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ORIGINAL RESEARCH

Estimating the circumference of a graph in terms of its leaf number

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Abstract Let $\mathcal{T}$ be the set of spanning trees of a graph G and let $L(T)$ be the number of leaves in a tree T. The leaf number $L(G)$ of G is defined as $L(G) = \max\{L(T)|T \in \mathcal{T}\}$. Let G be a connected graph of order n and minimum degree δ such that $L(G) \leq 2\delta - 1$. We show that the circumference of G is at least $n - 1$, and that if G is regular then G is hamiltonian.

Keywords Leaf number · Circumference · Hamiltonian

Mathematics Subject Classification 05C38 · 05C45

1 Introduction

We will deal with only finite nontrivial simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of a graph G are its number of vertices and edges, respectively. The notations $N_G(v)$ and $N_G[v]$ denote the neighborhood and closed neighborhood of $v \in V(G)$, respectively. The degree of v is $d_G(v) = |N_G(v)|$. $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of a graph G, respectively. If the graph G is clear from the context, we will omit it as subscript. For terminology and notations not explicitly described in this paper, readers can refer to related books [1, 19].

Let $\mathcal{T}$ be the set of spanning trees of G and let $L(T)$ be the number of leaves in a tree T, where a leaf means a vertex of degree 1. Then the leaf number $L(G)$ of G is defined as $L(G) = \max\{L(T)|T \in \mathcal{T}\}$. Many researchers have estimated the circumference of graphs by various invariants. The purpose of this paper is to estimate the circumference of a connected graph G by the two invariants $\delta(G)$ and $L(G)$.

DeLaViña's computer program, Graffiti.pc, posed attractive conjectures [3] and some of the conjectures speculate sufficient conditions for traceability based on the minimum degree and leaf number. In 2013, Mukwembi gave a partial solution to the Graffiti.pc 190a. He [14] showed that if G is a finite connected graph with minimum degree $\delta(G) \geq 5$, and leaf number $L(G)$ such that $\delta(G) \geq L(G) - 1$, then G is hamiltonian and thus traceable. In the same year, he [16] relaxed the condition $\delta(G) \geq 5$ to $\delta(G) \geq 3$. After that, Mukwembi [15] proved that if G is a connected claw-free graph with $\delta(G) \geq (L(G) + 1)/2$, then G is hamiltonian. In recent years, several authors reported on sufficient conditions for a graph to be hamiltonian or traceable based on the minimum degree and leaf number, see [9–13].

We state the following results, some of which will be used later in this paper.

Theorem 1 [12] *If G is a connected graph with $\delta(G) \geq (L(G) + 2)/2$, then G is hamiltonian.*

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Theorem 2 [11] *If G is a connected graph with $\delta(G) \geq (L(G)+1)/2$, then G is traceable.*

Theorem 3 [13] *Let G be a connected triangle-free graph with $L(G) \leq 2\delta(G)-1$. Then G is either hamiltonian or $G \in \mathcal{F}_2$, where $\mathcal{F}_2$ is the class of non-hamiltonian graphs with leaf number $2\delta(G)-1$.*

Let $p(G)$ and $c(G)$ be the order of a longest path and a longest cycle in a graph G, respectively. Note that $c(G)$ is equal to the circumference of a graph G. Many researchers have investigated the relation between $p(G)$ and $c(G)$ ([5],[8],[17],[18]). Motivated by Theorem 2, we obtain the following main result.

Theorem 4 *Let G be a connected graph of order n. If $L(G) \leq 2\delta(G)-1$, then $c(G) \geq n-1$. The bound is sharp and the condition cannot be relaxed.*

2 Main results

For a graph G, $\kappa(G)$ and $\alpha(G)$ denote the connectivity and independence number of G, respectively. For any integer $k \geq 1$, when $\alpha(G) \geq k$, let $\sigma_k(G) = \min\{\sum_{i=1}^{k} d(v_i) : v_1, v_2, \ldots, v_k$ are independent in $G\}$.

Lemma 5 [2] *Let G be a connected graph with at least three vertices. If $\kappa(G) \geq \alpha(G)$, then G is hamiltonian.*

Lemma 6 [5] *Let G be a 2-connected graph of order n. If $\sigma_3(G) \geq n+2$, then $c(G) \geq p(G)-1$.*

Given graphs G and H, the notation $G+H$ means the disjoint union of G and H. Then tG denotes the disjoint union of t copies of G. The notation $G \vee H$ means the join of G and H. For graphs we will use equality up to isomorphism, so $G = H$ means that G and H are isomorphic.

For any graph G, $G[S]$ denotes the subgraph of G induced by $S \subseteq V(G)$. Let $A, B \subseteq V(G)$ and $A \cap B = \emptyset$. Denote by $E(A, B)$ the set of edges of G with one end in A and the other end in B and $e(A, B) = |E(A, B)|$. K_n stands for the complete graph of order n.

Lemma 7 [5] *Let G be a connected graph with order $n \geq 3$. If $\sigma_3(G) \geq n$, then G satisfies $c(G) \geq p(G)-1$ or $G \in \mathcal{F}(n)$, where $\mathcal{F}(n)$ is the class of graphs defined below.*

$\mathcal{F}(n)$ consists six subclasses:

$$\mathcal{F}(n) = \mathcal{F}_1(n) \cup \mathcal{F}_2(n) \cup \mathcal{F}_3(n) \cup \mathcal{F}_4(n) \cup \mathcal{F}_5(n) \cup \mathcal{F}_6(n).$$

For any graph $G \in \mathcal{F}(n)$, we have $|V(G)| = n$ and $\sigma_3(G) \geq n$. The subclasses are defined as follows (more details can be found in [5]):

$\mathcal{F}_1(n)$: $G \in \mathcal{F}_1(n)$ if $V(G) = A \cup B$ with $A \cap B = \emptyset$, $G[A]$ and $G[B]$ are hamiltonian or isomorphic to K_2, and $e(A, B) = 1$.

$\mathcal{F}_2(n)$: $G \in \mathcal{F}_2(n)$ if $V(G) = A \cup B$ with $A \cap B = \{x\}$, $G[A]$ and $G[B]$ are both hamiltonian or both isomorphic to K_2, and $e(A \setminus \{x\}, B \setminus \{x\}) = 0$.

$\mathcal{F}_3(n)$: $G \in \mathcal{F}_3(n)$ if G is a 2-connected spanning subgraph of $K_2 \vee (K_a + K_b + K_c)$ with $a, b, c \geq 2$ $(n = a+b+c+2)$.

$\mathcal{F}_4(n)$: $G \in \mathcal{F}_4(n)$ if G is a 2-connected spanning subgraph of $K_3 \vee (aK_2 + bK_3)$ with $a, b \geq 0$ and $a+b = 4$ $(n = 2a+3b+3,\ 11 \leq n \leq 15)$.

$\mathcal{F}_5(n)$: $G \in \mathcal{F}_5(n)$ if G is a 2-connected spanning subgraph of $K_s \vee (sK_2 + K_3)$ with $s \geq 4$ $(n = 3s+3)$.

$\mathcal{F}_6(n)$: $G \in \mathcal{F}_6(n)$ if G is a 2-connected spanning subgraph of $K_s \vee (s+1)K_2$ with $s \geq 4$ $(n = 3s+2)$.

Lemma 8 [15] *Let G be a connected graph with $L(G) \leq 2\delta(G)-1$. Then G is 2-connected.*

Lemma 9 [16] *Let G be a connected graph of order n. If $\delta(G) = 2$ and $L(G) \leq 3$, then $c(G) \geq n-1$.*

Now we first show that the result of Theorem 4 is true when $n \leq 3\delta(G)$.

Theorem 10 *Let G be a connected graph with order $n \leq 3\delta(G)$. If $L(G) \leq 2\delta(G)-1$, then $c(G) \geq n-1$.*

Proof Let G be a connected graph with order $n \leq 3\delta(G)$ and $L(G) \leq 2\delta(G)-1$. By Lemma 8, G is 2-connected and hence $\delta(G) \geq 2$. Then by Lemma 9, the result of Theorem 10 is true when $\delta(G) = 2$. Thus it suffices to consider the case $\delta(G) \geq 3$.

By Lemma 5, the Theorem 10 holds when $\alpha(G) \leq 2$. Henceforth we may assume that $\alpha(G) \geq 3$. Note that $n > 3$ and $n \leq 3\delta(G) \leq \sigma_3(G)$, by Lemma 7, G satisfies $c(G) \geq p(G)-1$ or $G \in \mathcal{F}(n)$. Since

$L(G) \leq 2\delta(G) - 1$, by Theorem 2, $p(G) = n$ and hence $c(G) \geq n - 1$ or $G \in \mathcal{F}(n)$. Suppose to the contrary that $G \in \mathcal{F}(n)$.

Recall that G is 2-connected. This implies that $G \notin \mathcal{F}_1(n) \cup \mathcal{F}_2(n)$. First suppose $G \in \mathcal{F}_3(n)$. Then G is a 2-connected spanning subgraph of $K_2 \vee (K_a + K_b + K_c)$ with $a, b, c \geq 2$ ($n = a + b + c + 2$). Set $G' = K_2 \vee (K_a + K_b + K_c)$ and $V(G) = V(K_2) \cup V(K_a) \cup V(K_b) \cup V(K_c)$. Since G is a subgraph of G', for any vertex x of $G[V(K_a)]$, we have $d_{G[V(K_2)]}(x) \leq 2$ and hence $d_{G[V(K_a)]}(x) \geq \delta(G) - 2$ in G. So $|V(K_a)| \geq \delta(G) - 1$, that is, $a \geq \delta(G) - 1$. Similarly, $b \geq \delta(G) - 1$ and $c \geq \delta(G) - 1$. Then

$$n = a + b + c + 2 \geq 3(\delta(G) - 1) + 2 = 3\delta(G) - 1.$$

It implies that either $n = 3\delta(G) - 1$ or $n = 3\delta(G)$. For $n = 3\delta(G) - 1$, we have $a = b = c = \delta(G) - 1$. In fact, G contains $2K_1 \vee 3K_{\delta(G)-1}$ as a subgraph. It can easily be shown that G contains a spanning tree with leaf number at least $2\delta(G)$ since $\delta(G) \geq 3$, a contradiction. Hence we may assume that $n = 3\delta(G)$. Clearly, exactly one of a, b and c is equal to $\delta(G)$ and the rest are equal to $\delta(G) - 1$. Without loss of generality, suppose $a = \delta(G)$. Then G is a spanning subgraph of $K_2 \vee (K_{\delta(G)} + 2K_{\delta(G)-1})$. More precisely, $G[V(G) \setminus V(K_a)]$ contains $2K_1 \vee 2K_{\delta(G)-1}$ as a subgraph. Since $a = \delta(G) \geq 3$, we have at least two vertices of $V(K_a)$ have the same neighbor in $V(K_2)$. The subgraph of G induced by the vertex set of $V(G) \setminus V(K_a)$ with the two vertices of $V(K_a)$ has a spanning tree with leaf number $2\delta(G)$, contradicting $L(G) \leq 2\delta(G) - 1$. Thus $G \notin \mathcal{F}_3(n)$.

Next assume that $G \in \mathcal{F}_4(n)$. Then G is a 2-connected spanning subgraph of $K_3 \vee (aK_2 + bK_3)$ with $a, b \geq 0$ and $a + b = 4$ ($n = 2a + 3b + 3$, $11 \leq n \leq 15$). If $n \leq 3\delta(G) - 2$, then $n \leq \sigma_3(G) - 2$. By Lemmas 6 and 8, $c(G) \geq p(G) - 1$. Since $L(G) \leq 2\delta(G) - 1$, by Theorem 2, we have $p(G) = n$ and hence $c(G) \geq n - 1$, a contradiction. Next suppose $3\delta(G) - 1 \leq n \leq 3\delta(G)$. Since $11 \leq n \leq 15$, we have $4 \leq \delta(G) \leq 5$. When $\delta(G) = 4$, we have $11 \leq n \leq 12$. Note that $n = 2a + 3b + 3$ and $a + b = 4$. Then $a = 4$, $b = 0$ when $n = 11$ and $a = 3$, $b = 1$ when $n = 12$. More precisely, G is a spanning subgraph of $K_3 \vee 4K_2$ or $K_3 \vee (3K_2 + K_3)$. Recall that $\delta(G) = 4$. It is easy to check that $L(G) \geq 8 > 2\delta(G) - 1$ in both cases, a contradiction. Next assume that $\delta(G) = 5$. Since G is a spanning subgraph of $K_3 \vee (aK_2 + bK_3)$, we have $a = 0$. Otherwise, there is one vertex in $V(K_2)$ has degree at most 4, contradicting $\delta(G) = 5$. Note that $n = 2a + 3b + 3$, $a + b = 4$ and $a = 0$. Then $b = 4$ and $n = 15$. More precisely, G is a spanning subgraph of $K_3 \vee 4K_3$. Since $\delta(G) = 5$, we have G contains $3K_1 \vee 4K_3$ as a subgraph. Then $L(G) \geq 12 > 2\delta(G) - 1$ and hence $G \notin \mathcal{F}_4(n)$.

Now assume that $G \in \mathcal{F}_5(n)$. Then G is a 2-connected spanning subgraph of $K_s \vee (sK_2 + K_3)$ with $s \geq 4$ ($n = 3s + 3$). Set $G' = K_s \vee (sK_2 + K_3)$ and $V(G) = V(K_s) \cup V(K_2) \cup \ldots \cup V(K_2) \cup V(K_3)$. Since G is a subgraph of G', for any vertex x of $V(K_2)$, we have $d_{G[V(K_2)]}(x) \leq 1$ and hence $d_{G[V(K_s)]}(x) \geq \delta(G) - 1$ in G. So $|V(K_s)| \geq \delta(G) - 1$, that is, $s \geq \delta(G) - 1$. Then

$$n = 3s + 3 \geq 3(\delta(G) - 1) + 3 = 3\delta(G).$$

Since $n \leq 3\delta(G)$, then $s = \delta(G) - 1$ and $\delta(G) = s + 1 \geq 5$. Recall that G is a subgraph of G'. It implies that $G[V(K_{\delta(G)-1}) \cup V((\delta(G) - 1)K_2)]$ contains $(\delta(G) - 1)K_1 \vee (\delta(G) - 1)K_2$ as a subgraph. Then G contains a tree with leaf number at least $3\delta(G) - 5$. Since $\delta(G) \geq 5$, we have $3\delta(G) - 5 \geq 2\delta(G)$, contradicting $L(G) \leq 2\delta(G) - 1$. Thus $G \notin \mathcal{F}_5(n)$.

Hence we may assume that $G \in \mathcal{F}_6(n)$. Then G is a 2-connected spanning subgraph of $K_s \vee (s + 1)K_2$ with $s \geq 4$ ($n = 3s + 2$). Set $G' = K_s \vee (s + 1)K_2$ and $V(G) = V(K_s) \cup V(K_2) \cup \ldots \cup V(K_2)$. Since G is a subgraph of G', for any vertex x of $V(K_2)$, we have $d_{G[V(K_2)]}(x) \leq 1$ and hence $d_{G[V(K_s)]}(x) \geq \delta(G) - 1$ in G. So $|V(K_s)| \geq \delta(G) - 1$, that is, $s \geq \delta(G) - 1$. Recall that $n = 3s + 2 \leq 3\delta(G)$. Then $s = \delta(G) - 1$ and G is a spanning subgraph of $K_{\delta(G)-1} \vee \delta(G)K_2$. More precisely, G contains $K_1 \vee \delta(G)K_2$ as a subgraph. Thus, G has a spanning tree with leaf number at least $2\delta(G)$, a contradiction. This completes the proof of Theorem 10. □

Denote by P be the Petersen graph and $P^{\triangle}$ be a graph obtained from P by replacing one vertex of P by a triangle (see Figs. 1 and 2).

Lemma 11 [7] *Every 2-connected r-regular ($r \geq 3$) graph of order at most $3r + 3$ is hamiltonian except P and $P^{\triangle}$.*

Lemma 12 [10] *Let G be a connected graph of order n. If $L(G) \leq 2\delta(G) - 1$, then $n \leq \max\{2\delta(G) + 6, 3\delta(G)\}$.*

Before giving the proof of the main theorem, we prove a conclusion about regular graphs.

Theorem 13 *Let G be a r-regular connected graph. If $L(G) \leq 2r - 1$, then G is hamiltonian and the condition cannot be relaxed.*

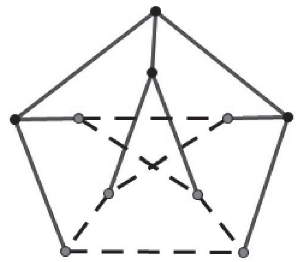

Fig. 1 The Petersen graph P

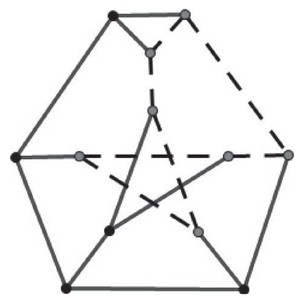

Fig. 2 The graph $P^{\triangle}$

Proof It is easy to check that the leaf number $L(P)$ of the Petersen graph P is equal to 6 (see Fig. 1). P is non-hamiltonian but satisfies $L(P) = 2r = 6$, so the condition cannot be relaxed.

Let G be a r-regular connected graph of order n with $L(G) \leq 2r - 1$. From Fig. 2, $L(P^{\triangle}) \geq 2r + 1 = 7$. In fact, it is easy to check that $L(P^{\triangle}) = 7$. Then $G \neq P$ and $G \neq P^{\triangle}$. Since $L(G) \geq 2$, then $r \geq 2$. Clearly, G is hamiltonian when $r = 2$. Next suppose that $r \geq 3$. Since $\delta(G) = r$, by Lemma 12, we have $n \leq \max\{2r+6, 3r\}$. Note that $3r + 3 \geq \max\{2r + 6, 3r\} \geq n$ when $r \geq 3$. By Lemmas 8 and 11, G is hamiltonian. This completes the proof of Theorem 13. □

The following lemmas play the key role in the proof of Theorem 4.

Lemma 14 *Let G be a connected graph with $\delta(G) \geq 3$ such that $L(G) \leq 2\delta(G) - 1$. If T is a tree in G such that $L(T) = 2\delta(G) - 1$, then $|V(G) \setminus V(T)| \leq 2$. In particular, if there is one vertex $x \in V(G)$ with degree $2\delta(G) - 1$, then $|V(G) \setminus N[x]| \leq 2$.*

Proof Let G be a connected graph with $\delta(G) \geq 3$ and $L(G) \leq 2\delta(G) - 1$. Let T be a tree in G with $L(T) = 2\delta(G) - 1$. Obviously, no interior vertex of T has a neighbor in $V(G) \setminus V(T)$ and each leaf of T has at most one neighbor in $V(G) \setminus V(T)$. Let X be the set of leaves of T. Then

$$E(V(T), V(G) \setminus V(T)) = E(X, V(G) \setminus V(T)),$$

and $|E(V(T), V(G) \setminus V(T))| \leq 2\delta(G) - 1$.

To the contrary, suppose $|V(G) \setminus V(T)| \geq 3$. By Lemma 8, G is 2-connected. Then there are at least two vertices $y_1, y_2 \in V(G) \setminus V(T)$ have neighbors in X. If $|N(y_i) \cap (V(G) \setminus V(T))| \geq 2$, then $L(G) \geq L(G[V(T) \cup N[y_i]]) \geq 2\delta(G)$ for $i \in \{1, 2\}$, a contradiction. Thus, we assume that each vertex of $\{y_1, y_2\}$ has at most one neighbor in $V(G) \setminus V(T)$ and hence at least $\delta(G) - 1$ neighbors in $V(T)$. Recall that each leaf of T has at most one neighbor in $V(G) \setminus V(T)$. Since $d(y_1) + d(y_2) \geq 2\delta(G) > |X|$, we have there is at least one vertex y_3 in $V(G) \setminus V(T)$ such that y_3 is adjacent to y_1 or y_2. Without loss of generality, assume that y_3 is adjacent to y_1. If $N(y_3) \cap V(T) = \emptyset$, then the subgraph of G induced by $V(T) \cup \{y_1\} \cup N[y_3]$ has a spanning tree with leaf number at least $(2\delta(G) - 1) - 1 + (\delta(G) - 1) = 3\delta(G) - 3$. Then G contains a spanning tree with leaf number at least $3\delta(G) - 3$. Since $\delta(G) \geq 3$, we have $3\delta(G) - 3 > 2\delta(G) - 1 = L(G)$, a contradiction. Next assume that $N(y_3) \cap V(T) \neq \emptyset$. Recall that $N(y_1) \cap N(y_2) \cap X = \emptyset$. Since $|E(V(T), V(G) \setminus V(T))| \leq 2\delta(G) - 1$ and $|N(y_i) \cap X| \geq \delta(G) - 1$ for $i \in \{1, 2\}$, we have $|N(y_3) \cap X| \leq 1$ and hence $|N(y_3) \cap (V(G) \setminus V(T))| \geq \delta(G) - 1$. Thus the subgraph of G induced by $V(T) \cup N[y_3]$ has a spanning tree with leaf number at least $3\delta(G) - 3 > 2\delta(G) - 1 = L(G)$ since $\delta(G) \geq 3$, a contradiction. This completes the proof of Lemma 14. □

Lemma 15 [4] *Let G be a 2-connected graph of order n and let C be a longest cycle in G. Then* $|V(C)| \geq \min\{n, 2\delta(G)\}$.

Lemma 16 [6] *Let G be a connected graph of order n.*

(1) If $\delta(G) \geq 4$, then $L(G) \geq \frac{2n+8}{5}$.
(2) If $\delta(G) \geq 5$, then $L(G) \geq \frac{n}{2} + 2$.

Lemma 17 *Let G be a connected graph of order n and let $C = c_1, c_2, \ldots, c_k, c_1$ be a longest cycle in G. The subscripts of the vertices c_t are taken modulo k.*

(1) The vertices c_i and c_{i+1} have no common neighbor in $V(G) \setminus V(C)$.
(2) Let $x, y \in V(G) \setminus V(C)$. If $c_i, c_j \in N(x)$ with $j > i+1$, then c_{i+1} and c_{j+1} cannot both belong to $N(y)$.
(3) Let $P_C = p_1, p_2, \ldots, p_s$ be a longest path in $G - V(C)$. If $c_i \in N(p_1)$ and $c_j \in N(p_s)$ $(i < j)$, then $min\{j-i, k-(j-i)\} \geq s+1$ and $s \leq \lfloor \frac{k}{2} \rfloor - 1$.

Proof (1) Obvious.

(2) Suppose that $c_{i+1}, c_{j+1} \in N(y)$. Replace (c_i, c_{j+1})-path on C with $c_i, x, c_j, c_{j-1}, \ldots, c_{i+1}, y, c_{j+1}$, we get a new cycle C'. Obviously, the length of C' is $k+2$, which contradicts that C is a longest cycle in G. This completes the proof of (2).

(3) Replace (c_i, c_j)-path on C with $c_i, p_1, p_2, \ldots, p_s, c_j$, we get a new cycle C' with length $k-(j-i)+s+1$. Replace (c_j, c_i)-path on C with $c_i, p_1, p_2, \ldots, p_s, c_j$, we get a new cycle C'' with length $(j-i)+s+1$. Since $c(G) = k$, we have $min\{j-i, k-(j-i)\} \geq s+1$.

Note that the length of C is k. Then one of (c_i, c_j)-path and (c_j, c_i)-path has length at least $\lceil \frac{k}{2} \rceil$. Without loss of generality, assume that (c_i, c_j)-path has length at least $\lceil \frac{k}{2} \rceil$. Then the length of C'' is at least $\lceil \frac{k}{2} \rceil + 2 + s - 1$. Since $c(G) = k$, $s \leq \lfloor \frac{k}{2} \rfloor - 1$. This completes the proof of (3). □

Now we prove Theorem 4.

Proof Let G be a connected graph with order n and $L(G) \leq 2\delta(G)-1$. By Lemma 12, we have $n \leq \max\{2\delta(G)+6, 3\delta(G)\}$. For $n \leq 3\delta(G)$, by Theorem 10, $c(G) \geq n-1$. Henceforth we may assume that $3\delta(G)+1 \leq n \leq 2\delta(G)+6$. Then we have $\delta(G) \leq 5$. By Lemma 8, G is 2-connected and hence $\delta(G) \geq 2$. By Lemma 9, the result of Theorem 4 is true when $\delta(G) = 2$. Denote by $\delta(G) = \delta$. Now, it suffices to consider the case $3 \leq \delta \leq 5$.

Case 1. $\delta = 3$

Note that $10 \leq n \leq 12$. Since $L(G) \leq 2\delta - 1 = 5$, we have $3 \leq \Delta(G) \leq 5$. If $\Delta(G) = 3$, by Theorem 13, G is hamiltonian and hence $c(G) \geq n-1$. If $\Delta(G) = 5$, there is one vertex $x \in V(G)$ with $d(x) = 5$. Since $d(x) = 2\delta - 1$, by Lemma 14, $n \leq |N[x]| + 2 = 8 < 10$, a contradiction. Next suppose that $\Delta(G) = 4$. We discuss it in two Subcases according to the order of G.

Subcase 1.1. Consider $n = 10$. Let $C = c_1, c_2, \ldots, c_k, c_1$ be a longest cycle in G and let P_C be a longest path in $G - V(C)$. By Lemmas 8 and 15, $k \geq 6$. Now we show that $k \geq n-1 = 9$. Suppose to the contrary that $6 \leq k \leq 8$.

For $k = 6$, by Lemma 17 (3), $|V(P_C)| \leq \lfloor \frac{k}{2} \rfloor - 1 = 2$. If $|V(P_C)| = 1$, then $G[V(G) \setminus V(C)] = 4K_1$ since $n = 10$. Let $x \in V(4K_1)$. Recall that $\delta = 3$. x has at least three neighbors in $V(C)$. More precisely, by Lemma 17 (1), we have x has exactly three neighbors in $V(C)$ and $N(x) = \{c_1, c_3, c_5\}$ or $\{c_2, c_4, c_6\}$. Without loss of generality, assume that $N(x) = \{c_1, c_3, c_5\}$. Since C is a longest cycle in G, by Lemma 17 (2), the neighbors of each vertex of $V(4K_1)$ are c_1, c_3, c_5. Then $d(c_1) \geq 2+4 = 6$, contradicting $\Delta(G) = 4$. Next suppose that $|V(P_C)| = 2$. Let $x, y \in V(G) \setminus V(C)$ and x is adjacent to y. Let $c_i \in N(x)$ and $c_j \in N(y)$ $(i < j)$. By Lemma 17 (3), $min\{j-i, k-(j-i)\} \geq 2+1 = 3$. Since $k = 6$, we have $j-i = 3$. Without loss of generality, assume that $c_1 \in N(x)$. Then $c_4 \in N(y)$. Since C is a longest cycle in G and $\delta = 3$, we have $N(x) = \{y, c_1, c_4\}$ and $N(y) = \{x, c_1, c_4\}$. Note that $|V(G) \setminus V(C)| = 4$ and $\Delta(G) = 4$. There is one vertex $z \in V(G) \setminus V(C)$ is adjacent to at least one of $\{c_2, c_3, c_5, c_6\}$. Suppose that z is adjacent to c_2. The subgraph induced by $\{c_6, c_1, x, y, c_2, c_3, z\}$ contains a tree with leaves c_6, x, y, z, c_3. Then G contains a tree with leaf number $2\delta - 1$. By Lemma 14, $n \leq 7+2 = 9$, a contradiction.

For $k = 7$, by Lemma 17 (3), $|V(P_C)| \leq \lfloor \frac{k}{2} \rfloor - 1 = 2$. If $|V(P_C)| = 1$, then $G[V(G) \setminus V(C)] = 3K_1$ since $n = 10$. Let $x, y, z \in V(G) \setminus V(C)$. By Lemma 17 (1) and (2), $d(x) = d(y) = d(z) = 3$ and $N(x) = N(y) = N(z)$. Then there is one vertex of $V(C)$ has degree at least 5, contradicting $\Delta(G) = 4$. Next suppose that $|V(P_C)| = 2$. Let $x, y \in V(G) \setminus V(C)$ and x is adjacent to y. Let $c_i \in N(x)$ and $c_j \in N(y)$ $(i < j)$. By Lemma 17 (3), $min\{j-i, k-(j-i)\} \geq 3$. Since $k = 7$, we have $3 \leq j-i \leq 4$. Without loss of generality,

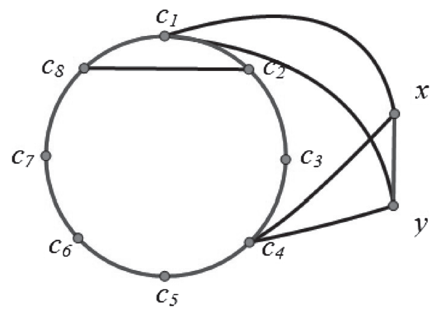

Fig. 3 The subgraph of G when c_2 is adjacent to c_8

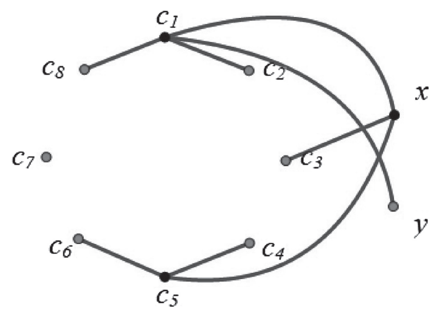

Fig. 4 The subgraph of G when y is adjacent to c_1

assume that $c_1 \in N(x)$. Then $c_4 \in N(y)$ or $c_5 \in N(y)$. First suppose $c_4 \in N(y)$. Since C is a longest cycle in G and $\delta = 3$, we have $N(x) = \{y, c_1, c_4\}$ and $N(y) = \{x, c_1, c_4\}$. Note that $|V(G) \setminus V(C)| = 3$ and $\Delta(G) = 4$. There is one vertex $z \in V(G) \setminus V(C)$ is adjacent to at least one of $\{c_2, c_3, c_5, c_7\}$. Suppose that z is adjacent to c_2. Using the same method as the case $k = 6$, we have $n \leq 7 + 2 = 9$, a contradiction. Now suppose $c_5 \in N(y)$. The proof is similar, so we omit it.

For $k = 8$, since $n = 10$, we have $|V(P_C)| \leq 2$. If $|V(P_C)| = 2$, then $G[V(G) \setminus V(C)] = K_2$. Let $x, y \in V(G) \setminus V(C)$. Let $c_i \in N(x)$ and $c_j \in N(y)$ $(i < j)$. By Lemma 17 (3), $min\{j - i, k - (j - i)\} \geq 3$. Since $k = 8$, we have $3 \leq j - i \leq 5$. Without loss of generality, suppose x is adjacent to c_1. Then y is adjacent to c_4 or c_5 or c_6. Obviously, we can split this into two cases. The first case is where y is adjacent to c_4. Since $\delta = 3$, by Lemma 17 (1) and (2), we have $d(x) = d(y) = 3$ and $N(x) \setminus \{y\} = N(y) \setminus \{x\} = \{c_1, c_4\}$. Consider the vertex c_2 with $d(c_2) \geq 3$. If c_2 is not adjacent to c_8, the subgraph induced by $N(c_1) \cup N(c_2)$ contains a tree with leaf number 5. By Lemma 14, $n \leq 7 + 2 = 9 < 10$, a contradiction. If c_2 is adjacent to c_8, G contains a cycle $c_2, c_8, c_7, c_6, c_5, c_4, y, x, c_1, c_2$ with length 9 (see Fig. 3), contradicting to $k = 8$. The second case is where y is adjacent to c_5. Similarly, by Lemma 17 (1) and (2), we have $d(x) = d(y) = 3$ and $N(x) \setminus \{y\} = N(y) \setminus \{x\} = \{c_1, c_5\}$. The following results which are derived from the above proof: c_2 is adjacent to c_8 and c_4 is adjacent to c_6. Then G contains a cycle $c_2, c_8, c_7, c_6, c_4, c_5, y, x, c_1, c_2$ with length 9, a contradiction.

Now suppose that $|V(P_C)| = 1$. Then $G[V(G) \setminus V(C)] = 2K_1$. Let $x, y \in V(G) \setminus V(C)$. Since $\delta = 3$ and $\Delta(G) = 4$, we have $3 \leq d(x) \leq 4$. If $d(x) = 4$, by Lemma 17 (1), we have $N(x) = \{c_1, c_3, c_5, c_7\}$ or $\{c_2, c_4, c_6, c_8\}$. Without loss of generality, assume that $N(x) = \{c_1, c_3, c_5, c_7\}$. Then the subgraph induced by $V(C) \cup \{x\}$ contains a tree with leaves $c_8, c_2, c_3, c_4, c_6, c_7$. Then G contains a tree with leaf number 2δ, a contradiction. Next suppose $d(x) = 3$. Similarly, suppose $d(y) = 3$. Since $d(x) = 3$, then the neighbors of x divide C into three parts. The lengths of the three parts of C are 2,2,4 or 2,3,3. Without loss of generality, suppose that $N(x) = \{c_1, c_3, c_5\}$ or $\{c_1, c_3, c_6\}$. Since $d(y) = 3$, by Lemma 17 (1) and (2), we have $N(y) \cap N(x) \neq \emptyset$. For the first case, if y is adjacent to c_1 or c_5, then G contains a tree with leaf number at least 6 (see Fig. 4), contradicting $L(G) \leq 2\delta - 1 = 5$. And if y is adjacent to c_3, the subgraph induced by the vertex set $\{c_1, c_2, c_3, c_4, c_5, x, y\}$ contains a tree with leaves c_1, c_2, y, c_4, c_5. Then G contains a tree with leaf number $2\delta - 1$. By Lemma 14, $n \leq 9$, a contradiction. For the second case, the proof method is similar to the first case, and will not be repeated here.

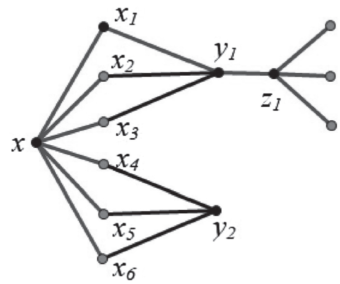

Fig. 5 The subgraph of G in Case 2 when $\Delta(G) = 6$

Subcase 1.2. Consider $n = 11$ or 12. Note that $\Delta(G) = 4$. Let $x \in V(G)$ with $d(x) = 4$. Set $N(x) = \{x_1, x_2, x_3, x_4\}$. Since $L(G) \leq 2\delta - 1 = 5$, we have $d_{G-N[x]}(x_i) \leq 2$ for $i \in \{1, 2, 3, 4\}$. If there is one vertex of $N(x)$ has exactly two neighbors in $V(G) \setminus N[x]$, then G contains a tree with leaf number 5. By Lemma 14, $n \leq 7 + 2 = 9 < 11$, a contradiction. Now suppose that $d_{G-N[x]}(x_i) \leq 1$ for $i \in \{1, 2, 3, 4\}$. Set $X = (N(x_1) \cup N(x_2) \cup N(x_3) \cup N(x_4)) \setminus N[x]$. Similarly, by Lemma 14, we can show that each vertex of X has at most one neighbor in $V(G) \setminus N[x]$. Since $\delta = 3$, we have each vertex of X has at least two neighbors in $N(x)$ and hence $e(N(x), X) \geq 2|X|$. Recall that $d_{G-N[x]}(x_i) \leq 1$ for $i \in \{1, 2, 3, 4\}$. Then $e(N(x), X) \leq 4$. Thus, $|X| \leq 2$. By Lemma 8, G is 2-connected. We have $|X| = 2$ and $d_{G-N[x]}(x_i) = 1$ for $i \in \{1, 2, 3, 4\}$. Set $X = \{y_1, y_2\}$. Without loss of generality, suppose that $N(y_1) \cap N(x) = \{x_1, x_2\}$ and $N(y_2) \cap N(x) = \{x_3, x_4\}$. Since $n \geq 11$ and $d_{G-N[x]}(y_i) = 1$ for $i \in \{1, 2\}$, we have y_1 is not adjacent to y_2. Let $z_1 = N(y_1) \setminus \{x_1, x_2\}$ and $z_2 = N(y_2) \setminus \{x_3, x_4\}$. Since G is 2-connected, we have $z_1 \neq z_2$. Note that $d_{G[N[x]]}(x_i) \geq 2$ for $i \in \{1, 2, 3, 4\}$. Then $G[N(x)]$ contains $2K_2$ and hence G contains a path of length 8 with endpoints z_1 and z_2. For $n = 11$, set $V(G) = \{x, x_1, x_2, x_3, x_4, y_1, y_2, z_1, z_2, w_1, w_2\}$. Since $\delta = 3$, we have $d(w_1) = d(w_2) = 3$ and $N(w_1) = \{w_2, z_1, z_2\}$, $N(w_2) = \{w_1, z_1, z_2\}$. Clearly, $c(G) = n$. For $n = 12$, set $V(G) = \{x, x_1, x_2, x_3, x_4, y_1, y_2, z_1, z_2, w_1, w_2, w_3\}$. One can easily show that $c(G) \geq n - 1$. So Case 1 is proven.

Case 2. $\delta = 4$

Note that $13 \leq n \leq 14$. For $n = 14$, by Lemma 16 (1), $L(G) \geq \frac{2n+8}{5} = \frac{36}{5} > 7$, contradicting $L(G) \leq 2\delta - 1 = 7$. Then we only need to consider the case $n = 13$.

Since $L(G) \leq 7$, we have $4 \leq \Delta(G) \leq 7$. If $\Delta(G) = 7$, there is one vertex $x \in V(G)$ with $d(x) = 7$. Since $d(x) = 2\delta - 1$, by Lemma 14, $n \leq |N[x]| + 2 = 10 < 13$, a contradiction. If $\Delta(G) = 6$, there is one vertex $x \in V(G)$ with $d(x) = 6$. Let X be the union of neighbors of vertices of $N(x)$. Since $n = 13$, by Lemma 14, we have each vertex of $N(x) \cup X$ has at most one neighbor in $V(G) \setminus N[x]$ and hence each vertex of X has at least three neighbors in $N(x)$ since $\delta = 4$. It implies that $e(N(x), X) \leq |N(x)| = 6$ and $e(N(x), X) \geq 3|X|$. Then $|X| \leq 2$. By Lemma 8, G is 2-connected. Then $|X| = 2$ and $e(N(x), X) = 6$. Set $X = \{y_1, y_2\}$. Without loss of generality, suppose that $N(y_1) \cap N(x) = \{x_1, x_2, x_3\}$ and $N(y_2) \cap N(x) = \{x_4, x_5, x_6\}$. Since $n = 13$, we have y_1 is not adjacent to y_2. Let $z_1 \in N(y_1)$. Then $|N(z_1) \setminus \{y_1\}| \geq 3$ since $\delta = 4$. From Fig. 5, we obtain G contains a tree with leaf number 8, a contradiction. For $\Delta(G) = 4$, by Theorem 13, $c(G) = n \geq n - 1$.

It remains the case of $\Delta(G) = 5$. Let $C = c_1, c_2, \ldots, c_k, c_1$ be a longest cycle in G and let P_C be a longest path in $G - V(C)$. By Lemmas 8 and 15, $k \geq 8$. Now we show that $k \geq n - 1 = 12$. Suppose to the contrary that $8 \leq k \leq 11$.

For $k = 8$, by Lemma 17 (3), $|V(P_C)| \leq \lfloor \frac{k}{2} \rfloor - 1 = 3$. Since $\delta = 4$ and $\Delta(G) = 5$, by Lemma 17 (1) and (2), there are at most three isolated vertices in $G - V(C)$. Note that $n = 13$. Then $2 \leq |V(P_C)| \leq 3$. If $|V(P_C)| = 2$, then $G - V(C) = 3K_1 + K_2$ or $K_1 + 2K_2$. Let $x, y \in V(G) \setminus V(C)$ and x is adjacent to y. Let $c_i \in N(x)$ and $c_j \in N(y)$ $(i < j)$. By Lemma 17 (3), $min\{j - i, k - (j - i)\} \geq 3$. Since $k = 8$, we have $3 \leq j - i \leq 5$. Without loss of generality, suppose x is adjacent to c_1. Then $N(y) \subseteq \{x, c_1, c_4, c_5, c_6\}$. By Lemma 17 (1), we have $N(y) = \{x, c_1, c_4, c_6\}$. Recall that C is a longest cycle. Then we have $N(x) = \{c_1, y\}$, contradicting $d(x) \geq 4$. Next suppose $|V(P_C)| = 3$. Let $P_C = x, y, z$. Let $c_i \in N(x)$ and $c_j \in N(z)$ $(i < j)$. By Lemma 17 (3), $min\{j - i, k - (j - i)\} \geq 4$. Since $k = 8$, we have $j - i = 4$. Without loss of generality, suppose x is adjacent to c_1. Then z is adjacent to c_5. Since $\delta(G) = 4$ and P_C is a longest path in $G - V(C)$, we have $N(x) = \{c_1, c_5, y, z\}$ and $N(z) = \{c_1, c_5, x, y\}$. Note that we have a new path $P'_C = y, x, z$ in $G - V(C)$. Then $N(y) = \{c_1, c_5, x, z\}$ and hence $G[V(G) \setminus V(C)] = 2K_1 + K_3$ or $K_2 + K_3$. Consider the vertices c_2 with $d(c_2) \geq 4$. If c_2 is not adjacent to c_8, then the subgraph induced by $N(c_1) \cup N(c_2)$ contains a tree with leaf

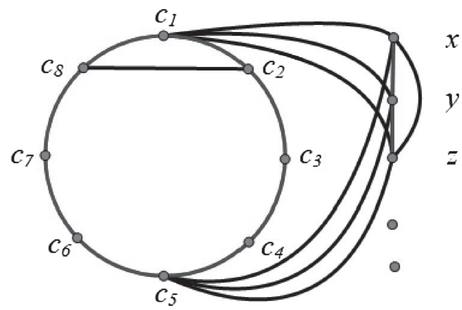

Fig. 6 The subgraph of G in Case 2 when $k = 8$ and $|V(P_c)| = 3$

number 7 and hence G contains a tree with leaf number $2\delta - 1 = 7$. By Lemma 14, $n \leq 9 + 2 = 11 < 13$, a contradiction. Thus, suppose that c_2 is adjacent to c_8. Then G contains a cycle $c_1, c_2, c_8, c_7, c_6, c_5, z, y, x, c_1$ with length 9 (see Fig. 6), contradicting $k = 8$.

For $k = 9$, by Lemma 17 (3), $|V(P_C)| \leq \lfloor \frac{k}{2} \rfloor - 1 = 3$. Then by Lemma 17 (1) and (2), there are at most three isolated vertices in $G - V(C)$ and hence $2 \leq |V(P_C)| \leq 3$. If $|V(P_C)| = 2$, then $G[V(G) \setminus V(C)] = 2K_1 + K_2$ or $2K_2$. Let $x, y \in V(G) \setminus V(C)$ and x is adjacent to y. Let $c_i \in N(x)$ and $c_j \in N(y)$ $(i < j)$. By Lemma 17 (3), $min\{j - i, k - (j - i)\} \geq 3$ and hence $3 \leq j - i \leq 6$. Without loss of generality, suppose x is adjacent to c_1. Then $N(y) \subseteq \{x, c_1, c_4, c_5, c_6, c_7\}$. By Lemma 17 (1), $|N(y) \cap \{c_4, c_5, c_6, c_7\}| \leq 2$. Since $d(y) \geq 4$, we have $|N(y) \cap \{c_4, c_5, c_6, c_7\}| = 2$ and hence $c_1 \in N(y)$. It follows that $N(x) \subseteq \{y, c_1, c_4, c_5, c_6, c_7\}$ and $|N(x) \cap \{c_4, c_5, c_6, c_7\}| = 2$. Recall that C is a longest cycle in G. Then $N(x) = \{y, c_1, c_4, c_7\}$ and $N(y) = \{x, c_1, c_4, c_7\}$. Let $z, w \in V(G) \setminus V(C)$. Since $d(z) \geq 4$, we have $|N(z) \cap V(C)| \geq 3$. If $N(z) \cap \{c_1, c_4, c_7\} \neq \emptyset$, the subgraph induced by $V(C) \cup \{x, y, z\}$ contains a tree with leaf number 8. Then G contains a spanning tree with leaf number at least 8, a contradiction. Now suppose that $N(z) \cap \{c_1, c_4, c_7\} = \emptyset$. Then $|N(z) \cap \{c_2, c_3, c_5, c_6, c_8, c_9\}| \geq 3$. By Lemma 17 (1), $|N(z) \cap \{c_2, c_3, c_5, c_6, c_8, c_9\}| = 3$ and hence z is adjacent to w. It follows that $G[V(G) \setminus V(C)] = 2K_2$. Similarly, we have $N(z) \cap N(w) \cap V(C) = \{c_2, c_5, c_8\}$ or $\{c_3, c_6, c_9\}$. Recall that $N(x) \cap N(y) \cap V(C) = \{c_1, c_4, c_7\}$. By Lemma 17 (2), we have a contradiction. Next suppose $|V(P_C)| = 3$. Let $P_C = x, y, z$. Without loss of generality, suppose x is adjacent to c_1. Using the same method as the case of $k = 8$ and $|V(P_C)| = 3$, we obtain $G[V(G) \setminus V(C)] = K_1 + K_3$ and $N(x) \cap V(C) = N(y) \cap V(C) = N(z) \cap V(C) = \{c_1, c_5\}$ or $\{c_1, c_6\}$. The two cases have the same proof. Hence we may assume that $N(x) \cap V(C) = \{c_1, c_5\}$. If c_2 is adjacent to c_9, G contains a cycle $c_1, c_2, c_9, c_8, c_7, c_6, c_5, z, y, x, c_1$ with length 10, a contradiction. If c_2 is not adjacent to c_9, the subgraph induced by $N(c_1) \cup N(c_2)$ contains a tree with leaf number 7. Then G contains a tree with leaf number $2\delta - 1 = 7$. By Lemma 14, $n \leq 9 + 2 = 11 < 13$, a contradiction.

For $k = 10$, since $n = 13$, we have $|V(P_C)| \leq 3$. For $|V(P_C)| = 3$, let $P_C = x, y, z$. Without loss of generality, suppose x is adjacent to c_1. By Lemma 17 (3), $N(z) \cap V(C) \subseteq \{c_1, c_5, c_6, c_7\}$. Then using case-by-case analysis, we obtain $N(z) \cap V(C) = N(x) \cap V(C) = \{c_1, c_5\}$ or $\{c_1, c_6\}$ or $\{c_1, c_7\}$. Since $\delta = 4$, we have x is adjacent to z. Then $G[V(G) \setminus V(C)] = K_3$. Note that we have a new path $P'_C = y, x, z$ in $G - V(C)$. Thus, $N(x) \cap V(C) = N(y) \cap V(C) = N(z) \cap V(C)$. If $N_C(x) = \{c_1, c_5\}$, we have $N(c_1) = \{x, y, z, c_2, c_{10}\}$. We consider the vertex c_2 with $d(c_2) \geq 4$. If c_2 is not adjacent to c_{10}, then the subgraph induced by $N(c_1) \cup N(c_2)$ contains a tree with leaf number 7. Hence G contains a tree with leaf number $2\delta - 1 = 7$. By Lemma 14, $n \leq 9 + 2 = 11 < 13$, a contradiction.

Thus, suppose that c_2 is adjacent to c_{10}. Then G contains a cycle $c_1, c_2, c_{10}, c_9, c_8, c_7, c_6, c_5, z, y, x, c_1$ with length 11 (see Fig. 7a), contradicting $k = 10$. If $N_C(x) = \{c_1, c_6\}$, we consider the vertex c_2 and c_5. If c_2 is not adjacent to c_{10} or c_5 is not adjacent to c_7, using the same method, we have done. Now suppose that c_2 is adjacent to c_{10} and c_5 is adjacent to c_7. Then G contains a cycle $c_1, c_2, c_{10}, c_9, c_8, c_7, c_5, c_6, z, y, x, c_1$ with length 11 (see Fig. 7b), contradicting $k = 10$. The case of $N(x) \cap V(C) = \{c_1, c_7\}$ can be proved in the same way, so we omit it. For $|V(P_C)| \leq 2$, it implies that there exists at least one isolated vertex in $G - V(C)$. Let x be an isolated vertex in $G - V(C)$. Note that $4 = \delta \leq d(x) \leq \Delta(G) = 5$. If $d(x) = 5$, by Lemma 17 (1), then $N(x) = \{c_1, c_3, c_5, c_7, c_9\}$ or $\{c_2, c_4, c_6, c_8, c_{10}\}$.

From Fig. 8a, G contains a tree with leaf number $2\delta - 1 = 7$. Then by Lemma 14, $n \leq 10 + 2 = 12 < 13$, a contradiction. Now suppose that $d(x) = 4$. Then the neighbors of x divide C into four parts and the lengths of the four parts of C are 2,2,2,4 or 2,2,3,3 or 2,3,2,3.

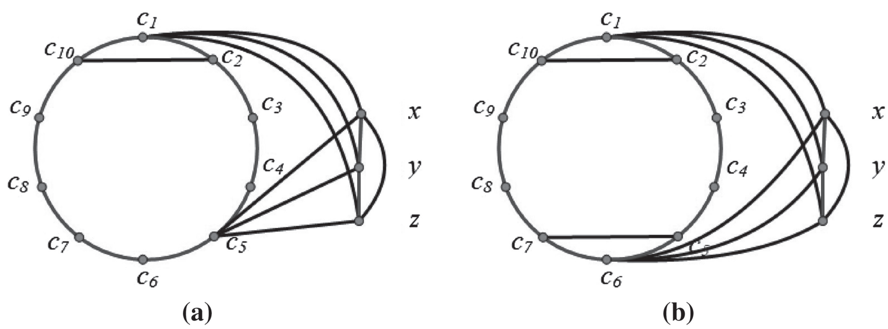

Fig. 7 **a** The subgraph of G when $k = 10$ and c_2 is adjacent to c_{10}. **b** The subgraph of G when $k = 10$ and c_2 is adjacent to c_{10}, c_5 is adjacent to c_7

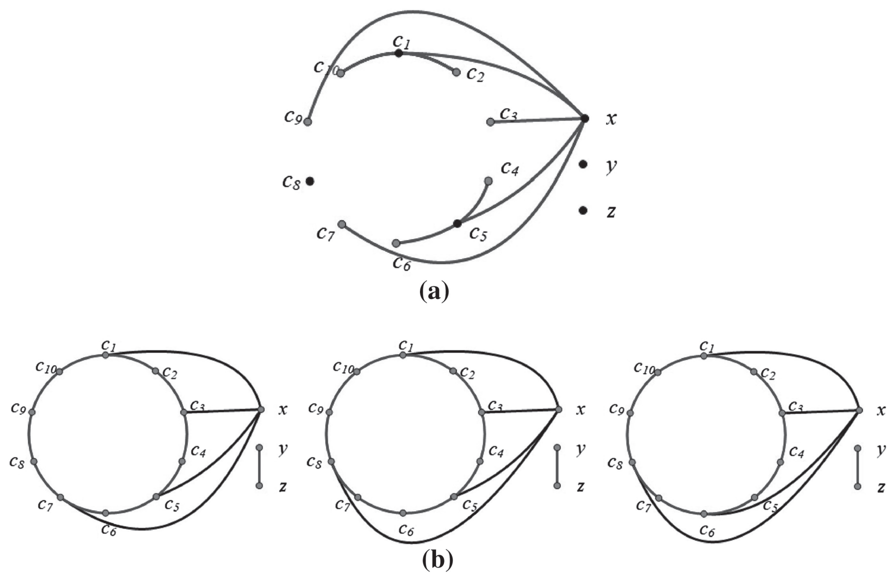

Fig. 8 **a** The subgraph of G when $k = 10$ and $d(x) = 5$. **b** The subgraph of G when $k = 10$ and $|V(P_c)| = 2$

Let $G - V(C) - \{x\} = \{y, z\}$. If any one of $\{y, z\}$ is adjacent to one of $N(x)$, one can easily show that G contains a tree of order 10 with leaf number 7. And by Lemma 14, we have a contradiction. Now suppose that $N(y) \cap N(x) = \emptyset$ and $N(z) \cap N(x) = \emptyset$. Then $|(N(y) \cup N(z)) \cap V(C)| \leq 10 - 4 = 6$. By Lemma 17 (1) and (2), we have y is adjacent to z and hence $|V(P_C)| = 2$ (see Fig. 8b). By Lemma 17 and case-by-case analysis, it is easy to show that in each case there is a contradiction, so we omit it.

For $k = 11$, since $n = 13$, we have $|V(P_C)| \leq 2$. Let $x, y \in V(G) \setminus V(C)$. For $|V(P_C)| = 2$, we have x is adjacent to y. Recall that $\delta = 4$ and $\Delta(G) = 5$. Then $3 \leq |N(x) \cap V(C)| \leq 4$. By Lemma 17 (1), the vertices of $N(x) \cap V(C)$ are independent on C. For $|N(x) \cap V(C)| = 4$, it is easy to check that G contains a tree of order 10 with leaf number $2\delta - 1 = 7$. By Lemma 14, $n \leq 10 + 2 = 12 < 13$, a contradiction. Now suppose that $|N(x) \cap V(C)| = 3$. Then the neighbors of x in $V(C)$ divide C into three parts and the lengths of the three parts of C are 2,2,7 or 2,3,6 or 2,4,5 or 3,3,5 or 3,4,4. Let $c_i \in N(x)$ and $c_j \in N(y)$ $(i < j)$. By Lemma 17 (3), we have then $min\{j - i, k - (j - i)\} \geq 3$. The proof methods for the first three cases are similar, so we only give the proof for the case 2,2,7. Without loss of generality, suppose $N(x) = \{y, c_1, c_3, c_5\}$. Then by Lemma 17 (3), $N(y) \subseteq \{x, c_8, c_9\}$. By Lemma 17 (1), we have $d(y) \leq 2$, contradicting $\delta = 4$. The proofs for the latter two cases are similar, we only give the proof for the case 3,3,5. Without loss of generality, suppose $N(x) = \{y, c_1, c_4, c_7\}$. Then by Lemma 17 (3), $N(y) \setminus \{x\} = N(x) \setminus \{y\} = \{c_1, c_4, c_7\}$. Consider the vertex c_3 with $d(c_3) \geq 4$. If $N(c_3) \cap \{c_1, c_5, c_7\} = \emptyset$, the subgraph induced by $N[c_3] \cup \{x, y, c_1, c_5, c_7\}$ contains a tree with leaf number 7 (see Fig. 9a) and by Lemma 14, $n \leq 12 < 13$, a contradiction. Now suppose that $N(c_3) \cap \{c_1, c_5, c_7\} \neq \emptyset$. If c_3 is adjacent to c_1 or c_7, G contains a tree with leaf number $2\delta - 1 = 7$

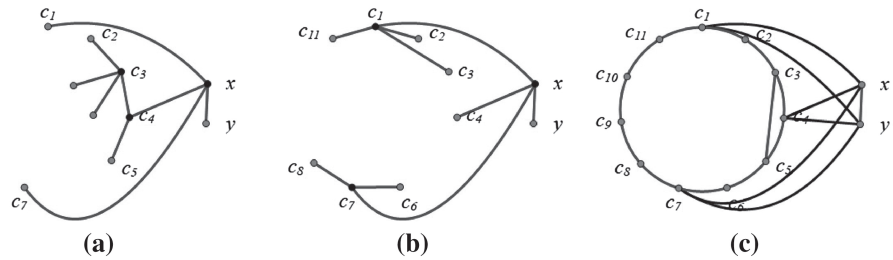

Fig. 9 **a** The subgraph of G when any one of $\{c_1, c_5, c_7\}$ is not adjacent to c_3. **b** The subgraph of G when c_3 is adjacent to c_1. **c** The subgraph of G when c_3 is adjacent to c_5

(see Fig. 9b). Then by Lemma 14, $n \leq 12 < 13$, a contradiction. If c_3 is adjacent to c_5, G contains a cycle $c_4, c_3, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_1, x, y, c_4$ of length 12 (see Fig. 9c), contradicting $k = 11$.

For $|V(P_C)| = 1$, we have x is not adjacent to y. If one of $\{x, y\}$ has degree 5 or $N(x) \cap N(y) \neq \emptyset$, one can easily show that G contains a tree of order 10 with leaf number 7. And by Lemma 14, we have a contradiction. Hence we may assume that $d(x) = d(y) = 4$ and $N(x) \cap N(y) = \emptyset$. Note that the neighbors of x in $V(C)$ divide C into four parts and hence we have four cases. The lengths of each parts of C are 2,2,2,5 or 2,2,3,4 or 2,3,2,4 or 2,3,3,3. Similarly, by Lemma 17 (1), (2) and case-by-case analysis, one can easily show that in each case there is a contradiction. Then Case 2 is proven.

Case 3. $\delta = 5$

Note that $n = 16$. By Lemma 16 (2), $L(G) \geq \frac{n}{2} + 2 = 10$, contradicting $L(G) \leq 2\delta - 1$. So Case 3 is proven.

For the sharpness, consider the following graph. The graph G_1 of order $n \geq 5$ is formed by taking the cycle $C_{n-1} = v_1, v_2, \ldots, v_{n-2}, v_{n-1}, v_1$ and add one vertex v_n together with edges $v_1 v_n$, $v_3 v_n$. Note that $\delta(G_1) = 2$ and $L(G_1) = 3$. Then G_1 satisfying $L(G_1) \leq 2\delta(G_1) - 1$ and $c(G_1) = n - 1$.

The condition $L(G) \leq 2\delta(G) - 1$ cannot be relaxed. The graph G_2 with order $n \geq 8$ is formed by taking the cycle $C_{n-2} = v_1, v_2, \ldots, v_{n-2}, v_1$ and add two vertices v_{n-1} and v_n together with edges $v_1 v_{n-1}$, $v_3 v_{n-1}$, $v_{n-5} v_n$, $v_{n-3} v_n$. Clearly, $\delta(G_2) = 2$ and $L(G_2) = 4$. Then G_2 satisfying $L(G_2) \leq 2\delta(G_2)$ but $c(G_2) = n - 2$. This completes the proof of Theorem 4. □

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Declarations

Competing interests The authors declare no competing financial interests.

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