Indian J Pure Appl Math (2026) 57:1660–1673
https://doi.org/10.1007/s13226-024-00610-7

ORIGINAL RESEARCH

Estimation of a heat source in a parabolic equation with nonlocal Wentzell boundary condition using a spectral technique

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Received: 30 October 2023 / Accepted: 20 May 2024 / Published online: 2 June 2024

Abstract We present a numerical method for approximating the temperature distribution and a time-dependent source function in the one-dimensional heat equation, considering integral overdetermination and non-local Wentzel-Neumann boundary conditions. Initially, we reformulate the problem as a non-classical parabolic equation with initial and homogeneous boundary conditions. We apply the θ-weighted finite difference method (FDM) to discretize the time derivative. Subsequently, the main problem is transformed into a system of second-order ordinary differential equations (ODEs), which is then solved using a spectral method. This approach ensures that the obtained approximation accurately satisfies the boundary conditions at each time level. Additionally, a regularization method is employed to find a stable approximation for the derivative of perturbed boundary data. We conduct stability analysis to address the solution of the considered problem, and three numerical tests are provided to demonstrate the effectiveness and accuracy of the proposed scheme.

Keywords Parabolic equation · Spectral method · Regularization technique · Wentzel-Neumann boundary condition

Mathematics Subject Classification 35R30 · 65M70 · 65Z05

1 Introduction

In this paper, we wish to approximate the unknown functions $r(t)$ and $v(x,t)$ such that the pair $(r(t), v(x,t))$ satisfies the following heat equation [17]

$$v_t(x,t) - v_{xx}(x,t) = r(t)f(x,t), \quad (x,t) \in \Omega_T, \tag{1.1}$$

supplied with the initial condition

$$v(x,0) = \rho(x), \quad 0 \le x \le 1, \tag{1.2}$$

Dirichelet boundary condition

$$v(0,t) = 0, \quad 0 \le t \le T, \tag{1.3}$$

nonlocal Wentzell-Neumann boundary condition

$$v_x(0,t) + \alpha v_{xx}(1,t) = 0 \quad t \in [0,T], \tag{1.4}$$

Communicated by NM Bujurke.

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and subject to the additional specification given in the integral form

$$\int_0^1 v(x,t)dx = E(t) \quad t \in [0,T]. \tag{1.5}$$

It is assumed that $\Omega_T = [0,1] \times [0,T]$ is a bounded domain in $\mathbb{R}^2$, α is a positive constant and

$$\rho(x),\ E(t),\ f(x,t),$$

are given sufficiently smooth functions. Moreover, we suppose that the following conditions hold for the input data:

$$\rho(0) = \rho'(0) + \alpha\rho''(1) = 0, \quad \int_0^1 \rho(x)dx = E(0), \quad A(t) := \int_0^1 f(x,t)dx \neq 0,\ \forall\, t \in\ [0,T]. \tag{1.6}$$

If we denote $v(x,t)$ and $E(t)$ as the concentration of absorbed heat and the total energy content of a dynamical system such as the process of microwave heating [17] in industrial applications, the problem (1.1)-(1.5) can show how designing $E(t)$ is possible by controlling the heat sources while the temperature distribution at the boundaries of the material is prescribed by the nonlocal Wentzell-Neumann condition (1.4) and Dirichlet condition (1.2). When the multiplicative term $r(t)$ is known, the problem of finding the function $v(x,t)$ satisfying the system of equations (1.1)-(1.4) is referred to as the direct problem. Mathematically speaking, it belongs to the class of second-order time-dependent parabolic partial differential equations (PDEs) with so-called dynamic boundary conditions. These problems play an important role in modeling physical phenomena in areas such as thermoelasticity, heat transfer and the problems related to fluid dynamics [9,15–18,27]. Unlike the classical boundary conditions in which we are dealing with either the Neumann, Dirichlet or Robin boundary conditions, the literature concerning simultaneous boundary conditions (1.4) and (1.5) is minuscule. The nonlocal Wentzell-Neumann boundary condition (1.4) involves motion on the boundary of the problem [17–19], which was originally observed while studying a Markov process in a closed bounded region [30]. It was discovered that the elliptic operator of the governing equation can also appear in boundary conditions. Later in [10,11,26], the generalized Wentzell boundary condition (GWBC)

$$a(x)\mathbf{A}v(x) + b(x)\frac{\partial v(x)}{\partial \nu} + c(x)v(x) = 0, \quad x \in \partial\Omega, \tag{1.7}$$

was investigated where $(a(x), b(x), c(x)) \neq (0,0,0)$ and $\Omega \subset \mathbb{R}^n$, ν, $\mathbf{A}$ were assumed as the bounded subset of $\mathbb{R}^n$, the outer normal derivative, and the elliptical operator respectively.

The evolution equations equipped with the generalized Wentzell boundary condition have been studied in several papers [4,12,13,17,18,20,28]. In [12,13,28], the authors considered linear parabolic systems of equations and elliptic equations supplied with the Wentzell boundary conditions and proved some results of regularity in spaces of Holder continuous functions by means of semigroup theory approach. In [18], the authors considered an inverse problem for recovering a time-dependent control parameter in a heat equation with Wentzell-Neumann boundary condition and proved the existence and uniqueness of the classical solution, and proposed a numerical technique based on the FDM. In [4,20], the authors discussed identification of some source terms in hyperbolic and parabolic equations from the overdetermination of the displacement and temperature at the final time. For the considered problems, an adjoint problem approach and a weak solution technique were employed and a conjugate gradient technique was used for the numerical solution. In [17], sufficient conditions for the existence and uniqueness of the classical solution to problem (1.1)-(1.5), and the continuous dependence of the solution upon data have been discussed. However, there is no discussion about the numerical solution of this inverse problem. This motivates us to provide an effective and easy to implement numerical algorithm that recovers the unknown functions accurately and also guarantees the stability of the solution when noisy input data are supplied.

The organization of this article is as follows: In Section 2, we present the computational scheme for solving the inverse problem. In Section 3, we discuss the stability analysis of the proposed method. In Section 4, we present the results of the numerical simulation. In Section 5, we conclude the paper.

2 Solution Method

First, we differentiate equation (1.5) with respect to t and use equation (1.1) to get

$$r(t) = \frac{E'(t) + v_x(0,t) - v_x(1,t)}{A(t)}. \tag{2.1}$$

Applying (2.1) in the governing equation (1.1) results the following PDE

$$v_t(x,t) - v_{xx}(x,t) = \frac{f(x,t)E'(t) + f(x,t)v_x(0,t) - f(x,t)v_x(1,t)}{A(t)}, \quad (x,t) \in \Omega_T, \tag{2.2}$$

which by denoting

$$G(x,t) := \frac{f(x,t)}{A(t)}, \quad F(x,t) := \frac{f(x,t)E'(t)}{A(t)},$$

and assuming that $E(t) \in C^1([0,T])$ and taking the conditions (1.6) into account, the following equivalent problem is yielded

$$v_t(x,t) - v_{xx}(x,t) = G(x,t)\Big\{v_x(0,t) - v_x(1,t)\Big\} + F(x,t), \quad (x,t) \in \Omega_T, \tag{2.3}$$

$$v(x,0) = \rho(x), \quad 0 \le x \le 1, \tag{2.4}$$

$$v(0,t) = 0, \quad 0 \le t \le T, \tag{2.5}$$

$$v_x(0,t) + \alpha v_{xx}(1,t) = 0 \quad t \in [0,T]. \tag{2.6}$$

By employing the gradient differential operator ∇, the given parameter $\theta \in [0,1]$ and the time step size h, and using the θ-weighted technique [5–8], the equation (2.3) is discretized as

$$\begin{aligned}\frac{v(x,t+h) - v(x,t)}{h} &= \theta\nabla^2 v(x,t+h) + (1-\theta)\nabla^2 v(x,t)\\ &+ F(x,t+h) + G(x,t+h)\Big\{v_x(0,t+h) - v_x(1,t+h)\Big\}, \quad x \in [0,1].\end{aligned} \tag{2.7}$$

By assuming that N is a positive integer and the following notations

$$t_k = kh,\ h = \frac{T}{N},\ v^k(x) = v(x,t_k),\ F^k(x) = F(x,t_k),\ G^k(x) = G(x,t_k),\ k \in \{0,1,\ldots,N\},$$

equation (2.7) is rewritten as

$$\begin{aligned}&v^{k+1}(x) - h\theta\nabla^2 v^{k+1}(x) - h\Big\{F^{k+1}(x) + G^{k+1}(x)v_x^{k+1}(0) - G^{k+1}(x)v_x^{k+1}(1)\Big\}\\ &= v^k(x) + h(1-\theta)\nabla^2 v^k(x), \quad x \in [0,1].\end{aligned} \tag{2.8}$$

Let $\Phi(x)$ be a vector, including the polynomial basis functions as $\Phi^\top(x) = [\phi_0(x), \phi_1(x), \ldots, \phi_M(x)]$ subject to $\{\phi_i(x)\}_{i=0}^\infty$ are complete in the Hilbert spaces $L^2[0,1]$. We approximate $\nabla^2 v^{k+1}(x)$ as

$$\nabla^2 v^{k+1}(x) \simeq C_{k+1}^\top \Phi(x), \tag{2.9}$$

where $C_{k+1}^\top = [c_0^{k+1}, \ldots, c_M^{k+1}]$ are unknown constants. Now, by doing simple calculations we have

$$v_x^{k+1}(x) - v_x^{k+1}(0) = \int_0^x v_{xx}^{k+1}(\tau)d\tau \Longrightarrow v^{k+1}(x) = \int_0^x \int_0^z v_{xx}^{k+1}(\tau)d\tau dz + x v_x^{k+1}(0) + v^{k+1}(0). \tag{2.10}$$

Using conditions (2.5)-(2.6) we get

$$v^{k+1}(0) = 0,\ v_x^{k+1}(0) = -\alpha v_{xx}^{k+1}(1), \tag{2.11}$$

then, from (2.10) and (2.11) we have

$$v^{k+1}(x) = \int_0^x \int_0^z v_{xx}^{k+1}(\tau) d\tau dz - x\alpha v_{xx}^{k+1}(1), \tag{2.12}$$

which implies that (2.12) fulfills the conditions (2.11) accurately at each time level. Now, by defining

$$\Phi^{\#}(x) := \int_0^x \Phi(\tau) d\tau, \quad \Phi^{*}(x) := \int_0^x \int_0^z \Phi(\tau) d\tau dz, \tag{2.13}$$

and substituting (2.9) in equations (2.12) and (2.8) we get the following relations for $0 \le k \le N-1$

$$v^{k+1}(x) \simeq C_{k+1}^{\top} \Big\{ \Phi^{*}(x) - x\alpha\Phi(1) \Big\}, \tag{2.14}$$

$$C_{k+1}^{\top} \Big(\underbrace{\Phi^{*}(x) - x\alpha\Phi(1) - h\theta\Phi(x) + hG^{k+1}(x)\Phi^{\#}(1)}_{:=S^{k+1}(x)} \Big) = \underbrace{hF^{k+1}(x) + v^k(x) + h(1-\theta)v_{xx}^k(x)}_{:=\chi^k(x)}, \tag{2.15}$$

where $S^{k+1}(x)$ is a column vector including known entries specified as

$$S^{k+1}(x) = [s_0^{k+1}(x), \dots, s_M^{k+1}(x)]^{\top},$$

$$s_p^{k+1}(x) = \int_0^x \int_0^z \phi_p(\tau) d\tau dz - x\alpha\phi_p(1) - h\theta\phi_p(x) + hG^{k+1}(x) \int_0^1 \phi_p(\tau) d\tau,\ p \in \{0, \dots, M\},$$

and $\chi^k(x)$ is a known function at the time level kth. By multiplying equation (2.15) by $\Phi^{\top}(x)$ and then integrating with respect to x over [0, 1], we achieve

$$C_{k+1}^{\top} \underbrace{\int_0^1 S^{k+1}(\tau)\Phi^{\top}(\tau) d\tau}_{:=H_{k+1}} = \underbrace{\int_0^1 \chi^k(\tau)\Phi^{\top}(\tau) d\tau}_{:=b_k}, \tag{2.16}$$

where the matrix H_{k+1} and vector b_k are specified as

$$H_{k+1} = \begin{pmatrix} \int_0^1 s_0^{k+1}(x)\phi_0(x)dx & \cdots & \int_0^1 s_0^{k+1}(x)\phi_M(x)dx \\ \vdots & & \vdots \\ \int_0^1 s_M^{k+1}(x)\phi_0(x)dx & \cdots & \int_0^1 s_M^{k+1}(x)\phi_M(x)dx \end{pmatrix},$$

$$b_k = [\int_0^1 \chi^k(x)\phi_0(x)dx, \dots, \int_0^1 \chi^k(x)\phi_M(x)dx]. \tag{2.17}$$

The solution of linear system of equations (2.16) is presented as

$$C_{k+1}^{\top} = b_k H_{k+1}^{-1}. \tag{2.18}$$

By taking $v^0(x) = \rho(x)$ from condition (2.4) and substituting it in equation (2.15) and following equations (2.16)-(2.18), we find the vector $C_{k+1}^{\top}$ for $k \in \{0, 1, \dots, N-1\}$ and this way the approximation (2.14) is determined. Finally, by denoting

$$r^{k+1} := r(t_{k+1}),\ A^{k+1} := A(t_{k+1}),\ (E')^{k+1} := E'(t_{k+1}),\ k \in \{0, \dots, N-1\},$$

and combining equation (2.10) with (2.1) we get $r(t_{k+1}) = \frac{E'(t_{k+1}) - \int_0^1 v_{xx}^{k+1}(\tau)d\tau}{A(t_{k+1})}$. Therefore, from (2.9) and (2.13) we obtain

$$r^{k+1} = \frac{(E')^{k+1} - C_{k+1}^{\top}\Phi^{\#}(1)}{A^{k+1}}. \tag{2.19}$$

Remark 2.1 Following the proposed instruction for computing the regularized approximation of the derivatives of noisy data in [29], we consider $E(t)$ and $E_\xi(t)$ as the exact and perturbed functions corresponding to condition (1.5), subject to

$$\frac{1}{M'}\sum_{p=1}^{M'}(E(t_p)-E_\xi(t_p))^2 \le \xi^2,$$

where $t_p \in [0, T]$, and ξ stands for the noise level of the collected data and M' represents the number of measured data namely, $E_\xi(t_p)$. It has been shown that the stable solution $E_\lambda(t)$ given by

$$E_\lambda(t) = \sum_{j=1}^{M'}\theta_j|t-t_j|^{2M''-1} + \sum_{j=1}^{M''} d_j t^{j-1}, \tag{2.20}$$

is obtained from solving the following minimization problem

$$\min_{E\in\Gamma_{M''}} \Omega(E) = \frac{1}{M'}\sum_{p=1}^{M'}(E(t_p)-E_\xi(t_p))^2 + \lambda\|\frac{d^{M''}E}{dt^{M''}}\|_{L^2(\mathbf{R})}, \tag{2.21}$$

such that

$$\Gamma_{M''} = \{E|E\in C^{M''-1}(\mathbf{R}),\ E^{(M^{"})}\in L^2(\mathbf{R})\},$$

and λ is the regularization parameter. The coefficients $\{\theta_j\}_{j=1}^{M'}$ and $\{d_j\}_{j=1}^{M''}$ fulfill the following system of equations

$$E_\lambda(t_i) + 2(2M''-1)!(-1)^{M''}\lambda M'\theta_i - E_\xi(t_i) = 0, \quad i = 1, \ldots, M', \tag{2.22}$$

$$\sum_{j=1}^{M''} d_j t_j^i = 0, \quad i = 0, \ldots, M''-1. \tag{2.23}$$

The parameter λ can be selected as $\lambda = \xi^2$ which is a priori rule or is found by a posteriori choice strategy, namely the Morozov's discrepancy principle from the following equation

$$\frac{1}{M'}\sum_{p=1}^{M'}(E_\lambda(t_p)-E_\xi(t_p))^2 = \xi^2. \tag{2.24}$$

Thus, given the integer number M', we set $M''=2$ in the linear system of equations (2.20)-(2.23) and solve it to find $E_\lambda(t)$. Then, $\frac{d}{dt}\bigg(E_\lambda(t)\bigg)$ is calculated to achieve the approximation of $E'_\xi(t)$.

3 Analysis

In this section some arguments regarding the stability [1] of the presented method are given. Moreover, an upper bound concerning the estimation error of (2.19) is proved.

Lemma 3.1 *Assume $v^{k+1}(x) \in H^2([0,1])$, $k \in \{0, 1, \ldots, N-1\}$ with $N = Th$ as the solution of equation (2.8) with boundary conditions (2.11) subject to $v^{k+1}(x)$, $v_x^{k+1}(x)$, $v_{xx}^{k+1}(x)$ be bounded on* $[0, 1]$. *Furthermore, suppose that $\gamma_1, \gamma_2, \gamma_3$ are positive constants such that the following inequalities hold for $m \in \{0, 1, \ldots, N-1\}$:*

$$\max\{v_x^m(0), v_x^m(1)\} \le \gamma_1,\ v^m(1) \le \gamma_2\|v^m(x)\|_2,\ \|v_x^m\|_2\|v_x^{m+1}\|_2 \le \gamma_3\|v^{m+1}(x)\|_2, \tag{3.1}$$

then

$$\|v^{k+1}(x)\|_2 \le \|v^0(x)\|_2 + h(k+1)\bigg\{\gamma_1\gamma_2 + (1-\theta)\gamma_3 + \Re_F + 2\gamma_1\Re_G\bigg\}, \tag{3.2}$$

where $\Re_F = \max_{1\le l\le k+1}\|F^l(x)\|_2$, and $\Re_G = \max_{1\le l\le k+1}\|G^l(x)\|_2$.

Proof Recalling the notations

$$\langle p(x), q(x)\rangle := \int_0^1 p(x)q(x)dx, \quad \|p(x)\|_2^2 := \int_0^1 p(x)^2 dx,$$

and multiplying equation (2.8) by $v^{k+1}(x)$ and integrating over the interval [0, 1] we derive the following

$$\begin{aligned}&\|v^{k+1}(x)\|_2^2 - h\theta\langle v_{xx}^{k+1}(x), v^{k+1}(x)\rangle = \langle v^k(x), v^{k+1}(x)\rangle + h(1-\theta)\langle v_{xx}^k(x), v^{k+1}(x)\rangle \\ &+h\langle F^{k+1}(x), v^{k+1}(x)\rangle + h\left\{v_x^{k+1}(0)\langle G^{k+1}(x), v^{k+1}(x)\rangle - v_x^{k+1}(1)\langle G^{k+1}(x), v^{k+1}(x)\rangle\right\}. \end{aligned} \tag{3.3}$$

By applying integration by parts and using the condition $v^{k+1}(0) = 0$, we acquire

$$\begin{aligned}&\|v^{k+1}(x)\|_2^2 - h\theta\left(v^{k+1}(1)v_x^{k+1}(1) - \|v_x^{k+1}(x)\|_2^2\right) = h(1-\theta)v^{k+1}(1)v_x^k(1) \\ &+\langle v^k(x), v^{k+1}(x)\rangle + h\langle F^{k+1}(x), v^{k+1}(x)\rangle - h(1-\theta)\langle v_x^{k+1}(x), v_x^k(x)\rangle \\ &+h\left\{v_x^{k+1}(0)\langle G^{k+1}(x), v^{k+1}(x)\rangle - v_x^{k+1}(1)\langle G^{k+1}(x), v^{k+1}(x)\rangle\right\}. \end{aligned} \tag{3.4}$$

Using assumptions (3.1) in (3.4) gives

$$\begin{aligned}\|v^{k+1}(x)\|_2^2 \ \le\ &\|v^{k+1}(x)\|_2\|v^k(x)\|_2 + h\theta\gamma_1\gamma_2\|v^{k+1}(x)\|_2 + h(1-\theta)\|v^{k+1}(x)\|_2 + h(1-\theta)\gamma_3\|v^{k+1}(x)\|_2 \\ &+h\|F^{k+1}(x)\|_2\|v^{k+1}(x)\|_2 + 2h\gamma_1\|G^{k+1}(x)\|_2\|v^{k+1}(x)\|_2, \end{aligned} \tag{3.5}$$

or equivalently

$$\|v^{k+1}(x)\|_2 \le \|v^k(x)\|_2 + h\gamma_1\gamma_2 + h(1-\theta)\gamma_3 + h\|F^{k+1}(x)\|_2 + 2h\gamma_1\|G^{k+1}(x)\|_2, \ k \in \{0, \dots, N-1\}. \tag{3.6}$$

From relation (3.6), it follows that

$$\|v^{k+1}(x)\|_2 \le \|v^0(x)\|_2 + h(k+1)(\gamma_1\gamma_2 + (1-\theta)\gamma_3) + h\underbrace{\sum_{j=1}^{k+1}\|F^j(x)\|_2 + 2\gamma_1\|G^j(x)\|_2}_{\le (k+1)(\Re_F + 2\gamma_1\Re_G)}, \tag{3.7}$$

which completes the proof. □

Theorem 3.2 *Suppose that the assumptions of Lemma 3.1 hold and* $0 < \frac{1}{\gamma_3}(\frac{1}{T} - \gamma_1\gamma_2 - \Re_F - 2\gamma_1\Re_G) < 1$. *Then by considering* $v^{k+1}(x)$ *and* $V^{k+1}(x)$ *as the exact and approximate solutions of equation (2.8) with boundary conditions (2.11) and selecting*

$$0 < \theta \in \left[1 - \frac{1}{\gamma_3}(\frac{1}{T} - \gamma_1\gamma_2 - \Re_F - 2\gamma_1\Re_G), 1\right], \tag{3.8}$$

we have

$$\|\Lambda^{k+1}(x)\|_2 \le \|\Lambda^0(x)\|_2 + 1, \tag{3.9}$$

where

$$\Lambda^{k+1}(x) = V^{k+1}(x) - v^{k+1}(x). \tag{3.10}$$

Proof If (3.8) holds, then

$$1-\theta \le \frac{1}{\gamma_3}(\frac{1}{T}-\gamma_1\gamma_2-\Re_F-2\gamma_1\Re_G) \Longrightarrow (1-\theta)\gamma_3+\gamma_1\gamma_2+\Re_F+2\gamma_1\Re_G \le \frac{1}{hN}. \tag{3.11}$$

Furthermore, the combination of (3.7) and (3.11) gives

$$\|v^{k+1}(x)\|_2 \le \|v^0(x)\|_2 + \underbrace{h(k+1)\frac{1}{hN}}_{\le 1}. \tag{3.12}$$

We take $V^{k+1}(x)$ as the approximation of equation (2.8) and recall from (2.12) that $V^{k+1}(0) = V_x^{k+1}(0) + \alpha V_{xx}^{k+1}(1) = 0$. Hence, denoting the difference between $v^{k+1}(x)$ and $V^{k+1}(x)$ as the exact and approximate solutions of (2.8) by $\Lambda^{k+1}(x)$, it is observed that

$$\Lambda^{k+1}(x) - h\theta\nabla^2\Lambda^{k+1}(x) - h\Big\{F^{k+1}(x) + G^{k+1}(x)\Lambda_x^{k+1}(0) - G^{k+1}(x)\Lambda_x^{k+1}(1)\Big\}$$
$$= \Lambda^k(x) + h(1-\theta)\nabla^2\Lambda^k(x), \quad x \in [0,1], \tag{3.13}$$
$$\Lambda^{k+1}(0) = 0, \quad \Lambda_x^{k+1}(0) = -x\alpha\Lambda_{xx}^{k+1}(1). \tag{3.14}$$

Now, by applying Lemma 3.1 and Theorem 3.2 we conclude

$$\|\Lambda^{k+1}(x)\|_2 \le \|\Lambda^0(x)\|_2 + 1.$$

□

Theorem 3.3 *Let $V_{xx}^{k+1}(x)$ be the Lagrange interpolating polynomial at the distinct nodes $x_i \in [0,1]$, $i \in \{0,\dots,M\}$, which is considered as the approximation of $\nabla^2 v^{k+1}(x)$ in equation (2.9). Moreover, assume that the function $E(t)$ is contaminated with errors subject to the perturbation $E_\xi(t)$ and the regularized approximation [14] of its derivative denoted by $E_\lambda^{'}(t)$, possess the following property*

$$\max\{|E(t)-E_\xi(t)|, |E'(t)-E_\lambda^{'}(t)|\} \le \lambda^*, \tag{3.15}$$

and $|A(t)| \ge \varrho > 0$. Then, considering $r^*(t)$ as the approximation of $r(t)$ presented in equation (2.1) we have

$$|r(t^{k+1}) - r^*(t^{k+1})| \le \frac{\lambda^*+\epsilon}{\varrho}, \tag{3.16}$$

where ϵ is the upper bound of the interpolation error [25].

Proof From equation (2.1) we get

$$|r(t_{k+1}) - r^*(t_{k+1})| = \frac{|E'(t_{k+1}) - E_\lambda^{'}(t_{k+1}) + v_x^{k+1}(0) - V_x^{k+1}(0) + V_x^{k+1}(1) - v_x^{k+1}(1)|}{|A(t)|} \tag{3.17}$$
$$\le \frac{\lambda^* + \overbrace{|v_x^{k+1}(0) - V_x^{k+1}(0)|}^{:=\bigstar_1} + \overbrace{|V_x^{k+1}(1) - v_x^{k+1}(1)|}^{:=\bigstar_2}}{\varrho}. \tag{3.18}$$

In view of condition (3.14), we have $\bigstar_1 = 0$. From (3.10) we find that $\bigstar_2 = \Lambda_x^{k+1}(1)$. Therefore, we have

$$\Lambda_x^{k+1}(1) = \underbrace{\Lambda_x^{k+1}(0)}_{=\bigstar_1=0} + \int_0^1 \Lambda_{zz}^{k+1}(z)dz = \int_0^1 \Big(V_{zz}^{k+1}(z) - v_{zz}^{k+1}(z)\Big)dz \le \int_0^1 \epsilon dz = \epsilon. \tag{3.19}$$

□

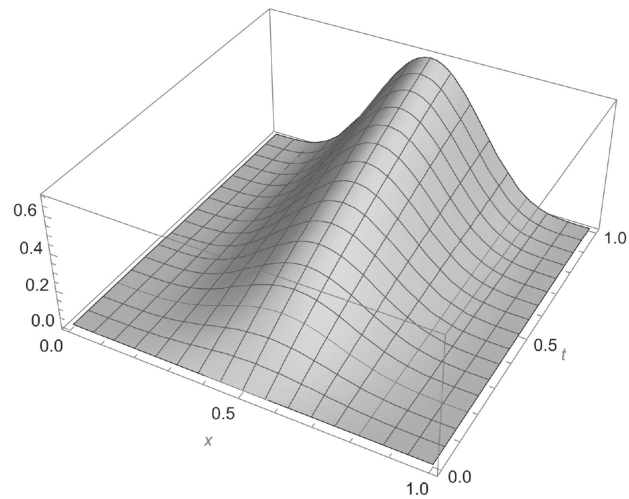

Fig. 1 Graph of the exact solution for $v(x, t)$, discussed in Example 4.1

4 Numerical Tests

We solve three problems in this section. The notations

$$\|e(v(x, t_i)\|_2 = \sqrt{\int_0^1 (v(x, t_i) - V(x, t_i))^2 dx}, \quad e(r(t)) = |r(t) - r^*(t)|, \quad \delta(r(t)) = \frac{e(r(t))}{r(t)}, \; r(t) \neq 0,$$

are utilized as criteria to measure the accuracy of computations. Numerical simulations are implemented in MATHEMATICA. We use commands such as, LinearSolve for solving the linear system of algebraic equations and $RandomReal[-1, 1]$ for producing random real numbers belonging to the interval $[-1, 1]$.

4.1 Example 1

Consider the inverse problem

$$v_t - v_{xx} = r(t) \frac{\frac{\sin^4(\pi x)}{1+t} - Ln(1+t)\left(3\pi^2 \sin^2(2\pi x) - 4\pi^2 \sin^4(\pi x)\right)}{1 + t^2}, \quad in \quad [0, 1] \times [0, 1], \tag{4.1}$$

with the initial condition

$$v(x, 0) = 0, \quad 0 \leq x \leq 1, \tag{4.2}$$

boundary conditions

$$v(0, t) = 0, \quad v_x(0, t) + 5v_{xx}(1, t) = 0, \quad 0 \leq t \leq 1, \tag{4.3}$$

and overspecification

$$\int_0^1 v(x, t) dx = \frac{3}{8} Ln(1 + t), \quad t \in [0, 1]. \tag{4.4}$$

We use the shifted Legendre basis functions and solve the problem by employing the numerical scheme presented in Section 2 with

$$M = 11, \; N = 25, \; h = 0.04, \; \theta = 0.5, \tag{4.5}$$

where the exact initial and boundary conditions are applied. Figures 1-5 depict the graphs of exact and approximate solutions of $v(x, t)$ and $r(t)$ at time levels t_i, the L^2 norm of absolute error of $v(x, t)$, i.e. $\|e(v(x, t_i)\|_2$, and the absolute and relative errors corresponding to the function $r(t)$. The approximations are in close agreement with the exact solutions.

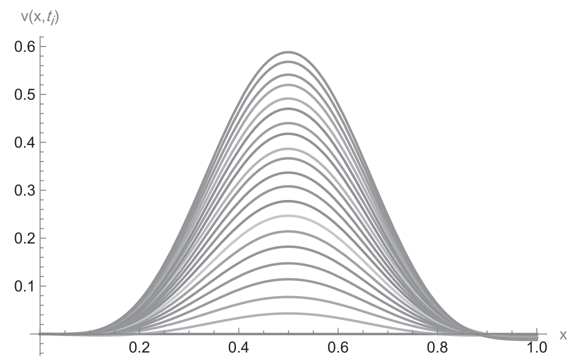

Fig. 2 This graph shows the approximate solution for $v(x,t)$ at time levels $t_i,\ i=0,\ldots,25$, obtained in the presence of exact input data by using the proposed method with $M=11,\ N=25,\ h=0.04,\ \theta=0.5$, discussed in Example 4.1

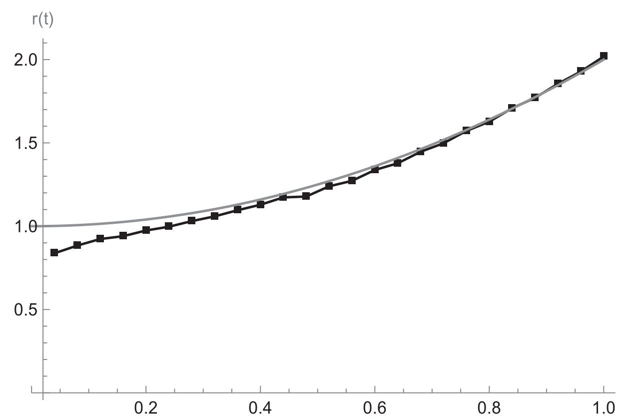

Fig. 3 This figure illustrates the exact (blue line) and approximate (■) solutions for $r(t)$ obtained by employing the proposed method in the presence of exact input data, discussed in Example 4.1

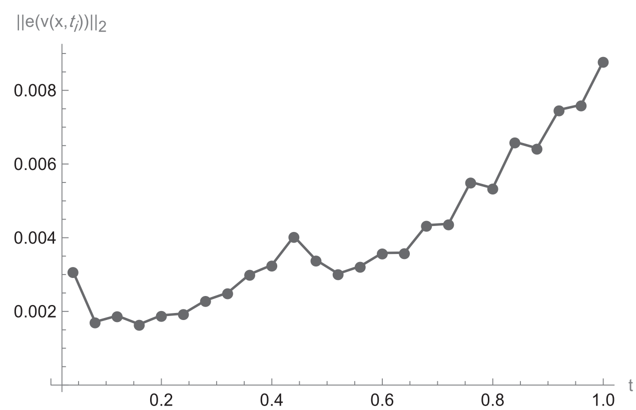

Fig. 4 This figure shows the L^2 norm of absolute error corresponding to the function $v(x,t)$ at time levels $t_i,\ i=1,\ldots,25$, discussed in Example 4.1

4.1.1 Example 2

Consider the problem given by equations (1.1)-(1.5) defined over the domain $\Omega_1=[0,1]\times[0,1]$ with the following properties:

$$f(x,t)=\sin^3(\pi x)e^{3t}-e^{2t}(1+e^t)(6\pi^2\sin(\pi x)\cos^2(\pi x)-3\pi^2\sin^3(\pi x)), \tag{4.6}$$

$$\alpha=5,\ E(t)=\frac{4(1+e^t)}{3\pi},\ \rho(x)=2\sin^3(\pi x), \tag{4.7}$$

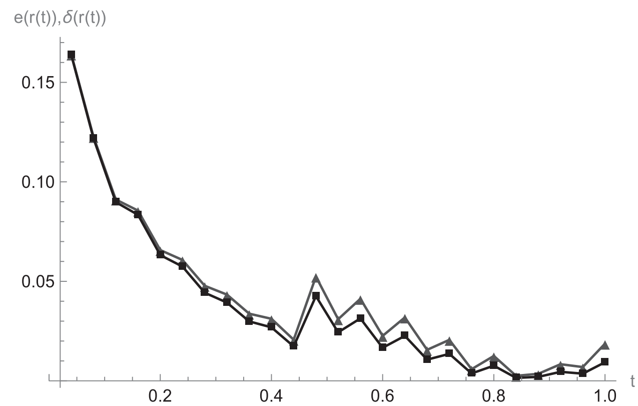

Fig. 5 This figure represents the graphs of the absolute (▲) and relative errors (■) corresponding to function $r(t)$ obtained by employing the proposed method in the presence of exact input data, discussed in Example 4.1

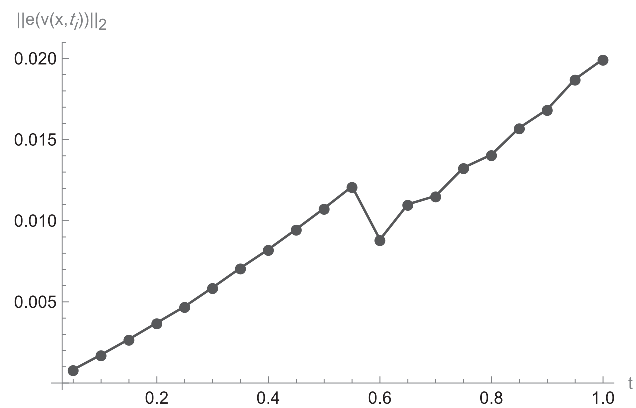

Fig. 6 This figure shows the L^2 norm of absolute error corresponding to the function $v(x, t)$ at time levels t_i, $i = 1, \ldots, 20$, discussed in Example 4.1.1

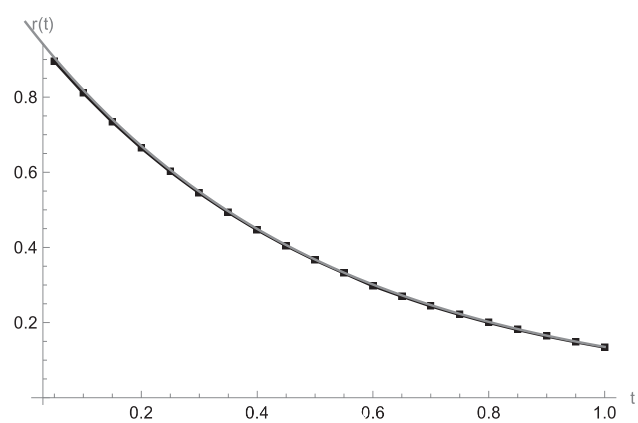

Fig. 7 This figure illustrates the exact (blue line) and approximate (■) solutions for $r(t)$ obtained by employing the proposed method in the presence of exact input data, discussed in Example 4.1.1

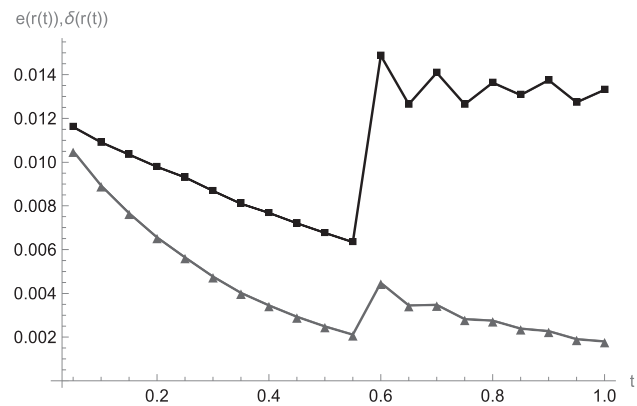

Fig. 8 This figure represents the graphs of the absolute (▲) and relative errors (■) corresponding to function $r(t)$, obtained by employing the proposed method in the presence of exact input data, discussed in Example 4.1.1

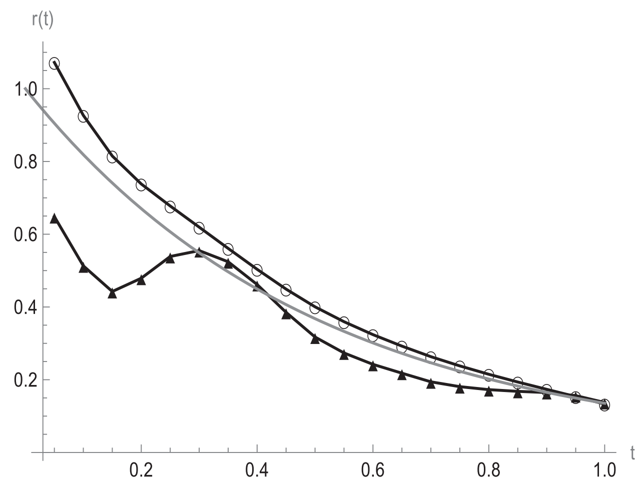

Fig. 9 The graph of the exact solution for $r(t)$, discussed in Example 4.1.1 is pictured by the blue line. Yielded approximations in the presence of the perturbed boundary data with different values ξ are depicted. The figure given by $\circ\circ\circ$ corresponds to the case $\xi = 0.02$ and ▲ represents the estimation when $\xi = 0.08$

where the exact solutions are

$$r(t) = e^{-2t}, \quad v(x,t) = \sin^3(\pi x)(1 + e^t).$$

By applying the presented technique in Section 2 with

$$M = 11,\ N = 20,\ h = 0.05,\ \theta = 0.5, \tag{4.8}$$

we derive the outcomes illustrated in Figures 6-8. Following the numerical findings, it can be seen that when no noise is introduced to the initial and boundary data, the proposed method provides accurate approximations.

We wish to observe the impact of inaccurate input data on the performance of the solution method. Hence, we contaminate the boundary conditions with artificial errors using the following rule

$$E_\xi(t_i) = E(t_i) + \xi \times RandomReal[-1, 1], \quad t_i \in [0, 1], \tag{4.9}$$

where $\xi \in \{2 \times 10^{-2}, 8 \times 10^{-2}\}$ is the percentage of noise. By solving the problem presented in Section 2 and utilizing the regularization technique given by equations (2.22)-(2.23) with the following parameters

$$M' = 25,\ M'' = 2,\ \lambda = \xi^2, \tag{4.10}$$

we get the results of approximating $r(t)$ depicted by Figure 9. It is seen that by applying the regularization method, numerical approximations agree well with the analytical solution when amount of noise, introduced into the boundary conditions are small (up to $\xi = 0.03$). In other words, for $\xi > 0.03$ the approximations deviate from the exact solutions and acceptable estimations are not obtained.

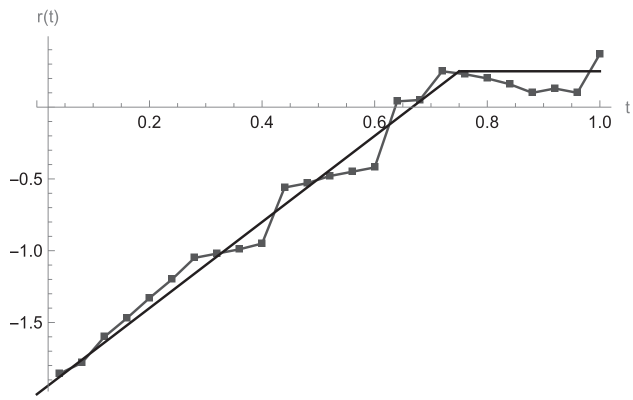

Fig. 10 This figure illustrates the exact (black line) and approximate (■) solutions for $r(t)$ obtained by employing the proposed method in the presence of exact input data, discussed in Example 4.1.2

4.1.2 Example 3

In this example, we aim to verify the efficiency of our method by solving an inverse problem with no analytical solution and non-smooth heat source function

$$r(t) = \begin{cases} -2+3t, & 0 \le t \le 0.75, \\ 0.25, & 0.75 < t \le 1, \end{cases} \quad f(x,t) = 1. \tag{4.11}$$

In this regard, by paying attention to conditions (1.6), we take

$$\rho(x) = x^2 - 2x, \quad \alpha = 1. \tag{4.12}$$

To get the measurement $E(t)$, we solve the direct problem (1.1)-(1.4) with data given by equations (4.11)-(4.12). That is we use the Ritz approximation [2,21–24]

$$v^{Dir}(x,t) = \sum_{i,j=0}^{9} tx\mu_{i,j} B_i(x) B_j(t) + \rho(x),$$

where $B_i(x)$, $B_j(t)$ are the basis functions of [22], and find the coefficients $\mu_{i,j}$ by minimizing the following functional

$$\varpi(v) = \left(\frac{\partial v^{Dir}(x,t)}{\partial t} - \frac{\partial^2 v^{Dir}(x,t)}{\partial x^2} - r(t) f(x,t) \right)^2 + \left(v_x^{Dir}(0,t) + v_{xx}^{Dir}(1,t) \right)^2.$$

Doing so, we get the approximation of $E(t)$ as

$$\begin{aligned} E(t) \simeq\ & -0.6666666666666667 - 0.001448286561721468t + 1.466944866075485t^2 \\ & -5.178386953305214t^3 + 26.502555817990853t^4 - 89.78659471997832t^5 + 173.12367111098956t^6 \\ & -185.03665045738035t^7 + 102.00561580854173t^8 - 22.625795706201302t^9. \end{aligned} \tag{4.13}$$

The inverse problem with $f(x,t) = 1$ and the data given by (4.12) and (4.13) is solved via the proposed method of Section 2. The features (4.5) are applied and the results are shown by Figure 10. One can see that the performance of our technique is good even for retrieving the non-smooth functions.

5 Conclusion

This paper explores the numerical solution of a linear inverse heat problem aimed at approximating a time-dependent heat source under integral overspecification and the presence of a Wentzell boundary condition. Initially, the problem is reformulated as a specific partial differential equation (PDE), and the θ-weighted finite difference method (FDM) is utilized to discretize the time derivative. Subsequently, the core problem is transformed into a system of second-order differential equations, which is then addressed through a spectral method ensuring compliance with all boundary conditions at each time step. The method incorporates regularization technique to approximate the first derivative of the perturbed boundary data. Three numerical tests are conducted, addressing the issue of numerical stability under slight noise introduced to the boundary condition (1.5). Results demonstrate the effectiveness of the proposed method, as it successfully recovers unknown functions when exact initial and boundary data are provided. Moreover, in cases of noisy boundary data, the obtained estimations align with analytical solutions. The versatility of the proposed algorithm allows for straightforward extension to tackle inverse problems with identical initial and boundary conditions as those examined in this study, governed by the following equation:

$$w_t - w_{xx} = r(t)w + f(x,t),$$

where recovering the pair $(r(t), w(x,t))$ is reduced to solve the studied problem (1.1)-(1.5), by employing the transformation $w(x,t) = v(x,t)e^{\int_0^t r(z)z}$. The method has the computational advantage of reducing the dimension of the main problem, and also can be considered as an efficient linearization technique to deal with non-linear problems.

Acknowledgements The author is very grateful to the referees for their valuable comments and helpful suggestions to improve the earlier version of this article.

Declarations

Conflict of interest This work does not have any conflicts of interest.

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