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ORIGINAL RESEARCH

Two weighted norm inequalities for generalized one-sided maximal function on one-sided weighted like Morrey space

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Abstract In this article, we study the boundedness of generalized one-sided maximal function, $\mathcal{M}_g^+$ on one-sided weighted like Morrey space, $M_{p,\alpha}^+$ for a pair of weights (u, v). We also discuss the Fefferman-Stein's type weighted inequalities for generalized one-sided maximal function on the same space. Finally, as a corollary, we obtain the Fefferman-Stein's type inequalities for generalized one-sided maximal function on one-sided weighted like Morrey space for a pair of weights.

Keywords One-sided maximal function · Generalized one-sided maximal function · Muckenhoupt weight · Weighted norm inequality · Morrey space · One-sided weighted like Morrey space

Mathematics Subject Classification 42B35 · 42B25 · 46E30

1 Introduction

Let $B_r = B(0, r)$ be the ball in $\mathbb{R}^n$ centered at the origin and radius r. For $f \in L^1_{loc}(\mathbb{R}^n)$, the classical Hardy-Littlewood maximal function is defined as

$$\mathcal{M}f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy, x \in \mathbb{R}^n$$

where the supremum is taken over all cubes Q having x inside and sides are parallel to the co-ordinate axes.

In $\mathbb{R}$, the classical Hardy-Littlewood maximal function is defined as

$$\mathcal{M}f(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y)|\, dy, x \in \mathbb{R}.$$

Considering the forward interval $[x, x + h)$, we can define the one-sided forward Hardy-Littlewood maximal function as

$$\mathcal{M}^+f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(y)|\, dy, x \in \mathbb{R}.$$

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Similarly, by considering the backward interval $[x-h, x)$ in the above definition we can define the one-sided backward Hardy-Littlewood maximal function as

$$\mathcal{M}^{-} f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)|\, dy,\ x \in \mathbb{R}.$$

Let g be a positive locally integrable function on $\mathbb{R}$. For $f \in L^1_{loc,g}(\mathbb{R})$ where $L^1_{loc,g}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} | \int_K |f|\, g\, dx < \infty, \forall K \text{ compact set in } \mathbb{R} \right\}$ and $x_0 \in \mathbb{R}$, the generalized one-sided maximal function is defined as

$$\mathcal{M}^{+}_{g}(f)(x) = \sup_{h>0} \left(\int_{x}^{x+h} g(y)dy \right)^{-1} \left(\int_{x}^{x+h} |f(y)|\, g(y)dy \right).$$

Similarly, replacing the forward interval with the backward interval $[x-h, x)$ in the above definition we get the generalized one-sided backward maximal function, defined as

$$\mathcal{M}^{-}_{g}(f)(x) = \sup_{h>0} \left(\int_{x-h}^{x} g(y)dy \right)^{-1} \left(\int_{x-h}^{x} |f(y)|\, g(y)dy \right).$$

One-sided maximal function plays an important role in ergodic theory. The weighted theory for one-sided maximal function was first studied by E. Sawyer [11]. Following Sawyer's work there were significant developments in the field of one-sided function [5,6,15]. In case of one-sided maximal function the problem is yet to be solved for $\mathbb{R}^n$, $n \geq 3$. The aim of this article is to study the boundedness of generalized one-sided maximal function on one-sided weighted like Morrey space.

The idea of Morrey space was first introduced by C.B.Morrey [7] in 1938 to investigate the local behaviour of solutions to the second order elliptic Partial Differential Equations. A function t is said to be in Morrey space $M^{p,\alpha}$ for $1 \leq p < \infty$ if t is measurable and t satisfies

$$\|t\|_{M^{p,\alpha}} = \left(\sup_{x \in \mathbb{R}^n} r^{\alpha - n} \int_{B(x,r)} |t(y)|^p\, dy \right)^{1/p} < \infty,$$

where $0 < \alpha \leq n$. For $\alpha = n$, Morrey space coincides with the Lebesgue space. For a weight function w on $\mathbb{R}^n$, the weighted Morrey space $M^{p,\alpha}_{w}$ is defined as the set of all measurable functions t such that

$$\|t\|_{M^{p,\alpha}_{w}} = \sup_{B} \left(\frac{1}{w(B)^{\alpha}} \int_{B} |t(x)|^p\, w(x)dx \right)^{1/p} < \infty,$$

where $0 < \alpha < 1$ and $1 \leq p < \infty$. Shi and Fu [13] introduced one-sided weighted like Morrey spaces and Camponato spaces in order to study the boundedness of one-sided Hardy-Littlewood maximal function on one-sided weighted like Morrey space. For $1 \leq p < \infty$, the one-sided weighted like Morrey space is defined as the set of all locally integrable functions t such that

$$\|t\|_{M^{+}_{p,\alpha}} = \sup_{x_0} \sup_{h>0} \frac{1}{h^{\alpha}} \left(\frac{1}{w(x_0 - h, x_0)} \int_{x_0}^{x_0+h} |t(y)|^p\, dy \right)^{1/p} < \infty.$$

Wang [14] studied the vector-valued inequalities for the maximal function as well as the weighted maximal function on weighted Morrey spaces. Recently, Duoandikoetxea and Rosenthal [2] proved that the Muckenhoupt type condition is necessary and sufficient for the boundedness of the maximal function on weighted local Morrey space.

The boundedness of generalized one-sided maximal function for a pair of weights is not yet studied on one-sided weighted like Morrey space which motivate us to study boundedness of this type of function. At the end of this article, we prove Fefferman-Stein's type weighted inequalities for generalized one-sided maximal function on one-sided weighted like Morrey space for a pair of weights.

Throughout this article, for a positive and locally integrable function u, $u(F)$ represents the integral $\int_F u(x)dx$, where F is any measurable set, C denotes an independent positive constant not necessarily same at each occurance. For $h, \lambda > 0$ and $x_0 \in \mathbb{R}$, we denote I^{-}, I^{+}, I, λI as $(x_0 - h, x_0)$, $(x_0, x_0 + h)$, $(x_0 - h, x_0 + h)$ and $(x_0, x_0 + \lambda h)$ respectively.

We organize the article as follows. In Section 2, we discuss some existing definitions, lemmas and theorems. In Section 3, we state and prove our main results.

2 Preliminaries

In this section we recall some basic definitions and results which are used in subsequent sections.

In 1970, Muckenhoupt [8] introduced A_p weight class to characterize the weight u for which the classical Hardy-Littlewood maximal function is bounded on weighted Lebesgue space. As an extension, Sawyer [11] introduced A_p^+ weight class to show the boundedness of one-sided Hardy-Littlewood maximal function on weighted Lebesgue space.

Definition 2.1 Let u be a non-negative locally integrable function on $\mathbb{R}^n$. Then

(i) u is said to be in A_p weight class if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|Q|}\int_Q u(x)dx\right)\left(\frac{1}{|Q|}\int_Q u(x)^{-1/p-1}dx\right)^{p-1} \leq C; 1 < p < \infty$$

for all cubes Q with sides parallel to the co-ordinate axes.
And u is said to satisfy A_1 condition if

$$\mathcal{M}u(x) \leq Cu(x) \text{ a.e.}$$

(ii) u is said to be in A_p^+ weight class if for all $x_0 \in \mathbb{R}$ and $h > 0$ there exists a constant $C > 0$ such that
$\left(\frac{1}{h}\int_{x_0-h}^{x_0} u(x)dx\right)\left(\frac{1}{h}\int_{x_0}^{x_0+h} u(x)^{-1/p-1}dx\right)^{p-1} \leq C; 1 < p < \infty, \forall x_0 \in \mathbb{R}.$
And the weight u is said to satisfy A_1^+ condition if

$$\mathcal{M}^- u(x) \leq Cu(x) \text{ a.e.}$$

(iii) u is said to be in A_p^- weight class if for all $x_0 \in \mathbb{R}$ and $h > 0$ there exists a constant $C > 0$ such that

$$\left(\frac{1}{h}\int_{x_0}^{x_0+h} u(x)dx\right)\left(\frac{1}{h}\int_{x_0-h}^{x_0} u(x)^{-1/p-1}dx\right)^{p-1} \leq C; 1 < p < \infty.$$

And the weight u is said to satisfy A_1^- condition if

$$\mathcal{M}^+ u(x) \leq Cu(x) \text{ a.e.}$$

Muckenhoupt [8] proved the following inequalities for the classical Hardy-Littlewood maximal function.

Theorem 2.1 *[8, Theorem 2] Let $f \in S(\mathbb{R}^n)$ and $1 \leq p < \infty$. Then*

(i) *the weighted strong (p, p) inequality holds*

$$\left(\int_{\mathbb{R}^n} \mathcal{M}f(x)^p u(x)dx\right)^{1/p} \leq C\left(\int_{\mathbb{R}^n} f(x)^p u(x)dx\right)^{1/p}$$

if and only if $u(x) \in A_p$.

(ii) *for $p = 1$, the weighted weak $(1, 1)$ inequality holds*

$$u\left(\{x : \mathcal{M}f(x) > \alpha\}\right) \leq \frac{C}{\alpha}\|f\|_{1,u}$$

if and only if $u(x) \in A_1$.

Here $S(\mathbb{R}^n)$ denotes the space of Schwartz class functions on $\mathbb{R}^n$.

Sawyer [11] proved the following boundedness result for the one-sided maximal function.

Theorem 2.2 [11, Theorem 1] *Let $1 < p < \infty$. Then the following weighted inequality holds*

$$\left(\int_{\mathbb{R}} \left(\mathcal{M}^+ f\right)(x)^p u(x)dx\right)^{1/p} \leq C\left(\int_{\mathbb{R}} |f(x)|^p u(x)dx\right)^{1/p}$$

if and only if $u(x) \in A_p^+$, where C is a non-zero positive constant.

Definition 2.2 Let u and v be two positive locally integrable functions on $\mathbb{R}^n$. Then

(i) the pair (u, v) is said to satisfy A_p condition if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|Q|}\int_Q u(x)dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-1/p-1}dx\right)^{p-1} \leq C; 1 < p < \infty.$$

for all cubes Q in $\mathbb{R}^n$.
And (u, v) is said to satisfy A_1 condition if there exists a constant $C > 0$ such that

$$\mathcal{M}u(x) \leq Cv(x) \text{ a.e.}$$

(ii) the pair (u, v) is said to satisfy A_p^+ condition if there exists a constant $C > 0$ such that

$$\left(\frac{1}{h}\int_{x_0-h}^{x_0} u(x)dx\right)\left(\frac{1}{h}\int_{x_0}^{x_0+h} v(x)^{-1/p-1}dx\right)^{p-1} \leq C; 1 < p < \infty.$$

And (u, v) is said to satisfy A_1^+ condition if there exists a constant $C > 0$ such that

$$\mathcal{M}^- u(x) \leq Cv(x) \text{ a.e.}$$

(iii) the pair (u, v) is said to satisfy A_p^- condition if there exists a constant $C > 0$ such that

$$\left(\frac{1}{h}\int_{x_0}^{x_0+h} u(x)dx\right)\left(\frac{1}{h}\int_{x_0-h}^{x_0} v(x)^{-1/p-1}dx\right)^{p-1} \leq C; 1 < p < \infty.$$

And (u, v) is said to satisfy A_1^- condition if there exists a constant $C > 0$ such that

$$\mathcal{M}^+ u(x) \leq Cv(x) \text{ a.e.}$$

Muckenhoupt [9] proved the following boundedness result for the classical Hardy-Littlewood maximal function in case of pair weights.

Theorem 2.3 [9, Theorem 1] *Given a pair of weights* (u, v) *and for* $1 < p < \infty$, $\lambda > 0$ *the following statements are equivalent*

(i) *the strong* (p, p) *inequality*

$$\left(\int_{\mathbb{R}} |\mathcal{M}f(x)|_q^p\, u(x)dx\right)^{1/p} \leq C\left(\int_{\mathbb{R}} |f(x)|_q^p\, v(x)dx\right)^{1/p}$$

holds if $(u, v) \in A_p$.

(ii) *the weak* (p, p) *inequality*

$$u\left(\left\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\right\}\right) \leq \frac{C}{\lambda^p}\int_{\mathbb{R}^n} |f(x)|^p\, v(x)dx$$

holds if and only if $(u, v) \in A_p$.

Martin-Reyes et al. [5] introduced two new weight classes $A_p^+(g)$ and $S_p^+(g)$ in order to study the boundedness of generalized one-sided maximal function $\mathcal{M}_g^+$ on weighted Lebesgue space.

Definition 2.3 Let u be a non-negative locally integrable function on $\mathbb{R}^n$. Then

(i) u satisfies $A_p^+(g)$ condition if there exists a constant $C>0$ such that $\left(\int_{I^-} u(x)dx\right)\left(\int_{I^+} g^{p'}(x)u^{-p'/p}(x)dx\right)^{p-1} \leq C\left(\int_I g(x)dx\right)^p$; $p > 1$, p' is the conjugate of p.
And u satisfies $A_1^+(g)$ condition if the following inequality holds

$$\mathcal{M}_g^-\left(\frac{u}{g}\right)(x) \leq C\left(\frac{u}{g}\right)(x) \text{ a.e.}$$

(ii) u satisfies $A_p^-(g)$ condition if there exists a constant $C > 0$ such that $\left(\int_{I^+} u(x)dx\right)\left(\int_{I^-} g^{p'}(x)u^{-p'/p}(x)dx\right)^{p-1} \leq C\left(\int_I g(x)dx\right)^p$; $p > 1$, p' is the conjugate of p.

And u satisfies $A_1^-(g)$ condition if the following inequality holds

$$\mathcal{M}_g^+\left(\frac{u}{g}\right)(x) \leq C\left(\frac{u}{g}\right)(x) \text{ a.e.}$$

Definition 2.4 Let (u, v) be a pair of weights on $\mathbb{R}$, $1 < p < \infty$, p' is the conjugate of p. Then

(i) (u, v) is said to satisfy $A_p^+(g)$ condition if there exists a constant $C > 0$ such that

$$\left(\int_{I^-} u(x)dx\right)\left(\int_{I^+} g^{p'}(x)v^{-p'/p}(x)dx\right)^{p-1} \leq C\left(\int_I g(x)dx\right)^p.$$

And the pair (u, v) is said to satisfy $A_1^+(g)$ condition if

$$\mathcal{M}_g^-\left(\frac{u}{g}\right)(x) \leq C\left(\frac{v}{g}\right)(x) \text{ a.e.}$$

(ii) (u, v) is said to satisfy $A_p^-(g)$ condition if there exists a constant $C > 0$ such that

$$\left(\int_{I^+} u(x)dx\right)\left(\int_{I^-} g^{p'}(x)v^{-p'/p}(x)dx\right)^{p-1} \leq C\left(\int_I g(x)dx\right)^p.$$

And the pair (u, v) is said to satisfy $A_1^-(g)$ condition if

$$\mathcal{M}_g^+\left(\frac{u}{g}\right)(x) \leq C\left(\frac{v}{g}\right)(x) \text{ a.e.}$$

Martin-Reyes et.al [5] proved the boundedness of generalized one-sided maximal function in case of a pair of weights. They proved that $A_p^+(g)$ condition is necessary and sufficient for weak (p, p) estimate of generalized one-sided maximal function for a pair of weights but the condition is not sufficient for strong (p, p) estimate of generalized one-sided maximal function with different weights. For this reason they introduced a new weight class $S_p^+(g)$ in order to study strong (p, p) estimate of generalized one-sided maximal function for a pair of weights and proved that $S_p^+(g)$ condition is necessary and sufficient for strong (p, p) estimate of generalized one-sided maximal function.

Definition 2.5 A pair of weight (u, v) is said to satisfy $S_p^+(g)$ condition if there exists a constant $C > 0$ such that $\int_I \left(\mathcal{M}_g^+\left(\chi_I g^{p'/p}(x)v^{-p'/p}(x)\right)(x)\right)^p u(x)dx \leq C\int_I g^{p'}(x)v^{-p'/p}(x)dx < \infty$, p' is the conjugate of p.

Theorem 2.4 [5, Theorem 1] *$\mathcal{M}_g^+$ is of weak type (p, p), $1 \leq p < \infty$ with respect to the measures vdx and udx if and only if the pair (u, v) satisfies $A_p^+(g)$ condition.*

Theorem 2.5 [5, Theorem 2] *$\mathcal{M}_g^+$ is of strong type (p, p), $p > 1$ from $L^p(v)$ to $L^p(u)$ if and only if (u, v) satisfies $S_p^+(g)$ condition.*

Theorem 2.6 [5, Theorem 3] *For $u = v$ and $p > 1$, $A_p^+(g)$ and $S_p^+(g)$ are equivalent conditions.*

Theorem 2.7 [5, Theorem 7] *If $f \in L^p(u)$ then $\mathcal{M}_g^+$ is bounded operator from $L^p(u)$ to itself i.e.*

$$\left\|\mathcal{M}_g^+ f\right\|_{L^p(u)} \leq C\|f\|_{L^p(u)}; 1 < p < \infty$$

if and only if $u \in A_p^+(g)$.

Definition 2.6 Let $1 \leq p < \infty$ and $-1/p \leq \alpha < 1$. A locally integrable function f is said to belong to Camponato space $C^{p,\alpha}(\mathbb{R})$ if

$$\|f\|_{C^{p,\alpha}(\mathbb{R})} = \sup_I \|f\|_{C^{p,\alpha}(I)} =: \sup_I \frac{1}{|I|^\alpha} \left(\frac{1}{|I|}\int_I |f - f_I|^p \, dx\right)^{1/p} < \infty,$$

where $f_I = \left(\frac{1}{|I|}\right)\int_I f(x)dx$, I denotes an interval contained in $\mathbb{R}$ and $|I|$ is the Lebesgue measure of I.

Definition 2.7 Let $1 \leq p < \infty$ and $-1/p \leq \alpha < 1$. A locally integrable function f is said to be in one-sided weighted Camponato space $C^+_{p,\alpha}(w)$ if

$$\|f\|_{C^+_{p,\alpha}(w)} = \sup_{x_0}\sup_{h>0} \frac{1}{h^\alpha}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h} \left|f(y) - f_{(x_0,x_0+h)}\right|^p dy\right)^{1/p} < \infty,$$

where $w(x_0-h,x_0) = \int_{x_0-h}^{x_0} w(x)dx,\ x_0 \in \mathbb{R}$.

Definition 2.8 Let $-1/p \leq \alpha < 0$ and $1 \leq p < \infty$. The one-sided weighted Morrey space $M^+_{p,\alpha}(w)$ is defined by the norm

$$\|f\|_{M^+_{p,\alpha}(w)} = \sup_{x_0}\sup_{h>0} \frac{1}{h^\alpha}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h} |f(y)|^p \, dy\right)^{1/p} < \infty.$$

It shows that $M^+_{p,\alpha}(w) \subseteq C^+_{p,\alpha}(w)$ in the sense that the following is true

$$\|f\|_{C^+_{p,\alpha}(w)} \leq C \sup_{h>0} \frac{1}{h^\alpha}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h} |f(y)-a|^p \, dy\right)^{1/p}, \forall a \in \mathbb{R}, \alpha < 0.$$

Theorem 2.8 [13, Theorem 7(a)] *Let $-1/p \leq \alpha < 0$ and $w \in A^+_p$. Then there exists a constant $C > 0$ such that the following inequality holds*

$$\left\|\mathcal{M}^+ f\right\|_{M^+_{p,\alpha}(w)} \leq C \|f\|_{C^+_{p,\alpha}(w)}$$

for $1 < p < 1/1+\alpha$.

Lemma 2.9 [3, Lemma 2] *Let $p > 1$ and f, u be two non-negative locally integrable functions on $\mathbb{R}^n$. If $f \in L^p(\mathcal{M}u)$ then there exists a constant $C_{FS} > 0$ for which the following inequality holds*

$$\|\mathcal{M}f\|_{L^p(u)} \leq C_{FS} \|f\|_{L^p(\mathcal{M}u)}.$$

In other words,

$$\left(\int_{\mathbb{R}^n} \mathcal{M}f(x)^p u(x)dx\right)^{1/p} \leq C_{FS}\left(\int_{\mathbb{R}^n} f(x)^p \mathcal{M}u(x)dx\right)^{1/p}.$$

Lemma 2.10 [3, Lemma 1] *Let f and u be two non-negative real valued functions on $\mathbb{R}^n$ and $r > 1$. Then there exists a constant $C_r > 0$ such that*

$$\int_{\mathbb{R}^n} \mathcal{M}f(x)^r u(x)dx \leq C_r \int_{\mathbb{R}^n} f(x)^r \mathcal{M}u(x)dx.$$

Lemma 2.9 has a wide application in A_p weight theory. It was first studied by Fefferman and Stein known as Fefferman-Stein's weighted inequality. Shrivastava [12] proved the Fefferman-Stein's weighted inequality for one-sided maximal function on weighted Lebesgue space then extended the result as an analogue in vector valued set up. Recently, Iida and Nakamura [4] introduced the necessary and sufficient condition for existence of Fefferman-Stein's type inequality in Morrey space. After Shrivastava the extended version of Fefferman-Stein's weighted inequality for generalized one-sided maximal function on weighted Lebesgue space was studied by Chutia and Haloi [1]. Peréz [10] extended Lemma 2.10. to vector valued setting and pointed out that l_r-valued extension of this lemma is not valid. We also obtain same version of Fefferman-Stein's type inequalities that extends the existing results.

3 Main Result

In this section we prove the main result of the article and begin with following.

Assumption 3.1 If $1 \leq p < \infty$, $-1/p \leq \alpha < 0$ and $(u, v) \in A_p^+(g)$ then there exists a constant $C > 0$ such that

$$\int_{x_0-h}^{x_0+h} v(x)dx \leq C \int_{x_0}^{x_0+h} u(x)dx$$

for any positive locally integrable function g.

Lemma 3.2 *Let $(u, v) \in A_p^+(g)$ for $p \geq 1$. If $g((\lambda I))^p \left(\int_{\lambda I} v(x) g(x)^{-p} dx \right) \leq u(I)$ then $u((\lambda I)^-) \leq C\lambda^p u(I)$ for $\lambda > 0$, where C depends on h.*

Proof Using Hölder's inequality, $1/p + 1/p' = 1$ and then applying $A_p^+(g)$ condition for a pair of weights (u, v) we have

$$\begin{aligned}
\left(\frac{1}{|I|}\int_I |f(x)|\,dx\right)^p &= \left(\frac{1}{|I|}\int_I |f(x)|\, v(x)^{1/p} g(x) v(x)^{-1/p} g(x)^{-1} dx\right)^p \\
&\leq \left(\frac{1}{|I|^p}\int_I |f(x)|^p\, v(x) g(x)^{-p} dx\right)\left(\int_I v(x)^{-p'/p} g(x)^{p'} dx\right)^{p/p'} \\
&= \left(\frac{1}{|I|}\int_{I^-} u(x)dx\right)\left(\frac{1}{|I|}\int_I g(x)^{p'} v(x)^{-p'/p} dx\right)^{p-1} \\
&\quad \times \left(\frac{1}{u(I^-)}\int_I |f(x)|^p\, v(x) g(x)^{-p} dx\right) \\
&\leq C\left(\int_I g(x)dx\right)^p \left(\frac{1}{u(I^-)}\int_I |f(x)|^p\, v(x) g(x)^{-p} dx\right).
\end{aligned}$$

Now, putting λI in place of I and χ_I in place of f and applying the given condition we get

$$\begin{aligned}
\left(\frac{1}{|\lambda I|}\int_{\lambda I} |\chi_I(x)|\,dx\right)^p &\leq C\left(\int_{\lambda I} g(x)dx\right)^p \times \left(\frac{1}{u((\lambda I)^-)}\int_{\lambda I} |\chi_I(x)|^p\, v(x) g(x)^{-p} dx\right) \\
\leftrightarrow \frac{1}{|\lambda I|^p}\left(\int_{\lambda I} dx\right)^p &\leq C\left(\int_{\lambda I} g(x)dx\right)^p \times \left(\frac{1}{u((\lambda I)^-)}\int_{\lambda I} v(x) g(x)^{-p} dx\right) \\
\leftrightarrow u((\lambda I)^-) &\leq C\,\lambda^p \left(\int_{\lambda I} g(x)dx\right)^p \times \left(\int_{\lambda I} v(x) g(x)^{-p} dx\right) \\
&= C\lambda^p (g(\lambda I))^p \times \left(\int_{\lambda I} v(x) g(x)^{-p} dx\right) \\
&\leq C\lambda^p u(I)
\end{aligned}$$

Thus we get the desired result. □

Theorem 3.3 *Let $1 \leq p < \infty$, $-1/p \leq \alpha < 0$. If $(u, v) \in A_p^+(g)$ then there exists a constant $C > 0$ such that*

$$\left\| \mathcal{M}_g^+ f \right\|_{M_{p,\alpha}^+(u)} \leq C \left\| f \right\|_{M_{p,\alpha}^+(v)}.$$

Proof We shall decompose f into two different functions and then estimate them separately. We decompose f as

$$f = f_A + f_B = f_{\chi_{2I}} + f_{\chi_{(2I)^c}}.$$

Now,

$$\begin{aligned}
\left\|\mathcal{M}_g^+ f(x)\right\|_{M^+_{p,\alpha}(u)} &= \frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p}\\
&= \frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ (f_A+f_B))(x)\right|^p dx\right)^{1/p}\\
&\le \frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f_A)(x)+(\mathcal{M}_g^+ f_B)(x)\right|^p dx\right)^{1/p}\\
&\le \frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f_A)(x)\right|^p dx\right)^{1/p}\\
&\quad+\frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f_B)(x)\right|^p dx\right)^{1/p}\\
&= I+II.
\end{aligned}$$

Applying Assumption 3.1,

$$\begin{aligned}
I &= \frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f_A)(x)\right|^p dx\right)^{1/p}\\
&\le \frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p}\\
&\le \frac{C_1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+2h}|f(x)|^p dx\right)^{1/p}\\
&= \frac{C_1}{h^\alpha}\left(\frac{v(x_0-2h,x_0)}{u(x_0-h,x_0)}\right)^{1/p}\left(\frac{1}{v(x_0-2h,x_0)}\int_{x_0}^{x_0+2h}|f(x)|^p dx\right)^{1/p}\\
&\le \frac{C}{h^\alpha}\left(\frac{1}{v(x_0-2h,x_0)}\int_{x_0}^{x_0+2h}|f(x)|^p dx\right)^{1/p}\\
&= C\,\|f\|_{M^+_{p,\alpha}(v)}.
\end{aligned}$$

Using Lemma 3.2 and condition $-1/p \le \alpha < 0$ we have

$$\begin{aligned}
II &= \frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f_B)(x)\right|^p dx\right)^{1/p}\\
&\le \frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p}\\
&\le \frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\right)^{1/p}\sum_{i=1}^{\infty}\left(\frac{1}{2^i}\int_{x_0-2h}^{x_0+2^{i+1}h}|f(x)|^p dx\right)^{1/p}\\
&\le \sum_{i=1}^{\infty}\frac{1}{2^i(1/p-\alpha-1)}\left(\frac{v(x_0-h,x_0)}{u(x_0-h,x_0)}\right)^{1/p}\times\frac{1}{h^\alpha}\left(\frac{1}{v(x_0-h,x_0)}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}\\
&\le C\,\|f\|_{M^+_{p,\alpha}(v)}.
\end{aligned}$$

Therefore, combining the estimates of I and II we get the desired result. □

Next we proof some Fefferman-Stein's type inequalities for generalized one-sided maximal function on one-sided weighted like Morrey space for a pair of weights.

Theorem 3.4 *For $1 \leq p < \infty$. Let u, v and g be three positive locally integrable functions. If g is a monotonically decreasing function and $u \in A_1^+(g)$ then there exists a constant $C_{FS} > 0$ such that*

$$\left\| \mathcal{M}_g^+ f \right\|_{M_{p,\alpha}^+(u)} \leq C_{FS} \|f\|_{M_{p,\alpha}^+(\mathcal{M}_g^-(v/g)g)}.$$

Proof It is sufficient to prove the following inequality,

$$\frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p} \leq C_{FS}\frac{1}{h^\alpha}\left(\frac{1}{\mathcal{M}_g^-(\frac{v}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}.$$

We have,

$$\begin{aligned}
&\frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p}\\
&= \frac{1}{h^\alpha}\left(\frac{\mathcal{M}_g^-(\frac{u}{g})(x_0-h,x_0)g}{u(x_0-h,x_0)\mathcal{M}_g^-(\frac{u}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p}\\
&= \frac{1}{h^\alpha}\left(\frac{\mathcal{M}_g^-(\frac{u}{g})(x_0-h,x_0)g}{u(x_0-h,x_0)}\right)^{1/p}\left(\frac{1}{\mathcal{M}_g^-(\frac{u}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p}\\
&\leq \frac{1}{h^\alpha}\left(\frac{\mathcal{M}_g^-(\frac{u}{g})(x_0-h,x_0)g}{u(x_0-h,x_0)}\right)^{1/p}\left(\frac{1}{\mathcal{M}_g^-(\frac{v}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}\\
&\leq C_{FS}\frac{1}{h^\alpha}\left(\frac{1}{\mathcal{M}_g^-(\frac{v}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}.
\end{aligned}$$

□

Theorem 3.5 *For $1 \leq p < \infty$. Let u, v and g be three positive locally integrable functions. If g be a monotonically decreasing function and $u \in A_1^-(g)$ then there exists a constant $C_{FS} > 0$ such that*

$$\left\| \mathcal{M}_g^+ f \right\|_{M_{p,\alpha}^+(u)} \leq C_{FS} \|f\|_{M_{p,\alpha}^+(\mathcal{M}_g^+(v/g)g)}.$$

Proof It is sufficient to prove the following inequality,

$$\frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p} \leq C_{FS}\frac{1}{h^\alpha}\left(\frac{1}{\mathcal{M}_g^+(\frac{v}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}.$$

We have,

$$\begin{aligned}
&\frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+f)(x)\right|^p dx\right)^{1/p}\\
&=\frac{1}{h^\alpha}\left(\frac{\mathcal{M}_g^+(\frac{u}{g})(x_0-h,x_0)g}{u(x_0-h,x_0)\mathcal{M}_g^+(\frac{u}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+f)(x)\right|^p dx\right)^{1/p}\\
&=\frac{1}{h^\alpha}\left(\frac{\mathcal{M}_g^+(\frac{u}{g})(x_0-h,x_0)g}{u(x_0-h,x_0)}\right)^{1/p}\left(\frac{1}{\mathcal{M}_g^+(\frac{u}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+f)(x)\right|^p dx\right)^{1/p}\\
&\le\frac{1}{h^\alpha}\left(\frac{\mathcal{M}_g^+(\frac{u}{g})(x_0-h,x_0)g}{u(x_0-h,x_0)}\right)^{1/p}\left(\frac{1}{\mathcal{M}_g^+(\frac{v}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}\\
&\le C_{FS}\frac{1}{h^\alpha}\left(\frac{1}{\mathcal{M}_g^+(\frac{v}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}.
\end{aligned}$$

□

Theorem 3.6 *For $1\le p<\infty$. Let u, v and g be three positive locally integrable functions. If g is a monotonically decreasing function and $(u,v)\in A_1^+(g)$ then there exists a constant $C_{FS}>0$ such that*

$$\left\|\mathcal{M}_g^+f\right\|_{M_{p,\alpha}^+(v)}\le C_{FS}\|f\|_{M_{p,\alpha}^+(\mathcal{M}_g^-(v/g)g)}.$$

Proof It is sufficient to prove the following inequality,

$$\begin{aligned}
&\frac{1}{h^\alpha}\left(\frac{1}{v(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+f)(x)\right|^p dx\right)^{1/p}\\
&\le C_{FS}\frac{1}{h^\alpha}\left(\frac{1}{\mathcal{M}_g^-(\frac{v}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}.
\end{aligned}$$

We have,

$$\begin{aligned}
&\frac{1}{h^\alpha}\left(\frac{1}{v(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+f)(x)\right|^p dx\right)^{1/p}\\
&=\frac{1}{h^\alpha}\left(\frac{\mathcal{M}_g^-(\frac{u}{g})(x_0-h,x_0)g}{v(x_0-h,x_0)\mathcal{M}_g^-(\frac{u}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+f)(x)\right|^p dx\right)^{1/p}\\
&=\frac{1}{h^\alpha}\left(\frac{\mathcal{M}_g^-(\frac{u}{g})(x_0-h,x_0)g}{v(x_0-h,x_0)}\right)^{1/p}\left(\frac{1}{\mathcal{M}_g^-(\frac{u}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+f)(x)\right|^p dx\right)^{1/p}\\
&\le\frac{1}{h^\alpha}\left(\frac{\mathcal{M}_g^-(\frac{u}{g})(x_0-h,x_0)g}{v(x_0-h,x_0)}\right)^{1/p}\left(\frac{1}{\mathcal{M}_g^-(\frac{v}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}\\
&\le C_{FS}\frac{1}{h^\alpha}\left(\frac{1}{\mathcal{M}_g^-(\frac{v}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}.
\end{aligned}$$

□

Theorem 3.7 *For $1\le p<\infty$. Let u, v and g be three positive locally integrable functions. If g is a monotonically decreasing function and $(u,v)\in A_1^-(g)$ then there exists a constant $C_{FS}>0$ such that*

$$\left\|\mathcal{M}_g^+f\right\|_{M_{p,\alpha}^+(v)}\le C_{FS}\|f\|_{M_{p,\alpha}^+(\mathcal{M}_g^+(v/g)g)}.$$

Proof It is sufficient to prove the following inequality,

$$\frac{1}{h^\alpha}\left(\frac{1}{v(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p} \le C_{FS}\frac{1}{h^\alpha}\left(\frac{1}{\mathcal{M}_g^+(\frac{v}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}.$$

We have,

$$\begin{aligned}
&\frac{1}{h^\alpha}\left(\frac{1}{v(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p}\\
&= \frac{1}{h^\alpha}\left(\frac{\mathcal{M}_g^+(\frac{u}{g})(x_0-h,x_0)g}{v(x_0-h,x_0)\mathcal{M}_g^+(\frac{u}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p}\\
&= \frac{1}{h^\alpha}\left(\frac{\mathcal{M}_g^+(\frac{u}{g})(x_0-h,x_0)g}{v(x_0-h,x_0)}\right)^{1/p}\left(\frac{1}{\mathcal{M}_g^+(\frac{u}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p}\\
&\le \frac{1}{h^\alpha}\left(\frac{\mathcal{M}_g^+(\frac{u}{g})(x_0-h,x_0)g}{v(x_0-h,x_0)}\right)^{1/p}\left(\frac{1}{\mathcal{M}_g^+(\frac{v}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}\\
&\le C_{FS}\frac{1}{h^\alpha}\left(\frac{1}{\mathcal{M}_g^+(\frac{v}{g})(x_0-h,x_0)g}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}.
\end{aligned}$$

□

Theorem 3.8 *For $1 \le p < \infty$. Let u, v and g be three positive locally integrable functions. If g is a monotonically decreasing function and $u \in A_1$ then there exists a constant $C_{FS} > 0$ such that*

$$\left\|\mathcal{M}_g^+ f\right\|_{M_{p,\alpha}^+(u)} \le C_{FS}\,\|f\|_{M_{p,\alpha}^+(\mathcal{M}v)}.$$

Proof It is sufficient to prove the following inequality,

$$\frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p} \le C_{FS}\frac{1}{h^\alpha}\left(\frac{1}{\mathcal{M}v(x_0-h,x_0)}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}.$$

We have,

$$\begin{aligned}
&\frac{1}{h^\alpha}\left(\frac{1}{u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p}\\
&= \frac{1}{h^\alpha}\left(\frac{\mathcal{M}u(x_0-h,x_0)}{u(x_0-h,x_0)\mathcal{M}u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p}\\
&= \frac{1}{h^\alpha}\left(\frac{\mathcal{M}u(x_0-h,x_0)}{u(x_0-h,x_0)}\right)^{1/p}\left(\frac{1}{\mathcal{M}u(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|(\mathcal{M}_g^+ f)(x)\right|^p dx\right)^{1/p}\\
&\le \frac{1}{h^\alpha}\left(\frac{\mathcal{M}u(x_0-h,x_0)}{u(x_0-h,x_0)}\right)^{1/p}\left(\frac{1}{\mathcal{M}v(x_0-h,x_0)}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}\\
&\le C_{FS}\frac{1}{h^\alpha}\left(\frac{1}{\mathcal{M}v(x_0-h,x_0)}\int_{x_0}^{x_0+h}|f(x)|^p dx\right)^{1/p}.
\end{aligned}$$

□

Theorem 3.9 *For $1 \leq p < \infty$. Let u, v and g be three positive locally integrable functions. If g is a monotonically decreasing function and $(u, v) \in A_1(g)$ then there exists a constant $C_{FS} > 0$ such that*

$$\left\| \mathcal{M}_g^+ f \right\|_{M^+_{p,\alpha}(v)} \leq C_{FS} \| f \|_{M^+_{p,\alpha}(\mathcal{M}_g(v/g)g)} .$$

Proof It is sufficient to prove the following inequality,

$$\frac{1}{h^\alpha} \left(\frac{1}{v(x_0 - h, x_0)} \int_{x_0}^{x_0+h} \left| (\mathcal{M}_g^+ f)(x) \right|^p dx \right)^{1/p} \leq C_{FS} \frac{1}{h^\alpha} \left(\frac{1}{\mathcal{M}_g(\frac{v}{g})(x_0 - h, x_0)g} \int_{x_0}^{x_0+h} |f(x)|^p \, dx \right)^{1/p} .$$

We have,

$$\begin{aligned}
&\frac{1}{h^\alpha} \left(\frac{1}{v(x_0 - h, x_0)} \int_{x_0}^{x_0+h} \left| (\mathcal{M}_g^+ f)(x) \right|^p dx \right)^{1/p} \\
&= \frac{1}{h^\alpha} \left(\frac{\mathcal{M}_g(\frac{u}{g})(x_0 - h, x_0)g}{v(x_0 - h, x_0)\mathcal{M}_g(\frac{u}{g})(x_0 - h, x_0)g} \int_{x_0}^{x_0+h} \left| (\mathcal{M}_g^+ f)(x) \right|^p dx \right)^{1/p} \\
&= \frac{1}{h^\alpha} \left(\frac{\mathcal{M}_g(\frac{u}{g})(x_0 - h, x_0)g}{v(x_0 - h, x_0)} \right)^{1/p} \left(\frac{1}{\mathcal{M}_g(\frac{u}{g})(x_0 - h, x_0)g} \int_{x_0}^{x_0+h} \left| (\mathcal{M}_g^+ f)(x) \right|^p dx \right)^{1/p} \\
&\leq \frac{1}{h^\alpha} \left(\frac{\mathcal{M}_g(\frac{u}{g})(x_0 - h, x_0)g}{v(x_0 - h, x_0)} \right)^{1/p} \left(\frac{1}{\mathcal{M}_g(\frac{v}{g})(x_0 - h, x_0)g} \int_{x_0}^{x_0+h} |f(x)|^p \, dx \right)^{1/p} \\
&\leq C_{FS} \frac{1}{h^\alpha} \left(\frac{1}{\mathcal{M}_g(\frac{v}{g})(x_0 - h, x_0)g} \int_{x_0}^{x_0+h} |f(x)|^p \, dx \right)^{1/p} .
\end{aligned}$$

□

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Declarations

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