserving Rational Quardratic Fractal Interpolation Surfaces, International Journal of Numerical Analysis and Modeling 13(1), 2016.
19. N. Vijender, Positivity and Stability of Rational Cubic Fractal Interpolation Surface. Mediterr. J. Math. 15(89), 2018.
20. A. K. B. Chand, P. Viswanathan, and N. Vijender, Bivariate shape preserving interpolation: A fractal-classical hybrid approach, Chaos, Solitons and Fractals 81, 330-344, 2015.
21. A. K. B. Chand, P. Viswanathan, and N. Vijender, Bicubic partially blended rational fractal surface for a constrained interpolation problem, Computational and Applied Mathematics 37,785-804, 2018.
22. J. S. Geronimo, D. P. Hardin, Fractal interpolation functions from $\mathbb{R}^n \to \mathbb{R}^m$ and their projections, Z. Anal. Anwend. 12, 535-548, 1993.
23. R. Malysz, The Minkowski dimension of the bivariate fractal interpolation surfaces, Chaos Solitons Fractals 27(5), 1147-1156, 2006.
24. P. Bouboulis, L. Dalla, Closed Fractal Interpolation Surfaces, J. Math. Anal. Appl. 327, 116-126, 2007.

25. A. K. B. Chand, G. P. Kapoor, Hidden variable bivatiate fractal interpolation surfcaes, Fractals 11(3), 277-288, 2003.
26. Z. Feng, Y. Feng, and Z. Yuan, Fractal interpolation surfaces with function vertical scaling factors, Appl. Math. Lett. 25(11), 1896-1900, 2012.
27. L. Dalla, Bivariate fractal interpolation functions on grids, Fractals 10(1), 53-58, 2002.
28. M. A. Navascués, R. N. Mohapatra, and M. N. Akhtar, Construction of fractal surfaces, Fractals 28(02), 2050033, 2020.
29. P. Viswanathan, A revisit to smoothness preserving fractal perturbation of a bivariate function: Self-Referential counterpart to bicubic splines, Chaos, Solitons and Fractals 157, 111885, 2022.
30. K. Brodlie, P. Mashwama, and S. Butt, Visualization of surface data to preserve positivity and other simple constraints, Computer Graphics 19(4), 585-594, 1995.
31. W. Boehm, G. Farin, and J. Kahmann, A Survey of curve and surface methods in CAGD, CAGD 1, 1-60, 1984.
32. R. Delbourgo and J. A. Gregory, Determination of derivative parameters for a monotonic rational quadratic interpolant, IMA J. Numer. Anal. 5, 397-406, 1985.
33. David R. Kincaid and E. Ward Cheney, Numerical Analysis: Mathematics of Scientific Computing. Pacific Grove, Calif.: Brooks/Cole, 1991.
34. G D Smith Numerical Solution of Partial Differential equations:Finite Difference Method Oxford 1992.