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ORIGINAL RESEARCH

Two highly efficient methods for soliton solutions of the modified Fornberg–Whitham equation

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Abstract In this study, exact solutions of the modified Fornberg-Whitham equation, which is popular for modeling wave breaking, have been studied using the Kudryashov method, while its numerical solutions have been studied by finite element method. To achieve these aims, general form of the Kudryashov method and septic B-spline basis functions have been employed as a polynomial approximation to the solution respectively. The error norms L_2 and L_∞ are illustrated for the adequacy and efficiency of the proposed method for the model problem. Von-Neumann stability analysis has been implemented which ensures that the schemes are unconditionally stable. The numerical results are illustrated, and comparisons are made in tabular form to demonstrate the effectiveness of the scheme studied in this article. Additionally, graphics are presented to compare the exact and numerical results to show the behavior of the solutions. The graphical comparisons guarantee the power of numerical scheme presented for the nonlinear problem, it also confirms that the exact solution process is efficient and handy to handle highly nonlinear problems.

Keywords Modified Fornberg–Whitham (mFW) equation · Kudryashov's method · Collocation · Septic B-splines

Mathematics Subject Classification 65N30 · 65D07 · 74S05 · 74J35 · 76B25

Contents

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1 Introduction

Travelling wave solution is an important type of solution for the nonlinear partial differential equations (NLPDEs) and many NLPDEs have been found to have a variety of travelling wave solutions. One of the equations that has been studied recently much on the solutions of the traveling wave is the Fornberg–Whitham (FW) equation, which is defined as:

$$u_t - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx},\ \ t > 0,\ \ x > 0. \tag{1}$$

This equation was first suggested by Whitham in 1967 to analyze the qualitative behavior of wave breaking [1].The traveling wave solutions of the equation are called a kink-like wave solution and anti kink-like wave solutions. Also the equation is a nonlinear dispersive wave equation. Since Eq. (1) was derived, little attention has been paid for studying it. To investigate travelling wave solutions to Eq. (1), various techniques can be found in recent literature, such as, homotopy analysis method (HAM) [2], phase portrait analytical technology [3], improved qualitative method [4], bifurcation method [5–7], factorization technique [8], polynomial method [9], and so on. Fornberg and Whitham procured a peaked solution $u(x,t) = Ae^{-\frac{1}{2}\left|x-\frac{4}{3}t\right|}$ with an arbitrary constant A, in 1978 [10]. Modifying the nonlinear term uu_x in Eq. (1) as u^2u_x, He et al. [11] suggested in mFW equation as follows

$$u_t - u_{xxt} + u_x + u^2u_x = 3u_x u_{xx} + uu_{xxx}\ \ t > 0,\ \ x > 0, \tag{2}$$

where $u(x,t)$ is fluid velocity, t is time and x is spatial co-ordinate. Several mathematicians extracted solutions for Eq. (2) by means of various techniques [12–18].

FEM is one of the most powerful method used in the solutions of partial differential equations (PDEs). In approximation theory, B-spline interpolation functions have attracted attention in solving boundary-value problems. These functions mentioned are quite useful for numerical calculations. Since spline functions are piecewise polynomials, they can be integrable and differentiable, and most integrals in numerical methods are zero. B-spline based on collocation method is a very effective technique because it has a programmable computation approach and is easy to implement [19]. Therefore, it requires less computational effort compared to other available methods. In this work, new exact and numerical solutions of mFW equation are analyzed to make a contribution to the literature. In the way of our aim, two effective methods, namely Kudryashov and finite element methods which are quite effective, have been applied.

2 General form of Kudryashov's method [20]

A nonlinear PDE is given as follows:

$$G(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \ldots) = 0. \tag{3}$$

Step 1: Applying the following traveling wave transformation where k is a constant and c is the wave velocity.

$$\eta = kx - ct, \quad u(x,t) = f(\eta). \tag{4}$$

Implementing the transformation (4) on (3), following ordinary differential equation (ODE) is obtained:

$$F(f', f'', f''', \ldots) = 0, \tag{5}$$

where the derivative depends on the η.

Step 2: Solution of the Eq. (5) will be searched in the form of a finite series as follows:

$$f(\eta) = \sum_{i=0}^{N} (A_i (Q(\eta))^i, \tag{6}$$

where A_i $(i = 0, 1, 2, \ldots, N) \neq 0$, $A_N \neq 0$, are constants. The positive integer N will be calculated by homogeneous balance technique.

Step 3: $Q(\eta)$ is defined by:

$$Q(\eta) = \frac{\delta\phi(\eta)}{\delta + \rho(\phi(\eta) - 1)}, \rho \neq 0, \tag{7}$$

where $A_0, A_1, A_2, \ldots, A_N$, δ and ρ are arbitrary constants. $\phi(\eta)$ provides the following ODE:

$$\frac{d\phi}{d\eta} = \vartheta \log(m)(\phi(\eta) - 1)\phi(\eta), \tag{8}$$

and the solution of (8) is obtained as

$$\phi(\eta) = \frac{1}{1 + dm^{\vartheta\eta}}. \tag{9}$$

Step 4: Using (6) together with (8) in the Eq. (5) yields a polynomial of $Q(\eta)$. A system of algebraic equation is obtained when all the coefficients of the like powers of $Q(\eta)$ equating to zero.

Step 5: Algebraic equation system found in the previous step is solved with the Mathematica program.

2.1 Application of the technique

If wave transformation (4) is used in Eq. (3), the following ODE is obtained:

$$k^2 f^{(3)}(\eta)(c - kf(\eta)) + f'(\eta)\left(-c - 3k^3 f''^2 + k\right) = 0. \tag{10}$$

Integrating with respect to η one obtains,

$$ck^2 f''^3 f(\eta) f''^3 \left(-f'^2\right) + \frac{1}{3}kf(\eta)^3 + kf(\eta) = 0. \tag{11}$$

Implementing the Balance principle $f(\eta)f''(\eta)$ with $u(\eta)^3$ in (11), $2N + 2 = 3N$ then $N = 2$ is procured.

2.1.1 Exact solutions of the problem

Equality (6) shows the solution in form:

$$u(\eta) = A_0 + A_1 Q(\eta) + A_2 Q(\eta)^2. \tag{12}$$

Replacing equality (12) in (11) with (8) and when the coefficients of $Q^i(\eta)$ with the same powers are set to zero, the following system of equations is obtained:

$$
\begin{aligned}
&-A_0c+\frac{A_0^3k}{3}+A_0k=0,\\
&A_1\left(b^2ck^2\log^2(m)-c+k\right)+A_0A_1\left(-b^2\right)k^3\log^2(m)+A_0^2A_1k=0,\\
&-3A_1b^2ck^2\log^2(m)+4A_2b^2ck^2\log^2(m)-2A_1^2b^2k^3\log^2(m)\\
&+3A_0A_1b^2k^3\log^2(m)-4A_0A_2b^2k^3\log^2(m)\\
&-A_2c+A_0A_1^2k+A_0^2A_2k+A_2k=0,\\
&2A_1b^2ck^2\log^2(m)-10A_2b^2ck^2\log^2(m)\\
&+5A_1^2b^2k^3\log^2(m)-2A_0A_1b^2k^3\log^2(m)\\
&+10A_0A_2b^2k^3\log^2(m)-9A_1A_2b^2k^3\log^2(m)+\frac{A_1^3k}{3}+2A_0A_1A_2k=0,\\
&6A_2b^2ck^2\log^2(m)-3A_1^2b^2k^3\log^2(m)\\
&-8A_2^2b^2k^3\log^2(m)-6A_0A_2b^2k^3\log^2(m)\\
&+21A_1A_2b^2k^3\log^2(m)+A_0A_2^2k+A_1^2A_2k=0,\\
&18A_2^2b^2k^3\log^2(m)-12A_1A_2b^2k^3\log^2(m)+A_1A_2^2k=0,\\
&\frac{A_2^3k}{3}-10A_2^2b^2k^3\log^2(m)=0.
\end{aligned}
$$

When the equalities (4), (7), (9) and (12) are used, the following two groups of solutions are obtained:

Group 1:

$$
\begin{aligned}
&A_0=0,\quad A_1=-3\left(\sqrt{15}+5\right),\quad A_2=3\left(\sqrt{15}+5\right),\\
&b=\mp\frac{\sqrt{9\sqrt{15}+35}}{c\log(m)},\quad k=-\frac{1}{10}\left(\sqrt{15}-5\right)c\\
&u_{1,2}(x,t)=A_0+\frac{A_1\delta}{\left(dm^{b\eta}+1\right)\left(\rho\left(\frac{1}{dm^{b\eta}+1}-1\right)+\delta\right)}\\
&\qquad+A_2\left(\frac{\delta}{\left(dm^{b\eta}+1\right)\left(\rho\left(\frac{1}{dm^{b\eta}+1}-1\right)+\delta\right)}\right)^2.
\end{aligned}
\tag{13}
$$

Group 2:

$$
\begin{aligned}
&A_0=0,\quad A_1=3\left(\sqrt{15}-5\right),\quad A_2=-3\left(\sqrt{15}-5\right),\\
&b=\mp\frac{\sqrt{35-9\sqrt{15}}}{c\log(m)},\quad k=\frac{1}{10}\left(\sqrt{15}+5\right)c.u_{3,4}(x,t)\quad=A_0+\frac{A_1\delta}{\left(dm^{b\eta}+1\right)\left(\rho\left(\frac{1}{dm^{b\eta}+1}-1\right)+\delta\right)}\\
&\qquad+A_2\left(\frac{\delta}{\left(dm^{b\eta}+1\right)\left(\rho\left(\frac{1}{dm^{b\eta}+1}-1\right)+\delta\right)}\right)^2.
\end{aligned}
\tag{14}
$$

2.2 Graphical illustrations

In this section, graphs related to the two groups of solutions obtained in the previous section are given. The graph of (13) using the method at $\rho=0.01$, $c=0.001$, $d=0.05$, $\delta=1$ is introduced in Fig. 1. In Fig. 2 the graph of (14) using the method at $\rho=0.25$, $c=0.2$, $d=2$, $\delta=0.5$ is presented.

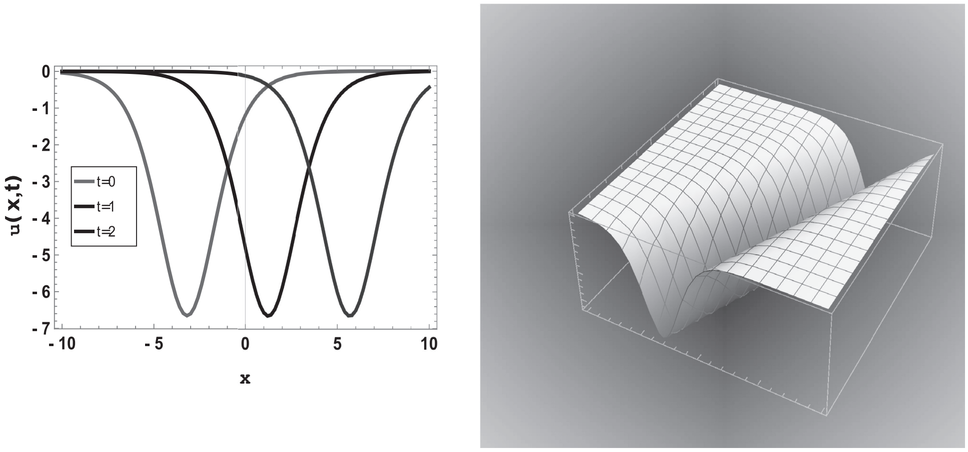

Fig. 1 Graph of (13) using the parameters $\rho = 0.01$, $c = 0.001$, $d = 0.05$, $\delta = 1$

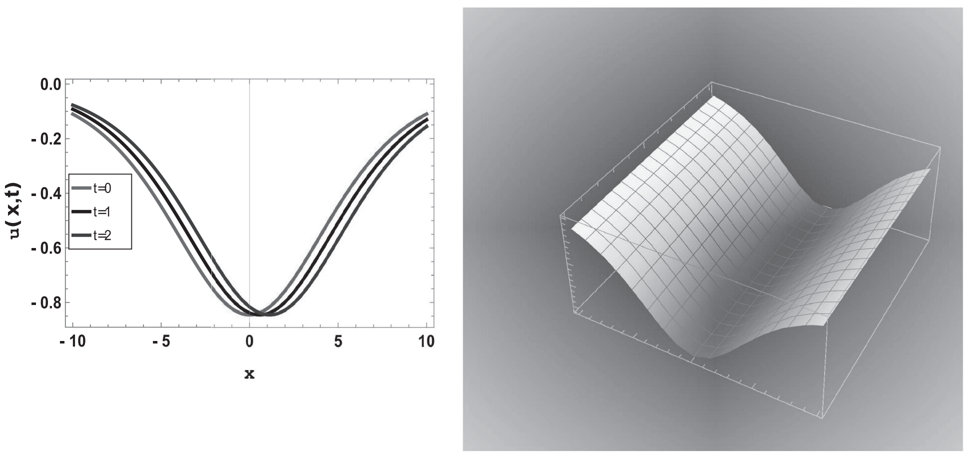

Fig. 2 Graph of (14) using the parameters $\rho = 0.25$, $c = 0.2$, $d = 2$, $\delta = 0.5$

3 Computer implementation of the numerical scheme

In this section, mFW equation has been solved using the septic B-spline collocation method. For this, we need to consider the following boundary and initial conditions;

$$\begin{array}{ll} u(x_L, t) = 0, & u(x_R, t) = 0, \\ u_x(x_L, t) = 0, & u_x(x_R, t) = 0, \\ u_{xx}(x_L, t) = 0, & u_{xx}(x_R, t) = 0, \\ u(x, 0) = f(x), & a \leq x \leq b. \end{array} \tag{15}$$

Septic B-spline functions $\phi_m(x)$, $m = -3(3)N$, at the nodes x_m are given over the solution interval $[x_L = a,\ x_R = b]$ in [19] . Seventh-order B-spline functions can be said to form a basis for functions defined in the range $[x_m, x_{m+1}]$. Among the others, collocation is well-known technique to improve numerical methods because of its various intended features [21–23]. In collocation method, $u_{numeric}(x, t)$ corresponding to the $u_{exact}(x, t)$ is written as a linear combination of septic B-splines as follows:

$$u_{numeric}(x, t) = \sum_{m=-3}^{N+3} \phi_m(x)\rho_m(t). \tag{16}$$

When $h\rho = x - x_m, \quad (0 \le \rho \le 1)$, transformation is made in the finite region $[x_m, x_{m+1}]$, the region turns to an interval of [0, 1]. Thus the septic B-spline functions in the new region [0, 1] are obtained as follows:

$$\begin{aligned}
\phi_{m-3} &= 1 - 7\rho + 21\rho^2 - 35\rho^3 + 35\rho^4 - 21\rho^5 + 7\rho^6 - \rho^7,\\
\phi_{m-2} &= 120 - 392\rho + 504\rho^2 - 280\rho^3 + 84\rho^5 - 42\rho^6 + 7\rho^7,\\
\phi_{m-1} &= 1191 - 1715\rho + 315\rho^2 + 665\rho^3 - 315\rho^4 - 105\rho^5 + 105\rho^6 - 21\rho^7,\\
\phi_{m} &= 2416 - 1680\rho + 560\rho^4 - 140\rho^6 + 35\rho^7,\\
\phi_{m+1} &= 1191 + 1715\rho + 315\rho^2 - 665\rho^3 - 315\rho^4 + 105\rho^5 + 105\rho^6 - 35\rho^7,\\
\phi_{m+2} &= 120 + 392\rho + 504\rho^2 + 280\rho^3 - 84\rho^5 - 42\rho^6 + 21\rho^7,\\
\phi_{m+3} &= 1 + 7\rho + 21\rho^2 + 35\rho^3 + 35\rho^4 + 21\rho^5 + 7\rho^6 - \rho^7,\\
\phi_{m+4} &= \rho^7.
\end{aligned} \tag{17}$$

Using the equalities given by (16) and (17), the following expressions are obtained:

$$\begin{aligned}
u_{numeric}(x_m, t) &= \rho_{m-3} + 120\rho_{m-2} + 1191\rho_{m-1} + 2416\rho_m + 1191\rho_{m+1} + 120\rho_{m+2} + \rho_{m+3},\\
u'_m &= \tfrac{7}{h}(-\rho_{m-3} - 56\rho_{m-2} - 245\rho_{m-1} + 245\rho_{m+1} + 56\rho_{m+2} + \rho_{m+3}),\\
u''_m &= \tfrac{42}{h^2}(\rho_{m-3} + 24\rho_{m-2} + 15\rho_{m-1} - 80\rho_m + 15\rho_{m+1} + 24\rho_{m+2} + \rho_{m+3}),\\
u'''_m &= \tfrac{210}{h^3}(-\rho_{m-3} - 8\rho_{m-2} + 19\rho_{m-1} - 19\rho_{m+1} + 8\rho_{m+2} + \rho_{m+3}),\\
u^{iv}_m &= \tfrac{840}{h^4}(\rho_{m-3} - 9\rho_{m-1} + 16\rho_m - 9\rho_{m+1} + \rho_{m+3}),\\
u^{v}_m &= \tfrac{2520}{h^5}(-\rho_{m-3} + 4\rho_{m-2} - 5\rho_{m-1} + 5\rho_{m+1} - 4\rho_{m+2} + \rho_{m+3}).
\end{aligned} \tag{18}$$

3.1 Analysis of the method on modified Fornberg–Whitham (mFW) equation

Now, putting (16) and (18) in Eq. (2) and simplifying, the following system of ODEs are reached:

$$\begin{aligned}
&\dot{\rho}_{m-3} + 120\dot{\rho}_{m-2} + 1191\dot{\rho}_{m-1} + 2416\dot{\rho}_m + 1191\dot{\rho}_{m+1} + 120\dot{\rho}_{m+2} + \dot{\rho}_{m+3}\\
&-\tfrac{42}{h^2}(\rho_{m-3} + 24\rho_{m-2} + 15\rho_{m-1} - 80\rho_m + 15\rho_{m+1} + 24\rho_{m+2} + \rho_{m+3})\\
&+Z_{m1}\tfrac{7}{h}(-\rho_{m-3} - 56\rho_{m-2} - 245\rho_{m-1} + 245\rho_{m+1} + 56\rho_{m+2} + \rho_{m+3})\\
&-Z_{m2}(\rho_{m-3} + 120\rho_{m-2} + 1191\rho_{m-1} + 2416\rho_m + 1191\rho_{m+1} + 120\rho_{m+2} + \rho_{m+3}) = 0,
\end{aligned} \tag{19}$$

where $\dot{\rho} = \frac{d\rho}{dt}$ and

$$Z_{m1} = 1 + u^2 - 3u_{xx},\quad Z_{m2} = u_{xxx}.$$

When the following Crank-Nicolson and forward finite difference approximation are written, in spatial and temporal directions, respectively,

$$\rho_i = \frac{\rho_i^{n+1} + \rho_i^n}{2}, \dot{\rho}_i = \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} \tag{20}$$

in the system of Eq. (19), following recurrence relation is obtained

$$\begin{aligned}
&\lambda_1^{n+1}\rho_{m-3}^{n+1} + \lambda_2^{n+1}\rho_{m-2}^{n+1} + \lambda_3\rho_{m-1}^{n+1} + \lambda_4\rho_m^{n+1} + \lambda_5\rho_{m+1}^{n+1} + \lambda_6\rho_{m+2}^{n+1} + \lambda_7\rho_{m+3}^{n+1}\\
&= \lambda_7\rho_{m-3}^{n} + \lambda_6\rho_{m-2}^{n} + \lambda_5\rho_{m-1}^{n} + \lambda_4\rho_m^{n} + \lambda_3\rho_{m+1}^{n} + \lambda_2\rho_{m+2}^{n} + \lambda_1\rho_{m+3}^{n}
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
&\lambda_1 = [1 - E - KZ_{m1} - TZ_{m2}], && \lambda_2 = [120 - 24E - 56KZ_{m1} - 120TZ_{m2}],\\
&\lambda_3 = [1191 - 15E - 245KZ_{m1} - 1191TZ_{m2}], && \lambda_4 = [2416 + 80E - 2416TZ_{m2}],\\
&\lambda_5 = [1191 - 15E + 245KZ_{m1} - 1191TZ_{m2}], && \lambda_6 = [120 - 24E + 56KZ_{m1} - 120TZ_{m2}],\\
&\lambda_7 = [1 - E + KZ_{m1} - TZ_{m2}],\\
&E = \tfrac{42}{h^2}, \quad K = \tfrac{7}{2h}\Delta t, \quad T = \tfrac{1}{2}\Delta t, \; m = 0, 1, \ldots, N-1.
\end{aligned} \tag{22}$$

To assure a unique solution, it is necessary to eliminate unknown parameters ρ_{-3}, ρ_{-2}, ρ_{-1}, ρ_{N+1}, ρ_{N+2} and ρ_{N+3} from the resulting system (21). This procedure can be easily done using the values of u and boundary conditions, and then

$$Pd^{n+1} = Qd^n \tag{23}$$

is obtained where $d^n = (\rho_0, \rho_1, \ldots, \rho_N)^T$.

4 Stability and accuracy analysis

In this study we analyze mFW equation with analytic solutions as well as numerical solutions. Analytic solutions have been generated by Kudryashov's method. For numerical approximation of the model problem higher order splines with collocation at space defined points have been employed followed by a finite difference scheme for time. It is to note that effectiveness of a numerical approximation depends on its convergence rate and numerical stability. First we start by the stability analysis and then we move on to the short accuracy concepts.

4.1 Stability analysis

For the stability analysis, Von Neumann technique has been used. For this Eq. (2) are linearized supposing that the expressions $1 + u^2 - 3u_{xx}$ and u_{xxx} in the nonlinear terms $u_x 1 + u^2 - 3u_{xx}$ and uu_{xxx} are locally constant, respectively. Amplification factor ξ of a typical Fourier mode of amplitude is determined as [24,25]:

$$\rho_m^n = \xi^n e^{imkh}, \tag{24}$$

here $i = \sqrt{-1}$. Using (24) into the (21), following growth factor

$$\xi = \frac{a_1 - ia_2}{a_1 + ia_2}, \tag{25}$$

is procured, in which

$$\begin{aligned} a_1 &= (2 - 2\varepsilon - 2\mu)\cos(3kh) + (240 - 48\varepsilon - 240\mu)\cos(2kh) + \\ &(2382 - 30\varepsilon - 2382\vartheta)\cos(kh) + \kappa, \\ a_2 &= -2\vartheta\sin(3kh) - 128\vartheta\sin(2kh) - 490\vartheta\sin(kh), \end{aligned} \tag{26}$$

where

$$\varepsilon = E, \quad \mu = TZ_{m2}, \quad \vartheta = KZ_{m1}, \ \kappa = 2416 + 80\varepsilon - 2416TZ_{m2}.$$

Modulus of Eq. (25) is 1. Thus, we show that scheme (21) is unconditionally stable under the present conditions.

4.2 Accuracy of the numerical scheme

Finite element analysis and approximation is firmly attached in a very elegant framework that enables accurate a priori and a posteriori estimates of convergence rates as well as discretization errors. Here we deploy some pertinent concepts and main outcomes. One may consult [26–29] and the references therein. The constants $C_i \geq 0$ has been considered, which are not necessarily the same in all the cases. Here we recall the following time dependent nonlinear PDE

$$u_t - u_{xxt} + u_x + u^2u_x = 3u_xu_{xx} + uu_{xxx} \ t > 0, \ x > 0,$$

and plan to demonstrate the error in an approximate the solutions. Usually spectral methods and pseudo-spectral methods are employed to approximate solutions of various mathematical models when the domain of computation is simple and when solution has been considered sufficiently smooth. However, most applied models are studied in a complex spatial domain and when the solutions are not considered to be sufficiently. For such problems

Table 1 The error norms for two values of h and at various times for $\Delta t = 0.01$

t	$\Delta t = 0.01, h = 0.1$		$\Delta t = 0.01, h = 0.05$	
	$L_2 \times 10^{-8}$	$L_\infty \times 10^{-8}$	$L_2 \times 10^{-9}$	$L_\infty \times 10^{-9}$
[12] 0.1	0.55596195	0.21513131	5.61116494	2.165036816
0.1	0.4795924957	0.7289484602	0.6147150885	0.8592975290
0.2	0.7146734342	1.2282706301	1.1920981590	1.6866911707
0.3	0.8140059445	1.5703011743	1.7337616436	2.4833650600
0.4	0.8504184873	1.8045885743	2.2414141347	3.2504597703
0.5	0.8663907231	1.9650730301	2.7165710560	3.9890735274
0.6	0.8804463856	2.0750032292	3.1608478005	4.7002637818
0.7	0.8963693908	2.1503042832	3.5756380049	5.3850487225
0.8	0.9124587223	2.2018840189	3.9623967289	6.0444087350
0.9	0.9270888596	2.2372168038	4.3225582214	6.6792878048
1.0	0.9401687927	2.2614194437	4.6573955922	7.2905948684

Table 2 The error norms for two values of h and at various times for $\Delta t = 0.001$

t	$\Delta t = 0.001, h = 0.1$		$\Delta t = 0.001, h = 0.05$	
	$L_2 \times 10^{-9}$	$L_\infty \times 10^{-9}$	$L_2 \times 10^{-8}$	$L_\infty \times 10^{-8}$
[12] 0.1	0.108863387	0.218611128	0.00886346	0.02186111
0.1	0.9151085851	1.2405697878	0.2442421645	0.3464755170
0.2	1.8624221348	2.4866531006	0.5037567136	0.6954613534
0.3	2.8305793401	3.7375629802	0.7538945167	1.0427055793
0.4	3.8154426879	4.9864519992	1.0198636081	1.3945310128
0.5	4.8523116503	6.2471677969	1.2861955902	1.7464191938
0.6	5.9033023130	7.5166325280	1.5667844517	2.1004267300
0.7	6.9538769753	8.7816372008	1.8494416275	2.4551205688
0.8	8.0638733792	10.0495234393	2.1351974134	2.8102944709
0.9	9.1545720456	11.3127065310	2.4311146656	3.1669172902
1.0	10.2590473470	12.5750790498	2.7394537328	3.5258744417

piecewise polynomial approximations play a key role and works very efficiently to approximate the solutions. Polynomials are differentiable functions which is a key property in approximation theory.

As stated above we use piecewise polynomials of degree seven (septic B-splines) collocation scheme for the space integration. As it is known that collocation technique has super-convergence at collocation points. Also it does not require an inner product as of the Galerkin inner product approach [30,31] which gives an advantage in solution computation.

Let there be $n + 1$ data values. Then there is exactly one polynomial of degree at most n passing through the data points and the error in the interpolating polynomial is proportional to a power of the distance between the data points [27,28,32]. Let $\mathcal{H}^r(\Omega)$ be the space of r times differentiable functions and $\|\cdot\|_n$ be the standard $\mathcal{H}^r(\Omega)$ norm. Let ϕ_h be an approximation to a function $\phi(x) \in \mathcal{H}^r(\Omega)$ in Ω. Here $\|\cdot\|_0$ represents $L_2(\Omega)$ norm. Let $h = x_{i+1} - x_i$ be the distance between the grids and $\Omega = \cup_i \Omega_i$, where $\Omega_i = [x_i,\ x_{i+1}]$. We observe [27,28,33,34] that

$$\|\phi(x) - \phi_h(x)\| \leq Ch^{k+1}\|\phi\|_{k+1} \text{ where } 1 \leq k < r,$$

and ϕ_h represents piecewise-polynomials of degree k (considering $\Omega = \cup_i \Omega_i$). The error is conserved by the Galerkin finite element approximation as well [26,28].

Thus from the above discussion it can be easily seen [26,32,34] that if ψ_h is a suitable B splines defined by a polynomial of degree less or equal m then also the following inequality holds:

$$\|\psi(x) - \psi_h(x)\| \leq Ch^{m+1}\|\psi\|_{m+1} \text{ where } 1 \leq m < r,$$

for any $\psi \in \mathcal{H}_r(\Omega)$. In this study septic B-splines have been considered for spatial integration. A forward difference scheme which is of $\mathcal{O}(\Delta t)$ in $L_2([0\ T])$ norm has been employed for some $T > 0$ [28]. Hence the accuracy the space time discrete scheme can be presented by

$$\|u(x,t) - u_h(x,t)\| \leq C_1 h^8 + C_2 \Delta t,$$

for a suitable $C_1 \geq 0$ and $C_2 \geq 0$.

Table 3 A comparison of the numeric results with Ref. [17] for the parameters $\Delta t = 0.001$ and $h = 0.5$

x	$t - value$	$u_{numeric}$ [17]	$u_{numeric}$ Present method
1	0.03	– 0.8236196600	– 0.8226733361
1	0.1	– 0.8268221520	– 0.8225845680
2	0.04	– 0.7612241946	– 0.7578271609
3	0.05	– 0.6677895714	– 0.6647276706
5	0.02	– 0.4504682776	– 0.4999263882
5	0.08	– 0.4529838898	– 0.4354901904
6	0.06	– 0.3576816406	– 0.3511456251
– 1	0.03	– 0.8200878398	– 0.8287304071
– 1	0.1	– 0.8150778480	– 0.8305607124
– 2	0.04	– 0.7527237054	– 0.7557343848
– 3	0.05	– 0.6566685000	– 0.6567131165
– 5	0.02	– 0.4456717220	– 0.4458292193
– 5	0.08	– 0.4344161099	– 0.4388962405
– 6	0.06	– 0.3458783593	– 0.3511423171

Table 4 A comparison of the present numeric results with Ref. [35] for the parameters $\Delta t = 0.001$ and $h = 0.5$

	$t = 0.02$		$t = 0.04$	
x	[35]	Present method	[35]	Present method
2.5	– 0.7144249843	– 0.7144249014	– 0.7165464263	– 0.7165463183
5	– 0.4504779212	– 0.4504779859	– 0.4528101955	– 0.4528101696
7.5	– 0.2349678714	– 0.2349678546	– 0.2364825099	– 0.2364825821
10	– 0.1107643868	– 0.1107643246	– 0.1115481917	– 0.1115481897
x	$t = 0.06$		$t = 0.08$	
	[35]	Present method	[35]	Present method
2.5	– 0.7186566715	– 0.7186565924	– 0.7207556337	– 0.7207555368
5	– 0.4551474661	– 0.4551474465	– 0.4574897077	– 0.4574897349
7.5	– 0.2380051612	– 0.2380051679	– 0.2395356954	– 0.2395356259
10	– 0.1123371196	– 0.1123371777	– 0.1131312117	– 0.1131313158
x	$t = 0.1$			
	[35]	Present method		
2.5	– 0.7228430209	– 0.7228429646		
5	– 0.4598370439	– 0.4598369517		
7.5	– 0.2410739190	– 0.2410739693		
10	– 0.1139306377	– 0.1139306312		

5 Numerical experiments and discussions

To examine the efficiency and validity of the proposed numerical method, we consider the mFW equation for various values of the time and space division. Firstly, an exact solution of the form of the mFW equation is

$$u(x,t) = \frac{3}{4}(\sqrt{15} - 5)\sec h^2[\frac{1}{20}\sqrt{10(5-\sqrt{15})}(x - (5-\sqrt{15})t)], \tag{27}$$

with the following initial and boundary conditons:

$$u(x,0) = \frac{3}{4}(\sqrt{15} - 5)\sec h^2[\frac{1}{20}\sqrt{10(5-\sqrt{15})}x], \tag{28}$$

where $u \to 0$ as $x \to \pm\infty$. To show the accuracy of our numerical scheme, the interval of the problem is chosen as $[x_L = -70, x_R = 70]$ and the final time of the processing is chosen as $t = 1$ considering the studies in the literature.

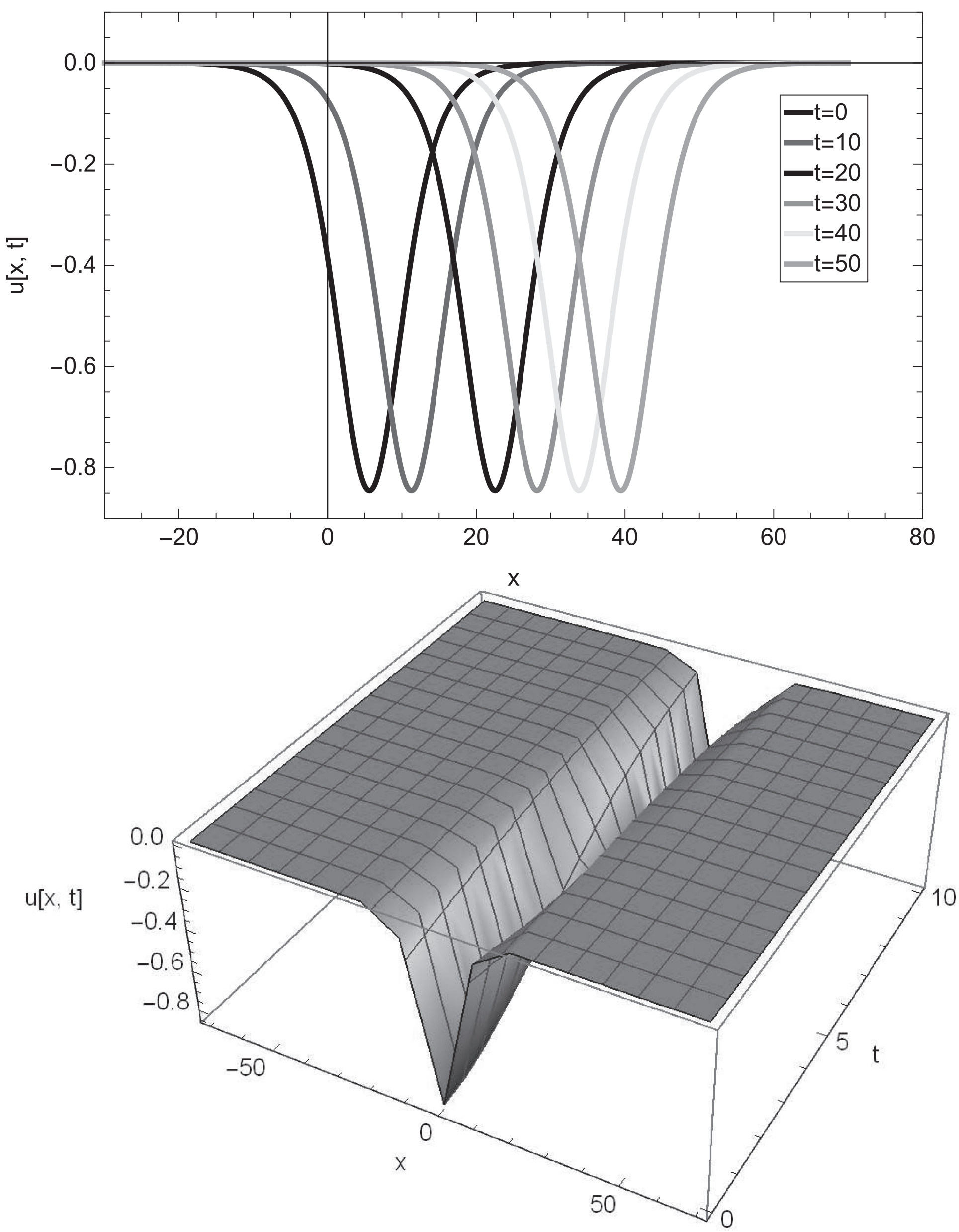

Fig. 3 Numerical behaviours of modified Fornberg–Whitham equation for $\Delta t = 0.001$ and $h = 0.05$

We will use the error norms, widely used in the literature, namely L_2 and L_∞ in order to check our method:

$$L_2 = \|u_{exact} - u_{numeric}\|_2 \simeq \sqrt{h \sum_{j=1}^{N} \left|(u_{exact})_j - (u_{numeric})_j\right|^2}, \tag{29}$$

$$L_\infty = \|u_{exact} - u_{numeric}\|_\infty \simeq \max_j \left|(u_{exact})_j - (u_{numeric})_j\right|, \quad j = 1, 2, \ldots, N. \tag{30}$$

In simulation calculations, in order to comply with the literature, as typical values $\Delta t = 0.01$; 0.001 with $h = 0.1$ and 0.05 were chosen. In Tables 1 and 2, values of the L_2 and L_∞ are calculated. Thus, the effect of amount of ranking points on the behavior of the numerical method can be easily noticed. When the tables were examined, the calculated error norms were found to be satisfactorily small. We can say that the increase in the number of time divisions has positive effects on the numerical results. In order to be able to compare with

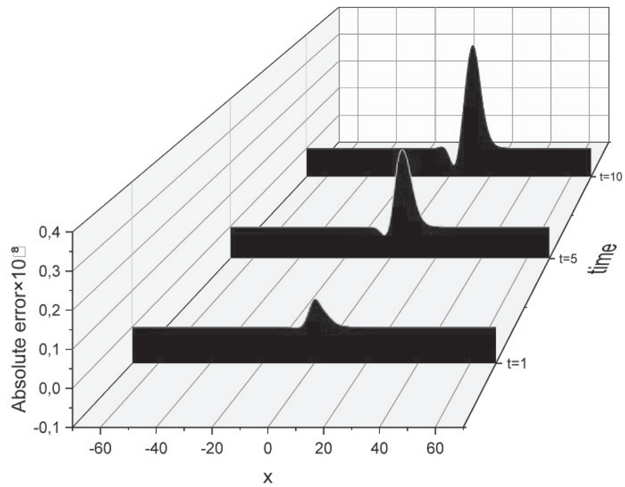

Fig. 4 Error distributions at $t = 0$, $t = 5$ and $t = 10$ for the parameters $\Delta t = 0.001$ and $h = 0.05$

previous articles, the method was studied up to $t = 1$. Secondly, some comparisons between different articles in the literature and newly obtained numerical results are presented in the tables. In Tables 1 and 2, comparisons of our results with reference [12] are made and listed. Tables clearly show that error norms procured by our method are in fairly perfect agreement with [12]. On the other hand, Table 3 presents a comparison of the numerical results between the current method and Ref. [17]. In Table 4, the numerical solution was calculated at selected x steps and t times and comparisons were made with Ref. [35]. In these tables, it is clearly seen that the collocation method combined with the division methods is better, faster and more reliable than the other methods. When Fig. 3 is examined, one can clearly observe the two and three-dimensional case of downward bell-shaped wave solutions generated at selected time intervals. The figure reveals the behavior of the wave as it moves slowly and does not change its shape. Also, in Fig. 4, numerical error distribution is plotted at $t = 1$, $t = 5$ and $t = 10$ time steps for different h and Δt values.

6 Conclusions

In this study, we propose an analytic solution procedure as well as a numerical solution procedure for the well known modified Fornberg-Whitham problem which is popular for modeling wave breaking. The analytic solutions show the exact wave pattern and guarentees that the analytic approach are very efficient for this type of nonlinear problems. Our numerical study guarantees that the proposed scheme for the model problem performs well, it is unconditionally stable and ensures convergence to the exact solutions. It is well demonstrated that our numerical results agree with various other previous published works and that confirms the compatibility of the scheme. We are sure that this study would help to solve similar high nonlinear problems.

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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