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ORIGINAL RESEARCH

Semi-commutants of toeplitz operators on the Fock-Sobolev space

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Abstract In this paper, we consider the semi-commutants of Toeplitz operators on the Fock-Sobolev space of the complex plane. This work generalizes several partial results in a novel manner.

Keywords Semi-commutants · Toeplitz operators · Fock-Sobolev space

Mathematics Subject Classification Primary 47B35

1 Introduction

Let $\mathbb{C}$ denote the complex plane and dA be the area measure. For any fixed non-negative integer m, L_m^2 is the space of Lebesgue measurable functions f on $\mathbb{C}$ such that the function $f(z)$ is in $L^2(\mathbb{C}, \frac{|z|^{2m}e^{-|z|^2}}{\pi m!}dA(z))$. It is well known that L_m^2 is a Hilbert space with the inner product

$$\langle f, g\rangle = \frac{1}{\pi m!}\int_{\mathbb{C}} f(z)\overline{g(z)}|z|^{2m}e^{-|z|^2}dA(z), \quad f, g \in L_m^2.$$

The Fock-Sobolev space $F^{2,m}$ consists of all entire functions f in L_m^2. Obviously, the Fock-Sobolev space $F^{2,m}$ is a closed subspace of the Hilbert space L_m^2. There is an orthogonal projection P from L_m^2 onto $F^{2,m}$, which is given by $Pf(z) = \langle f(w), K_m(w, z)\rangle$, where

$$K_m(z, w) = \sum_{k=0}^{\infty} \frac{m!}{(k+m)!}(z\overline{w})^k = m!\frac{e^{z\overline{w}} - q_m(z\overline{w})}{(z\overline{w})^m}$$

is the reproducing kernel of Fock-Sobolev space $F^{2,m}$. Here, $q_0 = 0$ and q_m is the Taylor polynomial of $e^{z\overline{w}}$ of order $m-1$ for all $m \geq 1$, see [10].

Let D be the set of all finite linear combinations of kernel functions of $F^{2,m}$. Suppose φ is a Lebesgue measurable function on $\mathbb{C}$ that satisfies

$$\int_{\mathbb{C}} |\varphi(w)||K_m(z, w)||w|^{2m}e^{-|w|^2}dA(w) < \infty. \tag{1}$$

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Since D is dense in $F^{2,m}$, so we can densely define the Toeplitz operator with the symbol φ on $F^{2,m}$ as follows:

$$T_\varphi f(z) = \frac{1}{\pi m!}\int_{\mathbb{C}} \varphi(w) f(w) K_m(z,w)|w|^{2m} e^{-|w|^2} dA(w).$$

Let $k_m(z,w)$ be the normalization of the reproducing kernel $K_m(z,w)$, i.e., $k_m(z,w) = \frac{K_m(z,w)}{\sqrt{K_m(z,z)}}$. Suppose that φ is a Lebesgue measurable function on $\mathbb{C}$ satisfying condition (1), the Berezin transform of T_φ is

$$\widetilde{T_\varphi}(z) = \langle T_\varphi k_m(w,z), k_m(w,z)\rangle.$$

If $m = 0$, the space is the Fock space F^2. For the Fock space F^2, there is a Weyl operator U_z on F^2 such that $U_z f(w) = f(w-z)k_z(w)$, where $k_z(w)$ is normalization of the reproducing kernel of F^2. If f satisfies condition (1), then

$$\widetilde{f}(z) = \int_{\mathbb{C}} f(z \pm w) e^{-|w|^2} dA(w)$$

by the property of U_z.

Up to our knowledge, the boundedness of a single Toeplitz operator on F^2 is still an open problem, see [1,4,5]. So, it is difficult to characterize the commuting Toeplitz operators on the Fock space. In fact, the commutativity of Toeplitz operators with general symbols on the Fock space is still an open problem, see [2].

Let $\epsilon(\mathbb{C})$ be the set of entire functions g on $\mathbb{C}$ such that $|g(z)|e^{-c|z|}$ is essentially bounded on $\mathbb{C}$ for some $c > 0$. If $T_f T_{\overline{g}} = T_{f\overline{g}}$ with $f, g \in \epsilon(\mathbb{C})$ on Fock space, then, by the elementary of ordinary equations with constant coefficients, the $\epsilon(\mathbb{C})$ can be replaced by the following set

$$\mathcal{A}_1 = \left\{ \sum_{j=1}^{N} p_j K_{a_j} : N \in \mathbb{N} \text{ and } p_j \in \mathcal{P}, a_j \in \mathbb{C} \text{ for } j = 1, \dots, N \right\},$$

where $\mathcal{P}$ is the space of all holomorphic polynomials on $\mathbb{C}$, see [2]. Based on these facts, the authors obtained the following result.

Theorem 1 *[2] Suppose $f, g \in \epsilon(\mathbb{C})$, then the following statements are equivalent on Fock space:*

(a) $T_f T_{\overline{g}} = T_{f\overline{g}}$.
(b) $\widetilde{f\overline{g}} = f\overline{g}$.
(c) *Either (1) or (2) holds;*

(1) *f or g is constant.*
(2) *There are finite collections $\{a_j\}_{j=1}^N$ and $\{b_l\}_{l=1}^M$ of distinct complex numbers such that*

$$f \in Span\{K_{a_1}, \cdots, K_{a_N}\}, \quad and \quad g \in Span\{K_{b_1}, \cdots, K_{b_M}\}$$

with $a_j \overline{b_l} = 2\eta\pi \mathrm{i}$ for each j and l. Here, η is any integer and K_z is the kernel of F^2.

The Weyl operator U_z is a key to consider the Operator Theory in F^2, the results of [1,2,4,5,9,12] are based on the property of U_z. However, the translations do not have such good property on the Fock-Sobolev space $F^{2,m}$. So the Berezin transform of the function can almost never be computed explicitly. In fact, we don't even know the specific form of $[\widetilde{K_m(z,a)\overline{K_m(z,b)}}]$ on the Fock-Sobolev space. The commutativity of Toeplitz operators with general symbols is left open on the Fock-Sobolev space.

So far, there are few results of semi-commuting Toeplitz operators with non-radial symbols on the Fock-Sobolev space. Recall that D is the set of all finite linear combinations of kernel functions of $F^{2,m}$. We have obtained the following result.

Theorem 2 *[15] Suppose m is positive integer, the following conditions are equivalent for any two functions f and g in D:*

(a) $T_f T_{\overline{g}} = T_{f\overline{g}}$ *on* $F^{2,m}$.
(b) $\widetilde{f\overline{g}} = f\overline{g}$.
(c) *At least one of f and g is a constant.*

There is a natural problem which arise from Theorem 1 and 2.
Problem Suppose that $f(z) = K_m(z,a)$ and $g(z) = e^{bz}$. What is the relationship between their symbols when $(T_f, T_{\overline{g}}] = 0$ on $F^{2,m}$

Clearly, $K_m(z,w) \notin \mathcal{A}_1$. In order to answer the problem, we define

$$\mathfrak{D} = D \cup \left\{ f(z) = ce^{az} + b : a, b, c \in \mathbb{C} \right\} \cup \{ f(z) = az + b : a, b \in \mathbb{C} \}.$$

The purpose of this work is to study the semi-commuting Toeplitz operators on the Fock-Sobolev space with $\mathfrak{D}-$ symbol. Moreover, we give a simple proof of the main result in [2], see the proof of Theorem 1. The main result of this paper is given by the following theorem.
Main Result Suppose that m is a positive integer and $f, g \in \mathfrak{D}$. Then the following statements are equivalent.

(a) $T_f T_{\overline{g}} = T_{f\overline{g}}$.
(b) $\widetilde{f\overline{g}} = f\overline{g}$.
(c) At least one of f and g is a constant.

As is well known, the Fock-sobolev space is the Fock space with the radial weight. So, some operators have the same properties in both the Fock and Fock-Sobolev space. For instance, Sarason's problem, the boundedness and compactness of composition operators, the commuting Toeplitz operators with radial symbols, Ha-plitz product and so on, see [3,6–9,11,13,16]. The main result of this paper says that there is a huge difference between the Fock and Fock-Sobolev space.

2 Proof of the Main Result

In this section, we prove the main result. We need the following lemma, which can be founded in [15].

Lemma 3 *[15] Let k, j and l be non-negative integers. Then*

$$T_{\overline{z}^k} z^l = \begin{cases} 0 & \text{if } l < k, \\ \frac{(l+m)!}{(l-k+m)!} z^{l-k} & \text{if } l \geq k; \end{cases}$$

$$T_{z^l} T_{\overline{z}^k} z^j = \begin{cases} 0 & \text{if } j < k, \\ \frac{(j+m)!}{(j-k+m)!} z^{j+l-k} & \text{if } j \geq k; \end{cases}$$

$$T_{z^l \overline{z}^k} z^j = \begin{cases} 0 & \text{if } j+l < k, \\ \frac{(j+l+m)!}{(j+l-k+m)!} z^{j+l-k} & \text{if } j+l \geq k. \end{cases}$$

Before we prove the main result, we establish the following special case first.

Proposition 4 *Suppose $f \in D$, m is a positive integer and $a, b \in \mathbb{C}$.*
(a) *If $T_{f(z)\overline{z}} = T_{f(z)} T_{\overline{z}}$, then f is a constant function.*
(b) *If $T_{e^{az}} T_{\overline{z}} = T_{e^{az}\overline{z}}$, then $a = 0$.*
(c) *If $T_{f(z)} T_{\overline{e^{az}}} = T_{f(z)\overline{e^{az}}}$, then f is a constant function or $a = 0$.*

Proof To prove (a). Since $T_{f(z)\overline{z}} = T_{f(z)} T_{\overline{z}}$ and $T_{f(z)} T_{\overline{z}} 1 = 0$. This shows that $T_{\overline{z}} f(z) = 0$. It follow that $\deg(f) < 1$. Then f is a constant function. similarly, $a = 0$ if (b) holds.

To prove (c). Let $f(z) = \sum_{i=1}^{N} b_i K_m(z, \lambda_i)$, where $N < \infty$. Without loss of generality, we assume

$$a \neq 0, \quad b_i \neq 0, \quad \lambda_i \neq 0, \quad \lambda_i \neq \lambda_\kappa (i \neq \kappa)$$

for $1 \leq i, \kappa \leq N$. By Lemma 3, we have

$$\sum_{i=1}^{N} b_i T_{K_m(z,\lambda_i) e^{\overline{az}}} z^l = \sum_{i=1}^{N} b_i \sum_{k_i=0}^{\infty} \sum_{\eta=0}^{k_i+l} \frac{m! \overline{\lambda_i}^{k_i} \overline{a}^{\eta}}{(k_i+m)! \eta!} \frac{(k_i+l+m)!}{(k_i+l-\eta+m)!} z^{k_i+l-\eta}$$

and

$$\sum_{i=1}^{N} b_i T_{K_m(z,\lambda_i)} T_{e^{\overline{az}}} z^l = \sum_{i=1}^{N} b_i \sum_{k_i=0}^{\infty} \sum_{\eta=0}^{l} \frac{m! \overline{\lambda_i}^{k_i} \overline{a}^{\eta}}{(k_i+m)! \eta!} \frac{(l+m)!}{(l-\eta+m)!} z^{k_i+l-\eta}.$$

Since $T_{f(z)}T_{\overline{e^{az}}} = T_{f(z)\overline{e^{az}}}$, then $T_{f(z)}T_{\overline{e^{az}}}z^l = T_{f(z)\overline{e^{az}}}z^l$ for every l. Let $j = k_i + l - \eta$, we have

$$\begin{aligned}&\sum_{i=1}^{N} b_i \sum_{k_i=\max\{0,j-l\}} \frac{m!}{(k_i+m)!}\frac{(k_i+l+m)!}{(k_i+l-j)!}\overline{\lambda_i}^{k_i}\bar{a}^{k_i+l-j}\\ &\quad= \sum_{i=1}^{N} b_i \sum_{k_i=\max\{0,j-l\}}^{j} \frac{m!}{(k_i+m)!}\frac{(j+m)!}{(k_i+l-j)!}\frac{(l+m)!}{(j-k_i+m)!}\overline{\lambda_i}^{k_i}\bar{a}^{k_i+l-j}\end{aligned} \tag{2}$$

for all j and l.

Now, assume $0 = l \leq j$, then, by (2), one can see that

$$\sum_{i=1}^{N} b_i \sum_{k_i=j}^{\infty} \frac{m!}{(k_i-j)!}\overline{\lambda_i}^{k_i}\bar{a}^{k_i-j} = \sum_{i=1}^{N} b_i m!\overline{\lambda_i}^{j}.$$

That is,

$$\sum_{i=1}^{N} b_i \overline{\lambda_i}^{j} e^{\overline{a\lambda_i}} = \sum_{i=1}^{N} b_i \overline{\lambda_i}^{j} \tag{3}$$

for all $j \geq 0$. Define

$$E_1 = \begin{pmatrix} e^{\overline{a\lambda_1}}-1 & \cdots & e^{\overline{a\lambda_N}}-1 \\ \vdots & \vdots & \vdots \\ \lambda_1^{N-1}(e^{\overline{a\lambda_1}}-1) & \cdots & \lambda_N^{N-1}(e^{\overline{a\lambda_N}}-1) \end{pmatrix}, \quad X_1 = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}.$$

Using (3), we have $E_1X_1 = 0$, which implies that $|E_1| = 0$. By the property of Vandermonde determinant,

$$0 = |E_1| = \prod_{i=1}^{N}(e^{\overline{a\lambda_i}}-1)\prod_{1\leq \iota < i \leq N}(\lambda_i - \lambda_\iota).$$

Since $\{\lambda_i\}$ is a finite collection of distinct non-zero complex numbers, there is a ι so that

$$e^{\overline{a\lambda_\iota}} - 1 = 0.$$

By (3) again, we can obtain

$$e^{\overline{\lambda_i a}} = 1 \tag{4}$$

for all $1 \leq i \leq N$.

If $l = j = 1$, then

$$\sum_{i=1}^{N} b_i \sum_{k_i=1}^{\infty} \frac{m!}{k_i!}(k_i+m+1)\overline{\lambda_i}^{k_i}\bar{a}^{k_i} = \sum_{i=1}^{N} b_i (m+1)!\overline{\lambda_i}\bar{a}.$$

So we have

$$\sum_{i=1}^{N} m!b_i \left\{e^{\overline{\lambda_i a}}\overline{\lambda_i}\bar{a} + (m+1)\left(e^{\overline{\lambda_i a}}-1\right)\right\} = \sum_{i=1}^{N} b_i(m+1)!\overline{\lambda_i}\bar{a}.$$

Using (4),

$$\sum_{i=1}^{N} m!b_i\overline{\lambda_i}\bar{a} = \sum_{i=1}^{N} b_i(m+1)!\overline{\lambda_i}\bar{a}.$$

Since $m \neq 0$ and $a \neq 0$, the above equation becomes

$$\sum_{i=1}^{N} b_i \overline{\lambda_i} = 0. \tag{5}$$

Now assume $j \geq l = 2$, it follows that

$$\begin{aligned}
&\sum_{i=1}^{N} b_i \sum_{k_i \geq j-2} \frac{m!}{(k_i+m)!} \frac{(k_i+2+m)!}{(k_i+2-j)!} \overline{\lambda_i}^{k_i} \bar{a}^{k_i+2-j} \\
&= \sum_{i=1}^{N} b_i \sum_{k_i \geq j-2}^{j} \frac{m!}{(k_i+m)!} \frac{(j+m)!}{(k_i+2-j)!} \frac{(2+m)!}{(j-k_i+m)!} \overline{\lambda_i}^{k_i} \bar{a}^{k_i+2-j}.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{i=1}^{N} \frac{m!}{2} b_i (m+1)(m+2) \overline{\lambda_i}^{j} \bar{a}^2 + \sum_{i=1}^{N} b_i m! (j+m)(2+m) \overline{\lambda_i}^{j-1} \bar{a} \\
&= \sum_{i=1}^{N} b_i \Bigg\{ \sum_{k_i = j+1} \frac{m!(k_i+m+1)(k_i+m+2)}{(k_i+2-j)!} \overline{\lambda_i}^{k_i} \bar{a}^{k_i+2-j} \\
&\quad + m!(j+m)(j+m+1) \overline{\lambda_i}^{j-1} \bar{a} + \frac{m!}{2} (j+m+1)(j+m+2) \overline{\lambda_i}^{j} \bar{a}^2 \Bigg\}.
\end{aligned} \tag{6}$$

In fact,

$$(k+m+1)(k+m+2) = (k+2-j)(k+1-j) + 2(m+j)(k+2-j) + (m+j)(m+j-1). \tag{7}$$

Using (7) and the fact $e^{\overline{\lambda_i a}} = 1$ for all $1 \leq i \leq N$, one can see that

$$\begin{aligned}
&\sum_{i=1}^{N} b_i \sum_{k_i \geq j+1} \frac{m!(k_i+m+1)(k_i+m+2)}{(k_i+2-j)!} \overline{\lambda_i}^{k_i} \bar{a}^{k_i+2-j} \\
&= \sum_{i=1}^{N} b_i m! (e^{\overline{\lambda_i a}} - 1) \overline{\lambda_i}^{j} \bar{a}^2 + \sum_{i=1}^{N} 2 b_i m! (m+j) (e^{\overline{\lambda_i a}} - 1 - \overline{\lambda_i} \bar{a}) \overline{\lambda_i}^{j-1} \bar{a} \\
&\quad + \sum_{i=1}^{N} b_i m! (j+m)(j+m-1) (e^{\overline{\lambda_i a}} - 1 - \overline{\lambda_i} \bar{a} - \frac{\overline{\lambda_i}^2 \bar{a}^2}{2}) \lambda_i^{j-2} \\
&= -\sum_{i=1}^{N} 2 b_i m! (m+j) \overline{\lambda_i}^{j} \bar{a}^2 - \sum_{i=1}^{N} b_i m! (m+j)(m+j-1) \overline{\lambda_i}^{j-1} \bar{a} \\
&\quad - \frac{1}{2} \sum_{i=1}^{N} b_i m! (m+j)(m+j-1) \overline{\lambda_i}^{j} \bar{a}^2 \\
&= -\frac{1}{2} \sum_{i=1}^{N} b_i m! (m+j)(j+m+3) \overline{\lambda_i}^{j} \bar{a}^2 - \sum_{i=1}^{N} b_i m! (m+j)(m+j-1) \overline{\lambda_i}^{j-1} \bar{a}.
\end{aligned}$$

Putting this in (6), we get

$$\begin{aligned}
&\sum_{i=1}^{N} \frac{m!}{2} b_i (m+1)(m+2) \overline{\lambda_i}^{j} \bar{a}^2 + \sum_{i=1}^{N} b_i m! (j+m)(2+m) \overline{\lambda_i}^{j-1} \bar{a} \\
&= -\frac{1}{2} \sum_{i=1}^{N} b_i m! (m+j)(j+m+3) \overline{\lambda_i}^{j} \bar{a}^2 - \sum_{i=1}^{N} b_i m! (m+j)(m+j-1) \overline{\lambda_i}^{j-1} \bar{a}
\end{aligned}$$

$$+\sum_{i=1}^{N} b_i m!(j+m)(j+m+1)\overline{\lambda_i}^{j-1}\bar{a} + \frac{1}{2}\sum_{i=1}^{N} m! b_i (j+m+1)(j+m+2)\overline{\lambda_i}^{j}\bar{a}^2$$
$$= \sum_{i=1}^{N} b_i m! \overline{\lambda_i}^{j}\bar{a}^2 + \sum_{i=1}^{N} 2b_i m!(j+m)\overline{\lambda_i}^{j-1}\bar{a}.$$

So we have

$$\sum_{i=1}^{N} b_i m!\left(1 - \frac{(m+1)(m+2)}{2}\right)\overline{\lambda_i}^{j}\bar{a}^2 = \sum_{i=1}^{N} m b_i m!(j+m)\overline{\lambda_i}^{j-1}\overline{a}.$$

It is easy to check that

$$2 - (m+1)(m+2) \neq 0$$

for all $m > 0$. This gives

$$\sum_{i=1}^{N} b_i \overline{\lambda_i}^{j} = \frac{m(j+m)}{\bar{a}(1 - \frac{(m+1)(m+2)}{2})} \sum_{i=1}^{N} m b_i (j+m)\overline{\lambda_i}^{j-1} \tag{8}$$

Setting $j = 2$ in (8), it follows from (5) that

$$\sum_{i=1}^{N} b_i \overline{\lambda_i}^{2} = 0.$$

Using iteration method in (8), we can see that

$$\sum_{i=1}^{N} b_i \overline{\lambda_i}^{j} = 0$$

for all $j \geq 1$. Since $\{\lambda_i\}$ is a finite collection of distinct non-zero complex numbers, then

$$\left|\begin{pmatrix} \lambda_1 & \cdots & \lambda_N \\ \vdots & \vdots & \vdots \\ \lambda_1^N & \cdots & \lambda_N^N \end{pmatrix}\right| = \prod_{1\leq i\leq N} \lambda_i \prod_{1\leq \kappa < \iota \leq N} (\lambda_\iota - \lambda_\kappa) \neq 0.$$

So, $b_i = 0$ for all $1 \leq i \leq N$. This contradiction shows that f is a constant or $a = 0$. This proof is complete. □

The following Lemma has been considered in [15] by a direct calculation. Now, we prove it by the Berezin transform.

Proposition 5 *Let $m > 0$. Suppose $f(z) = e^{Az}$ and $g(z) = e^{Bz}$ with $A, B \in \mathbb{C}$. Then $T_{f\overline{g}} = T_f T_{\overline{g}}$ on $F^{2,m}$ if and only if $A\overline{B} = 0$.*

Proof If $A\overline{B} = 0$, then $T_{f\overline{g}} = T_f T_{\overline{g}}$. Conversely, assume $A\overline{B} \neq 0$, we will get a contradiction.
we define

$$Q(z, \overline{z}) = \frac{1}{m!\pi}\int_{\mathbb{C}} e^{Aw+\overline{B}\overline{w}} |K_m(z, w)|^2 |w|^{2m} e^{-|w|^2} dA(w).$$

Then

$$Q(z, \overline{z}) = \frac{1}{m!\pi}\int_{\mathbb{C}} e^{Aw+\overline{B}\overline{w}} |K_m(z, w)|^2 |w|^{2m} e^{-|w|^2} dA(w)$$
$$= \frac{m!}{\pi|z|^{2m}}\int_{\mathbb{C}} e^{Aw+\overline{B}\overline{w}} [e^{\overline{z}w} - q_m(\overline{z}w)][e^{\overline{w}z} - q_m(\overline{w}z)] e^{-|w|^2} dA(w)$$

$$= \frac{m!}{\pi|z|^{2m}}\int_{\mathbb{C}} e^{Aw+\overline{B}\overline{w}}\Big\{e^{\overline{z}w}e^{\overline{w}z} - q_m(\overline{w}z)e^{\overline{z}w}$$

$$- q_m(\overline{z}w)e^{\overline{w}z} + q_m(\overline{z}w)q_m(\overline{w}z)\Big\}e^{-|w|^2}dA(w).$$

Define

$$p(z,\overline{z}) = \frac{1}{\pi}\int_{\mathbb{C}} e^{Aw+\overline{B}\overline{w}}q_m(\overline{z}w)q_m(\overline{w}z)e^{-|w|^2}dA(w),$$

note that p is polynomial in z and $\overline{z}$. In fact, if $A=0$, then

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{C}} e^{\overline{B}\overline{w}}q_m(\overline{z}w)q_m(\overline{w}z)e^{-|w|^2}dA(w) &= \sum_{k=0}^{m-1}\sum_{l=0}^{k}\frac{1}{\pi}\int_{\mathbb{C}}\frac{(\overline{z}w)^k}{k!}\frac{\overline{B}^l z^{k-l}\overline{w}^k}{l!(k-l)!}e^{-|w|^2}dA(w)\\ &= \sum_{k=0}^{m-1}\sum_{l=0}^{k}\frac{\overline{z}^k\overline{B}^l z^{k-l}}{l!(k-l)!}\\ &= \sum_{k=0}^{m-1}\frac{\overline{z}^k}{k!}\sum_{l=0}^{k}\frac{k!\overline{B}^l z^{k-l}}{l!(k-l)!}\\ &= q_m[\overline{z}(\overline{B}+z)].\end{aligned}$$

By a simple computation,

$$\begin{cases}\frac{1}{\pi}\int_{\mathbb{C}} e^{Aw+\overline{B}\overline{w}}e^{\overline{z}w+\overline{w}z}e^{-|w|^2}dA(w) = e^{(A+\overline{z})(\overline{B}+z)};\\ \frac{1}{\pi}\int_{\mathbb{C}} e^{Aw+\overline{B}\overline{w}}q_m(\overline{w}z)e^{\overline{z}w}e^{-|w|^2}dA(w) = e^{\overline{B}(A+\overline{z})}q_m(zA+|z|^2);\\ \frac{1}{\pi}\int_{\mathbb{C}} e^{Aw+\overline{B}\overline{w}}q_m(\overline{z}w)e^{\overline{w}z}e^{-|w|^2}dA(w) = e^{A(\overline{B}+z)}q_m(\overline{z}\overline{B}+|z|^2).\end{cases}\tag{9}$$

Since $T_{f\overline{g}} = T_f T_{\overline{g}}$, then $\widetilde{f\overline{g}} = f\overline{g}$. It follows that

$$m!\frac{e^{|z|^2} - q_m(|z|^2)}{|z|^{2m}}e^{Az+\overline{B}\overline{z}} = Q(z,\overline{z}).\tag{10}$$

By (9) and (10),

$$[e^{|z|^2} - q_m(|z|^2)][e^{Az+\overline{B}\overline{z}}] = \Big\{e^{(A+\overline{z})(\overline{B}+z)} - e^{\overline{B}(A+\overline{z})}q_m(zA+|z|^2)$$

$$- e^{A(\overline{B}+z)}q_m(\overline{z}\overline{B}+|z|^2) + p(z,\overline{z})\Big\}.\tag{11}$$

This shows that

$$e^{|z|^2}e^{Az+\overline{B}\overline{z}} = e^{(A+\overline{z})(\overline{B}+z)}.$$

So we have $e^{A\overline{B}} = 1$. It follows from (11) that

$$-q_m(|z|^2)e^{Az+\overline{B}\overline{z}} = \Big\{-e^{\overline{B}\overline{z}}q_m(zA+|z|^2) - e^{Az}q_m(\overline{z}\overline{B}+|z|^2) + p(z,\overline{z})\Big\}.$$

Now, we set $A\overline{B}\neq 0$, then $\{e^{Az+\overline{B}\overline{z}}, e^{Az}, e^{\overline{B}\overline{z}}, 1\}$ is linearly independent over the polynomials. This implies that $q_m(|z|^2) = 0$, which contradicts the fact that

$$q_m(|z|^2) > 0$$

for $m > 0$. We thus have $A\overline{B} = 0$. This complete the proof. □

Remark 6 Let D_1 be the set of all finite linear combinations of kernel function of the Fock space. In fact, we can rewrite $|K_m(z,w)|^2|w|^{2m}$ by

$$|K_m(z,w)|^2|w|^{2m} = \frac{(m!)^2}{|z|^{2m}}|e^{z\overline{w}} - q_m(z\overline{w})|^2.$$

Then we can almost get the form of $\widetilde{f\overline{g}}$ by the property of the reproducing kernel of the Fock space if $f, g \in D_1$. So the above Proposition also holds if $f, g \in D_1$.

Recall that

$$\mathfrak{D} = D \cup \left\{f(z) = ce^{az} + b : a, b, c \in \mathbb{C}\right\} \cup \{f(z) = az + b : a, b \in \mathbb{C}\}.$$

Next, we will consider the semi-commuting Toeplitz operator with the $\mathfrak{D}-$symbol on $F^{2,m}$.

Theorem 7 *Suppose that m is a positive integer and $f, g \in \mathfrak{D}$. Then the following statements are equivalent.*

(a) $T_f T_{\overline{g}} = T_{f\overline{g}}$.
(b) $\widetilde{f\overline{g}} = f\overline{g}$.
(c) *At least one of f and g is a constant.*

Proof Suppose f and g are functions in $\mathfrak{D}$. It is easy to check that $T_{z\bar{z}} \neq T_z T_{\bar{z}}$. Using Theorem 2, Proposition 4 and 5, we have

$$T_{f\overline{g}} - T_f T_{\overline{g}} = 0, \quad f, g \in \mathfrak{D}$$

if and only if one of f and g is a constant function if and only if $\widetilde{f\overline{g}} = f\overline{g}$. □

Let $\mathbb{N}$ denote the set of integer. We say that $a \in 2\mathbb{N}\pi\mathrm{i}$ means that $a = 2n\pi\mathrm{i}$ for any integer n. By the proof of the main result, we can see that $e^{a\overline{b}} = 1$, if $(T_{e^{az}}, T_{\overline{e^{\overline{b}z}}}] = 0$ on Fock space. So, we can provide a simpler proof for the main result in [2](Theorem 1 in this paper).

Proof of Theorem 1 In fact, we only need to consider the "Step 1"(the proof of the main result in [15]). Now, we assume $f, g \in \epsilon(\mathbb{C})$ and $(T_f, T_{\overline{g}}] = 0$. Then, by Theorem 3.5 in [2], one can see that

$$f = \sum_{j=1}^{N} c_j K_{a_j}, \quad g = \sum_{l=1}^{M} d_l K_{b_l},$$

where $\{a_j\}_{j=1}^N$ and $\{b_l\}_{l=1}^M$ are finite collections of distinct complex numbers, K_z is the kernel of F^2. By the "Step 1" and the proof of (c) in Proposition 4(see (4)), we have $e^{a_j\overline{b_l}} = 1$ for each j and l. It follows that $a_j\overline{b_l} \in 2\mathbb{N}\pi\mathrm{i}$ for each j and l.

Conversely, if f and g satisfy the conditions in (2), then $f\overline{g} = \widetilde{f\overline{g}}$ by a straightforward calculus argument. Thus, (a), (b) and (c) are equivalent. □

3 Open problems and Remarks

In this work, we show that $(T_{K_m(z,b)}, T_{\overline{e^{az}}}] \neq 0$ if $ab \neq 0$. The main result gives the fundamental difference between the geometries of Fock and Fock-Sobolev space. In fact, the semi-commutativity of Toeplitz operators with general symbols is still an open problem on the Fock and Fock-Sobolev space, so it is difficult to complete characterize the bounded semi-commutant on Fock spaces and Fock-Sobolev spaces. Setting $m \neq 0$, we define the symbol space $\mathcal{A}$:

$$\mathcal{A} = \{f(z) : \sum_{i=1}^{N} p_i(z) K_m(z, A_i), \quad p_i \in \mathcal{P}\}.$$

Since $e^{\overline{a}z} = (\overline{a}z)^m K_m(z,a) + q_m(\overline{a}z)$, one can see that

$$\mathfrak{D} \subset \mathcal{A}, \quad \mathcal{A}_1 \subseteq \mathcal{A}, \quad \mathcal{A} \nsubseteq \mathcal{A}_1.$$

There is a a natural problem, which arise from the main results of [2,12,15].

Problem Let $m \neq 0$, and suppose $f, g \in \mathcal{A}$. What is the relationship between their symbols when two Toeplitz operators T_f and $T_{\overline{g}}$ commute on $F^{2,m}$?

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