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ORIGINAL RESEARCH

A new generalization of the Stirling numbers of the first kind

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Abstract In this paper, we study a generalization of the Stirling numbers of the first kind (classical and analogues versions) related to an extension of elementary symmetric function. These numbers appear as the coefficients of x^k in the expression $\prod_{j=0}^{n-1}(x^s + jx^{s-1} + \cdots + j^{s-1}x + j^s)$. The array formed by these coefficients can be seen as a natural generalization of the first Stirling triangle. We also give a combinatorial interpretation of the classical numbers in terms of s-tuples of permutations of $[n]$ with k cycles and using the inversion and co-inversion statistics on the cycles in the case of the analogues numbers, which also allows us to establish as a particular case the analogues of the Stirling numbers of the first kind. In addition, using the Legendre–Stirling numbers and the Stirling numbers of the first kind new formulas and useful properties are proposed.

Keywords Stirling numbers of the first kind · q-Stirling numbers of the first kind · Legendre–Stirling numbers of the first kind · Symmetric functions

Mathematics Subject Classification 05A05 · 05A15 · 05A30 · 11B73

1 Introduction

The Stirling numbers of first kind, denoted by $c(n, k)$ or $\left[{n \atop k}\right]$ (see, Knuth [10]), enumerates the number of permutations of $[n] := \{1, \ldots, n\}$ with exactly k cycles.

They satisfy the recurrence relation

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k) \tag{1}$$

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with initial conditions

$$c(n,0)=\delta_{n,0},\quad c(0,k)=\delta_{0,k}.$$

They are also considered as the coefficients of the following product

$$x(x+1)\cdots(x+n-1)=\sum_{k=0}^{n}c(n,k)x^k. \tag{2}$$

If $n\in\mathbf{N}$ then its standard q-analogue is

$$[n]_q=1+q+\cdots+q^{n-1}$$

and its standard p,q-analogue is

$$[n]_{p,q}=p^{n-1}+p^{n-2}q+\cdots+q^{n-1},$$

where $[0]_q=[0]_{p,q}=0$.

So we can define the q-Stirling numbers (resp. p,q-Stirling numbers) of the first kind $c_q[n,k]$ (resp. $c_{p,q}[n,k]$) using a q-analogue (resp. p,q-analogue) of equation (1)

$$c_q[n,k]=c_q[n-1,k-1]+[n-1]_q c_q[n-1,k], \tag{3}$$

$$c_{p,q}[n,k]=c_{p,q}[n-1,k-1]+[n-1]_{p,q}c_{p,q}[n-1,k], \tag{4}$$

where $c_q[0,k]=c_{p,q}[0,k]=\delta_{0,k}$.

The q-Stirling numbers of the first kind were introduced in Gould's paper [8] and given an interpretation in terms of inversions by Gessel [7]. The first proposition contains the most elementary results generalizing properties of p,q-Stirling numbers (see de Médicis and Leroux [15]). Many generalizations of Stirling numbers can be found in the literature. For ewample, we refer the reader to [14].

Egge [5] gave a recursive definition for the Legendre–Stirling numbers of the first kind, denoted by $\left[\left[{n \atop k}\right]\right]$

$$\left[\left[\begin{matrix}n\\k\end{matrix}\right]\right]=\left[\left[\begin{matrix}n-1\\k-1\end{matrix}\right]\right]+n(n-1)\left[\left[\begin{matrix}n-1\\k\end{matrix}\right]\right], \tag{5}$$

where $\left[\left[{n \atop 0}\right]\right]=\delta_{n,0}$, $\left[\left[{0 \atop k}\right]\right]=\delta_{0,k}$, and a combinatorial interpretation in terms of pairs of permutations of $[n]$ with k cycles, in which $\left[\left[{n \atop k}\right]\right]$ is firstly introduced by Andrews and Littlejohn [1] via

$$<x>_n=\sum_{k=0}^{n}(-1)^{n+k}\left[\left[\begin{matrix}n\\k\end{matrix}\right]\right]x^k, \tag{6}$$

where $<x>_n=x(x-2)(x-6)\cdots(x-n(n-1))$.

The bi^{s}nomial coefficient or ordinary multinomial number is defined as the k-th coefficient in the development

$$(1+x+x^2+\cdots+x^s)^n=\sum_{k\geq 0}\binom{n}{k}_s x^k, \tag{7}$$

which satisfies the generalized Pascal relation

$$\binom{n}{k}_s=\sum_{j=0}^{s}\binom{n-1}{k-j}_s \tag{8}$$

A natural extension of the bi^{s}nomial coefficient is the q-bi^{s}nomial coefficient, which satisfies the following recursion

$$\left[\begin{matrix}n\\k\end{matrix}\right]_q^{(s)}=\sum_{j=0}^{s}q^{(n-1)j}\left[\begin{matrix}n-1\\k-j\end{matrix}\right]_q^{(s)} \tag{9}$$

Table 1 The first Legendre–Stirling triangle $\left[\left[{n \atop k}\right]\right]$.

n/k	0	1	2	3	4	5
0	1					
1	0	1				
2	0	2	1			
3	0	12	8	1		
4	0	144	108	20	1	
5	0	2880	2304	508	40	1

and which appear as the coefficients of the following product

$$\prod_{j=0}^{n-1}\left(1+q^jz+\cdots+(q^jz)^s\right)=\sum_{k\geq 0}\begin{bmatrix}n\\k\end{bmatrix}_q^{(s)}z^k. \tag{10}$$

For more details, we refer the reader to [2,3].

Let $X=\{x_1,x_2,\ldots\}$ be a countably infinite set of variables. The elementary symmetric function of degree k in $x_1,x_2,\ldots,x_n$ is defined by

$$e_k(n):=e_k(x_1,x_2,\ldots,x_n)=\sum_{1\leq i_1<i_2<\cdots<i_k\leq n}x_{i_1}x_{i_2}\cdots x_{i_k},$$

where $e_0(n)=1$, $e_k(0)=\delta_{0,k}$ and $e_k(n)=0$ unless $0\leq k\leq n$. Then for $n\geq 1$ and $k\in Z$,

$$e_k(n)=e_k(n-1)+x_ne_{k-1}(n-1). \tag{11}$$

For more details, see for instance [12,16].

The Stirling numbers of the first kind with their analogues can be expressed as specializations of elementary symmetric function

$$\begin{aligned}&c(n,k)=e_{n-k}(1,2,\ldots,n-1),\\&c_q[n,k]=e_{n-k}([1]_q,[2]_q,\ldots,[n-1]_q),\\&c_{p,q}[n,k]=e_{n-k}([1]_{p,q},[2]_{p,q},\ldots,[n-1]_{p,q}).\end{aligned}$$

In [2], Bazeniar et *al.* proposed a new symmetric function to interpret the bi^{s}nomial coefficients and their analogues. This function is a generalization of elementary symmetric function given as follows

$$E_k^{(s)}(n):=E_k^{(s)}(x_1,....,x_n)=\sum_{\substack{\alpha_1+\alpha_2+\cdots+\alpha_n=k\\0\leq\alpha_1,\alpha_2,\ldots,\alpha_n\leq s}}x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}, \tag{12}$$

where $E_0^{(s)}(n)=1$, $E_k^{(s)}(n)=0$ unless $0\leq k\leq sn$, and satisfy the following recurrence relation

$$E_k^{(s)}(n)=\sum_{j=0}^{s}x_n^jE_{k-j}^{(s)}(n-1). \tag{13}$$

If $s\geq k$ then $E_k^{(s)}(x_1,....,x_n)$ is the complete homogeneous symmetric function, so $E_k^{(s)}$ interpolates between the elementary and the complete homogeneous symmetric functions.

To illustrate,

$$E_4^{(2)}(3)=x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2+x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2.$$

Using this function and relation (13) a simple induction proves.

1. $E_k^{(s)}(1,1,\ldots,1)=\binom{n}{k}_s$,

Table 2 The first 2-Stirling triangle $c^{(2)}(n,k)$.

n/k	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	0	1								
2	0	0	1	1	1						
3	0	0	4	6	7	3	1				
4	0	0	36	66	85	54	25	6	1		
5	0	0	576	1200	1660	1270	701	250	65	10	1

2. $E_k^{(s)}(1,q,\ldots,q^{n-1}) = \begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)}$.

The generalized symmetric functions $E_k^{(s)}$ are not essentially a new generalization of the elementary symmetric functions e_k. An equivalent definition of these symmetric functions already exists in the paper of Doly and Walker [4] with another name called *modular complete symmetric function*. Fu and Mei [6] and Grinberg [9] independently introduced the generalized symmetric functions $E_k^{(s)}$. Grinberg called these functions the *Petrie symmetric functions* while Fu and Mei referred to them as *truncated homogeneous symmetric functions*.

In the present paper, we give an extension of the Stirling numbers of the first kind and their analogues. In Sect. 2 we give a recursive definition and the ordinary generating function of the generalized Stirling numbers of the first kind, which we interpret as specialization of the symmetric function given by the relation (12) and denote by $c^{(s)}(n,k)$. Then we establish that these numbers are linked to the classical Stirling numbers of the first kind. In Sect. 3 we give a combinatorial interpretation of $c^{(s)}(n,k)$ as the number of s-tuple permutations of $[n]$ having together k cycles. A connection between $c^{(2)}(n,k)$ and the Legendre–Stirling number of the first kind $\left[\begin{bmatrix} n \\ k \end{bmatrix}\right]$ and some identities will be established. In the last two sections we generalize some aforementioned results for the analogues of the generalized Stirling numbers of the first kind.

2 Generalized Stirling number of the first kind

In this section we propose a generalization of Stirling numbers of the first kind using the symmetric function defined by (12), which we give a recursive definition and an ordinary generating function.

Definition 1 For all $n \geq 0$ we write $c^{(s)}(n,k)$ to denote the generalized Stirling numbers of the first kind, which are given by the initial conditions

$$c^{(s)}(n,0) = \delta_{n,0}, \quad c^{(s)}(0,k) = \delta_{0,k}$$

and recurrence relation

$$c^{(s)}(n,k) = \sum_{j=0}^{s} (n-1)^j c^{(s)}(n-1, k-s+j), \tag{14}$$

where $c^{(s)}(n,k) = 0$ unless $s \leq k \leq sn$.

Comparing this definition with the generalized elementary symmetric function $E_k^{(s)}(n)$ and relation (13) a simple induction proves.

Proposition 1 *For $n \geq 1$ and $s \leq k \leq sn$,*

$$c^{(s)}(n,k) = E_{sn-k}^{(s)}(1,2,\ldots,n-1). \tag{15}$$

These numbers, as for usual Stirling numbers of the first kind, are built through the first Stirling triangle, known as "the first s-Stirling triangle".

As an illustration of recurrence relation (14), we establish the first 2-Stirling triangle.

According to the relation (14), we will show below that $c^{(s)}(n,k)$ is the coefficient of the k-th term of the product.

Theorem 1 *We have,*

$$\prod_{j=0}^{n-1}(x^s + jx^{s-1} + \cdots + j^{s-1}x + j^s) = \sum_{k=0}^{sn} c^{(s)}(n,k)x^k.$$

Proof Taking $n = 1$, we have

$$x^s = 0 + 0 \times x + \cdots + 0 \times x^{s-1} + 1 \times x^s = \sum_{k=0}^{s} c^{(s)}(1,k)x^k.$$

Suppose that this hypothesis is true for n. We show that it remains true for $n + 1$:

$$\begin{aligned}\prod_{j=0}^{n}(x^s + jx^{s-1} + \cdots + j^{s-1}x + j^s) &= \sum_{j=0}^{s} n^j x^{s-j} \sum_{k=0}^{sn} c^{(s)}(n,k)x^k \\ &= \sum_{k=0}^{s(n+1)} \left[\sum_{j=0}^{s} n^j c^{(s)}(n, k-s+j)\right] x^k \\ &= \sum_{k=0}^{s(n+1)} c^{(s)}(n+1,k)x^k.\end{aligned}$$

Hence, by induction hypothesis and relation (14), the proof is completed.

Now, we are able to give the connection between these numbers and the Stirling numbers of the first kind.

Theorem 2 *We have,*

$$c^{(s)}(n,k) = \sum_{j_1+\cdots+j_s=k} \left[\prod_{r=1}^{s} c(n, j_r)\right] a^{\sum_{r=1}^{s} r j_r},$$

where $a = e^{i\frac{2\pi}{s+1}}$.

Proof For $a = e^{i\frac{2\pi}{s+1}}$, we can see that

$$1 + x + \cdots + x^s = \prod_{r=1}^{s}(x - a^r).$$

So by Theorem 1, we have

$$\begin{aligned}\sum_{k=0}^{sn} c^{(s)}(n,k)x^k &= \prod_{j=0}^{n-1}(x^s + jx^{s-1} + \cdots + j^{s-1}x + j^s) \\ &= x^{ns}\prod_{j=0}^{n-1}\left(1 + \left(\frac{j}{x}\right) + \cdots + \left(\frac{j}{x}\right)^{s-1} + \left(\frac{j}{x}\right)^s\right) \\ &= x^{ns}\prod_{j=0}^{n-1}\prod_{r=1}^{s}\left(\frac{j}{x} - a^r\right) \\ &= \prod_{r=1}^{s}\prod_{j=0}^{n-1}(j - a^r x).\end{aligned}$$

It follows from relation (2) that

$$
\begin{aligned}
\sum_{k=0}^{sn} c^{(s)}(n,k)x^k &= \prod_{r=1}^{s}\sum_{k=0}^{n} c(n,k)\left(-a^r x\right)^k \\
&= \sum_{j_1=0}^{n} c(n,j_1)(-1)^{j_1}a^{j_1}x^{j_1} \times \cdots \times \sum_{j_s=0}^{n} c(n,j_s)(-1)^{j_s}a^{sj_s}x^{j_s} \\
&= \sum_{k=0}^{sn}\left[\sum_{j_1+\cdots+j_s=k} c(n,j_1)\times\cdots\times c(n,j_s)(-1)^k a^{\sum_{r=1}^{s} rj_r}\right]x^k
\end{aligned}
$$

By identification we get

$$
c^{(s)}(n,k) = \sum_{j_1+\cdots+j_s=k}\left[\prod_{r=1}^{s} c(n,j_r)\right]a^{\sum_{r=1}^{s} rj_r}.
$$

3 Combinatorial interpretations

The Stirling numbers of the first kind count the permutations of $[n]$ with k cycles. With this in mind, it is natural to ask for a similar combinatorial interpretation of the generalized Stirling number of the first kind. Indeed, Egge [5] has given a combinatorial interpretation of the Legendre–Stirling number in terms of pairs of permutations; one might call these Legendre–Stirling permutation pairs which are an ordered pairs (π_1, π_2) with $\pi_1 \in S_{n+1}$ and $\pi_2 \in S_n$, such that π_1 has one more cycle than π_2 and the cycle maxima of π_1 which are less than $n+1$ are exactly the cycle maxima of π_2. In this section we generalize Legendre–Stirling permutation pairs still further, to obtain objects we will call generalized Stirling permutation s-tuples. In connection with these s-tuples, it will be useful to note that the cycle maxima of a given permutation are the numbers which are largest in their cycles. For example, if $\pi = (1,3,5)(2,6,8)(7)(4,9)(10)$ is a permutation in S_{10}, written in cycle notation, then its cycle maxima are 5, 7, 8, 9 and 10.

Definition 2 The generalized Stirling permutation s-tuple of length n is an ordered s-tuple $(\pi_1, \ldots, \pi_s)$ with $\pi_1, \ldots, \pi_s \in S_n$ for which the following hold:

1. For each j, $1 \le j \le s-1$, the number of cycles in π_j is greater than or equal to the number of cycles in π_{j+1}.
2. For each j, $1 \le j \le s-1$, the cycle maxima of π_{j+1} is included in the cycle maxima of π_j.

Example 1 For $n = 3$ and $s = 2$, the possible generalized Stirling permutation pairs (π_1, π_2) of length 3 are

Number of cycles	Cycles
2	(123, 123); (123, 132); (132, 123); (132, 132)
3	((1)(23), 123); ((1)(23), 132); ((13)(2), 123); ((13)(2), 132); ((12)(3), 123); ((12)(3), 132)
4	((1)(2)(3), 123); ((1)(2)(3), 132); ((12)(3), (12)(3)); ((13)(2), (13)(2)); ((1)(23), (1)(23)); ((13)(2), (12)(3)); ((12)(3), (13)(2))
5	((1)(2)(3), (12)(3)); ((1)(2)(3), (13)(2)); ((1)(2)(3), (1)(23))
6	((1)(2)(3), (1)(2)(3))

Theorem 3 *For $n \ge 1$ and all k with $s \le k \le sn$, the number of generalized Stirling permutation s-tuples $(\pi_1, \ldots, \pi_s)$ of length n having together exactly k cycles is $c^{(s)}(n,k)$.*

Proof Let $p^{(s)}(n,k)$ denote the number of generalized Stirling permutation s-tuples $(\pi_1, \ldots, \pi_s)$ of length n having together exactly k cycles. It is clear that $p^{(s)}(n,0) = \delta_{n,0}$ and $p^{(s)}(0,k) = \delta_{0,k}$, so in view of (14) it is sufficient to show that if $n > 0$ and $k > 0$ then $p^{(s)}(n,k) = p^{(s)}(n-1,k-s) + \sum_{j=1}^{s}(n-1)^j p^{(s)}(n-1,k-s+j)$.

To do this, first note that by condition 2 of Definition 2, if $(\pi_1, \ldots, \pi_s)$ is a generalized Stirling permutation s-tuple of length n, then 1 is a fixed point in π_s if and only if it is a fixed point in each $\pi_i \in \{\pi_1, \ldots, \pi_{s-1}\}$. s-tuples $(\pi_1, \ldots, \pi_s)$ in which 1 is a fixed point in each $\pi_i \in \{\pi_1, \ldots, \pi_s\}$ are in bijection with s-tuples $(\sigma_1, \ldots, \sigma_s)$ of length $n-1$ having together exactly $k-s$ cycles by removing 1 from each permutation and decreasing all other entries by 1. Each s-tuple $(\pi_1, \ldots, \pi_s)$ in which 1 is a fixed point only in $\pi_i \in \left\{\pi_1, \ldots, \pi_{s-j}\right\}$ may be constructed uniquely by choosing s-tuple $(\sigma_1, \ldots, \sigma_s)$ of length $n-1$ having together $k-(s-j)$ cycles, increasing each entry of each permutation by 1 and inserting 1 as a fixed point in each permutation $\sigma_i \in \left\{\sigma_1, \ldots, \sigma_{s-j}\right\}$ and as a new entry in each permutation $\sigma_i \in \left\{\sigma_{s-j+1}, \ldots, \sigma_s\right\}$. There are $p^{(s)}(n-1, k-s+j)$ s-tuples $(\sigma_1, \ldots, \sigma_s)$, there is one way to insert 1 as a fixed point in each permutation $\sigma_i \in \left\{\sigma_1, \ldots, \sigma_{s-j}\right\}$, and there are $(n-1)^j$ ways to insert 1 as a new entry in each permutation $\sigma_i \in \left\{\sigma_{s-j+1}, \ldots, \sigma_s\right\}$.

According to the previous theorem, the generalized Stirling numbers, $c^{(s)}(n, k)$, satisfy the following identities.

Proposition 2 *We have,*

(i) $c^{(s)}(n, s) = [(n-1)!]^s = [c(n, 1)]^s$;
(ii) $c^{(s)}(n, s+1) = [(n-1)!]^{s-1} c(n, 2) = [(n-1)!]^s H_{n-1}$ *where H_{n-1} is the $(n-1)$-th Harmonic number;*
(iii) $c^{(s)}(n, sn) = 1$;
(iv) $c^{(s)}(n, sn-1) = c(n, n-1) = \binom{n}{2}$;
(v) *For $n \geq 2$, $\sum_{k=0}^{sn}(-1)^{sn-k}c^{(s)}(n, k) = 0$ if s odd;*
(vi) $\left[\begin{bmatrix} n \\ 1 \end{bmatrix}\right] = n \times c^{(2)}(n, 2)$.

Proof (i) For each permutation $\pi_i \in (\pi_1, \ldots, \pi_s)$, there are $(n-1)!$ ways to permute π_i in one cycle. Thus, there are

$$c^{(s)}(n, s) = [(n-1)!]^s = [c(n, 1)]^s$$

ways to permute $(\pi_1, \ldots, \pi_s)$ together in s cycles.

(ii) It is easy to see that π_1 has exactly two cycles and each $\pi_i \in \{\pi_2, \ldots, \pi_s\}$ has one cycle because $\pi_1, \pi_2, \ldots$ and π_s have together exactly $s+1$ cycles. Hence, there are $c(n, 2)$ ways to permute π_1 in two cycles and $(n-1)!$ ways to permute each $\pi_i \in \{\pi_2, \ldots, \pi_s\}$ in one cycle. Thus,

$$c^{(s)}(n, s+1) = [(n-1)!]^{s-1} c(n, 2),$$

since $c(n, 2) = (n-1)! H_{n-1}$ we get

$$c^{(s)}(n, s+1) = [(n-1)!]^s H_{n-1}.$$

(iii) The proof that $c^{(s)}(n, sn) = 1$ is trivial since each $\pi_i \in \{\pi_1, \ldots, \pi_s\}$ has exactly n cycles.

(iv) Each $\pi_i \in \{\pi_1, \ldots, \pi_{s-1}\}$ has n cycles and π_s has $n-1$ cycles because $\pi_1, \pi_2, \ldots$ and π_s have together exactly $sn-1$ cycles. Hence, by (iii) there is one way to permute all $\pi_i \in \{\pi_1, \ldots, \pi_{s-1}\}$ together in $(s-1)n$ cycles and $c(n, n-1)$ ways to permute π_s in $n-1$ cycles. Thus,

$$c^{(s)}(n, sn-1) = c(n, n-1) = \binom{n}{2}.$$

(v) From Theorem 1, it is not difficult to have

$$\sum_{k=0}^{sn}(-1)^{sn-k}c^{(s)}(n, k)x^k = x^s \prod_{j=1}^{n-1}(x^s - jx^{s-1} + \cdots + (-1)^s j^s).$$

Hence, for $n \geq 2$, by setting $x = 1$ and s odd in the above equation we obtain

$$\begin{aligned}\sum_{k=0}^{sn}(-1)^{sn-k}c^{(s)}(n, k) &= 1 \times \prod_{j=1}^{n-1}(1 - j + \cdots + (-1)^s j^s) \\ &= 0 \times \prod_{j=2}^{n-1}(1 - j + \cdots + (-1)^s j^s) = 0.\end{aligned}$$

(vi) Replacing $s = 2$ in (i) we obtain

$$c^{(2)}(n, 2) = [(n-1)!]^2 .$$

On the other hand, from relations (5) and (6) we can see that

$$\left[\begin{bmatrix} n \\ 1 \end{bmatrix}\right] = n!(n-1)! = n\,[(n-1)!]^2 .$$

Thus,

$$\left[\begin{bmatrix} n \\ 1 \end{bmatrix}\right] = n \times c^{(2)}(n, 2).$$

The relation between the generalized Stirling numbers for $s = 2$ and the Legendre–Stirling numbers are given by the following proposition.

Proposition 3 *The number of Legendre–Stirling permutation pairs (σ_1, σ_2) of length n is exactly the number of generalized Stirling permutation pairs (π_1, π_2) of length n, i.e:*

$$\sum_{k=0}^{2n} c^{(2)}(n, k) = \sum_{k=0}^{n} \left[\begin{bmatrix} n \\ k \end{bmatrix}\right].$$

Proof From relation (6), we have

$$\sum_{k=0}^{n} \left[\begin{bmatrix} n \\ k \end{bmatrix}\right] x^k = \prod_{j=0}^{n-1} (x + j(j+1)) \tag{16}$$

and taking $s = 2$ in Theorem 1, we get

$$\sum_{k=0}^{2n} c^{(2)}(n, k) x^k = \prod_{j=0}^{n-1} (x^2 + jx + j^2) = \prod_{j=0}^{n-1} (x^2 + j(j+x)). \tag{17}$$

Hence, by setting $x = 1$ in relations (16) and (17) we obtain

$$\sum_{k=0}^{2n} c^{(2)}(n, k) = \sum_{k=0}^{n} \left[\begin{bmatrix} n \\ k \end{bmatrix}\right].$$

4 Analogues of the generalized Stirling numbers

Now, we are able to propose a definition of the analogues of the generalized Stirling numbers of the first kind.

Definition 3 For all $n \geq 0$ we write $c_q^{(s)}[n, k]$ (resp. $c_{p,q}^{(s)}[n, k]$) to denote the generalized q-Stirling numbers (resp. p, q-Stirling numbers) of the first kind, which are given by the initial conditions

$$c_q^{(s)}[n, 0] = c_{p,q}^{(s)}[n, 0] = \delta_{n,0}, \quad c_q^{(s)}[0, k] = c_{p,q}^{(s)}[0, k] = \delta_{0,k}$$

and recurrence relations

$$c_q^{(s)}[n, k] = \sum_{j=0}^{s} [n-1]_q^j c_q^{(s)}[n-1, k-s+j], \tag{18}$$

$$c_{p,q}^{(s)}[n, k] = \sum_{j=0}^{s} [n-1]_{p,q}^j c_{p,q}^{(s)}[n-1, k-s+j], \tag{19}$$

where $c_q^{(s)}[n, k] = c_{p,q}^{(s)}[n, k] = 0$ unless $s \leq k \leq sn$.

Table 3 The q-analogue of the first 2-Stirling triangle.

n/k	0	1	2	3	4	5	6
0	1						
1	0	0	1				
2	0	0	1	1	1		
3	0	0	$1+2q+q^2$	$2+3q+q^2$	$3+3q+q^2$	$2+q$	1

Table 4 The p, q-analogue of the first 2-Stirling triangle.

n/k	0	1	2	3	4
0	1				
1	0	0	1		
2	0	0	1	1	1
3	0	0	$p^2+2pq+q^2$	$p^2+q^2+2pq+p+q$	$p^2+q^2+2pq+p+q+1$

These numbers, as for usual the generalized Stirling numbers of the first kind, are built through the first s-Stirling triangle, known as "q-analogue (resp. p, q-analogue) of the first s-Stirling triangle".

As an illustration of recurrence relation (18), we establish the q-analogue of the first 2-Stirling triangle.

Similarly for recurrence relation (19), we establish the p, q-analogue of the first 2-Stirling triangle.

Comparing Definition 3 with the generalized elementary symmetric function $E_k^{(s)}(n)$ and relation (18) (resp. (19)) a simple induction proves.

Proposition 4 *For* $n \geq 1$ *and* $s \leq k \leq sn$,

$$c_q^{(s)}[n, k] = E_{sn-k}^{(s)}\left([1]_q, [2]_q, \ldots, [n-1]_q\right), \tag{20}$$

$$c_{p,q}^{(s)}[n, k] = E_{sn-k}^{(s)}\left([1]_{p,q}, [2]_{p,q}, \ldots, [n-1]_{p,q}\right). \tag{21}$$

Remark 1 The generalized q-Stirling (resp. p, q-Stirling) number of the first kind $c_q^{(s)}[n, k]$ (resp. $c_{p,q}^{(s)}[n, k]$) is defined to be the sum of the $\binom{n-1}{sn-k}_s$ possible products, each factor appears at most s times in different product, which may be formed from the first $(n-1)$ q-natural numbers $[1]_q, [2]_q, \ldots, [n-1]_q$ (resp. p, q-natural numbers $[1]_{p,q}, [2]_{p,q}, \ldots, [n-1]_{p,q}$).

To illustrate,

$$\begin{aligned}&c_q^{(2)}[3, 2] = [1]_q^2[2]_q^2; \quad c_q^{(2)}[3, 3] = [1]_q^2[2]_q + [1]_q[2]_q^2;\\ &c_q^{(2)}[3, 4] = [1]_q^2 + [2]_q^2 + [1]_q[2]_q; \quad c_q^{(2)}[3, 5] = [1]_q + [2]_q; \quad c_q^{(2)}[3, 6] = 1.\end{aligned}$$

Moreover the following results appear as natural analogues of Theorem 1.

Theorem 4 *We have,*

$$\prod_{j=0}^{n-1}(x^s + [j]_q x^{s-1} + \cdots + [j]_q^{s-1} x + [j]_q^s) = \sum_{k=0}^{sn} c_q^{(s)}[n, k] x^k,$$

$$\prod_{j=0}^{n-1}(x^s + [j]_{p,q} x^{s-1} + \cdots + [j]_{p,q}^{s-1} x + [j]_{p,q}^s) = \sum_{k=0}^{sn} c_{p,q}^{(s)}[n, k] x^k.$$

Proof According to the relations (20) and (21) given in Proposition 4, we use the same method of the proof proposed in Theorem 1.

The method of the proof used in Theorem 2 can be carried over verbatim to their analogues. Here we omit the details for brevity.

Theorem 5 *We have,*

$$c_q^{(s)}[n,k] = \sum_{j_1+\cdots+j_s=k} \left[\prod_{r=1}^{s} c_q[n, j_r]\right] a^{\sum_{r=1}^{s} r j_r}$$

and

$$c_{p,q}^{(s)}[n,k] = \sum_{j_1+\cdots+j_s=k} \left[\prod_{r=1}^{s} c_{p,q}[n, j_r]\right] a^{\sum_{r=1}^{s} r j_r},$$

where $a = e^{i\frac{2\pi}{s+1}}$.

5 Combinatorial interpretation of the analogues of the generalized Stirling number

The q-Stirling numbers of the first kind $c_q(n,k)$ are introduced by Gould [8], and interpreted as generating function of the inversion statistic on cycles of permutations by Gessel [7]. Using the same statistic, we give in this section the combinatorial interpretation and some identities for the analogues of the generalized Stirling numbers of the first kind.

We know that the cycle minima of a given permutation are the numbers which are smallest in their cycles. For example, if $\pi = (1,3,5)(2,6,8)(7)(4,9)$ is a permutation in S_9, written in cycle notation, then its cycle minima are 1, 2, 4, and 7.

For $\pi = p_1 \dots p_n \in S_n$ the number of inversions is $inv(\pi) = \{(i,j) : i < j \text{ and } p_i > p_j\}$, and the number of co-inversions is $coinv(\pi) = \{(i,j) : i < j \text{ and } p_i < p_j\}$ where S_n is the group of permutations on n elements. The inversion statistic goes back to work of Cramer (1750), B Lezout (1764) and Laplace (1772). See the discussion in [11, page 92]. Netto enumerated the elements of the symmetric group by the inversion statistic in 1901 [11, Chapter 4, Sections 54 and 57], and in 1916 MacMahon [13, page 318] gave the q-factorial expansion $\sum_{\pi\in S_n} q^{inv(\pi)} = [n]_q!$.

Now, firstly we arrange the cycles of each permutation π_i, of the generalized Stirling permutation s-tuple $(\pi_1,\dots,\pi_s)$ of length n given in Definition 2, in increasing order from left to right according to cycle minima. Secondly, after removing the parentheses of the cycles we consider all cycles of each $\pi_i \in (\pi_1,\dots,\pi_s)$ only one cycle and we denote these new permutations by $\pi_i^{(\leq,c)}$.

Example 2 For the permutation $\pi = (1,3,5)(2,6,8)(7)(4,9)$ we have $\pi^{(\leq,c)} = (135268497)$.

Definition 4 Let $\mathcal{Q}_s(n,m)$ be the set of all permutation s-tuples of type $\left(\pi_1^{(\leq,c)},\dots,\pi_s^{(\leq,c)}\right)$ associated with all generalized Stirling permutation s-tuples $(\pi_1,\dots,\pi_s)$ of length n having together exactly m cycles.

Theorem 6 *For $n \geq 1$ and all k with $s \leq k \leq sn$, the generalized q-Stirling number of the first kind $c_q^{(s)}[n,k]$ is given by*

$$c_q^{(s)}[n,k] = \sum_{\left(\pi_1^{(\leq,c)},\dots,\pi_s^{(\leq,c)}\right)\in\mathcal{Q}_s(n,k)} q^{\sum_{i=1}^{s} inv\left(\pi_i^{(\leq,c)}\right)}. \tag{22}$$

Proof In view of (18) it is sufficient to show that if $n > 0$ and $k \geq s$ then

$$\sum_{\left(\pi_1^{(\leq,c)},\dots,\pi_s^{(\leq,c)}\right)\in\mathcal{Q}(n,k)} q^{\sum_{i=1}^{s} inv\left(\pi_i^{(\leq,c)}\right)} = c_q^{(s)}[n-1,k-s] + \sum_{j=1}^{s}[n-1]_q^j c_q^{(s)}[n-1,k-s+j].$$

To do this, first note that by condition 2 of Definition 2, if $(\pi_1,\dots,\pi_s)$ is a generalized Stirling permutation s-tuple of length n then n is a fixed point in π_s if and only if it is a fixed point in each $\pi_i \in \{\pi_1,\dots,\pi_{s-1}\}$. s-tuples $(\pi_1,\dots,\pi_s)$ in which n is a fixed point in each $\pi_i \in \{\pi_1,\dots,\pi_s\}$ are in bijection with s-tuples $(\sigma_1,\dots,\sigma_s)$ of length $n-1$ having together $k-s$ cycles by removing n from each permutation. Hence, by Definition 4 the s-tuples permutation $\left(\pi_1^{(\leq,c)},\dots,\pi_s^{(\leq,c)}\right)$ of length n are also in bijection with $\left(\sigma_1^{(\leq,c)},\dots,\sigma_s^{(\leq,c)}\right)$ of length $n-1$ and so that will contribute by $c_q^{(s)}[n-1,k-s]$. Each s-tuple $(\pi_1,\ldots,\pi_s)$ in which n is a fixed point only in $\pi_i \in \{\pi_1,\ldots,\pi_{s-j}\}$ may be constructed uniquely by choosing s-tuple $(\sigma_1,\ldots,\sigma_s)$ of length $n-1$ having together $k-(s-j)$ cycles, inserting n as a fixed point in each permutation $\sigma_i \in \{\sigma_1,\ldots,\sigma_{s-j}\}$ and as a new entry in each permutation $\sigma_i \in \{\sigma_{s-j+1},\ldots,\sigma_s\}$. There are $c^{(s)}(n-1,k-s+j)$ s-tuples $(\sigma_1,\ldots,\sigma_s)$ and therefore $\left(\sigma_1^{(\leq,c)},\ldots,\sigma_s^{(\leq,c)}\right)$ will contribute by $c_q^{(s)}[n-1,k-s+j]$, there is one way to insert n as fixed point in each permutation $\sigma_i \in \{\sigma_1,\ldots,\sigma_{s-j}\}$ that implies the total weight contribution of n in $\left(\sigma_1^{(\leq,c)},\ldots,\sigma_{s-j}^{(\leq,c)}\right)$ is $q^{\overbrace{0+\cdots+0}^{s-j \text{ times}}}=1$, and there are $(n-1)^j$ ways to insert n as a new entry in all permutations $\sigma_i \in \{\sigma_{s-j+1},\ldots,\sigma_s\}$ that implies the weight contribution of n in each permutation $\sigma_i^{(\leq,c)} \in \left\{\sigma_{s-j+1}^{(\leq,c)},\ldots,\sigma_s^{(\leq,c)}\right\}$ is $1+q+\cdots+q^{n-2}=[n-1]_q$. The total weight for all permutations $\left(\sigma_{s-j+1}^{(\leq,c)},\ldots,\sigma_s^{(\leq,c)}\right)$ is $[n-1]_q^j$. Finally, the last equality is the recurrence (18).

Example 3 From Example 1, the generalized Stirling permutation pairs (π_1,π_2) of length 3 having together three cycles are

((1)(23), 123) − ((1)(23), 132) − ((13)(2), 123) − ((13)(2), 132) − ((12)(3), 123)− ((12)(3), 132).

Then the permutation pairs $\left(\pi_1^{(\leq,c)},\pi_2^{(\leq,c)}\right)$ are: (123, 123) − (123, 132) − (132, 123) − (132, 132) − (123, 123) − (123, 132). Thus,

$$\begin{aligned} c_q^{(2)}[3,3] &= q^{inv(123)+inv(123)}+q^{inv(123)+inv(132)}+q^{inv(132)+inv(123)} \\ &\quad +q^{inv(132)+inv(132)}+q^{inv(123)+inv(123)}+q^{inv(123)+inv(132)} \\ &= 2+3q+q^2. \end{aligned}$$

By setting $s=1$, we obtain immediately the following result due to Gessel [7].

Corollary 1 *The q-Stirling numbers of the first kind $c_q[n,k]$ is given by*

$$c_q[n,k]=\sum_{\pi^{(\leq,c)}\in\mathcal{Q}(n,k)} q^{inv\left(\pi^{(\leq,c)}\right)}, \tag{23}$$

where the sum is over all permutations $\pi^{(\leq,c)}$ associated with π of length n having exactly k cycles.

Similar to Proposition 2, it is easy to see from Theorems 4 and 6 that the analogues of the generalized Stirling numbers satisfy the following identities.

Proposition 5 *We have,*

(i) $c_q^{(s)}[n,s]=\left[[n-1]_q!\right]^s=\left[c_q[n,1]\right]^s$;
(iii) $c_q^{(s)}[n,sn]=1$;
(iv) *For* $n\geq 2$, $\sum_{k=0}^{sn}(-1)^{sn-k}c_q^{(s)}[n,k]=0$ *if s odd.*

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Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest

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