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ORIGINAL RESEARCH

Wavelet Packet analysis associated with the generalized Weinstein operator

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Abstract In this present paper, we consider the generalized Weinstein operator $\Delta_W^{d,\alpha,n}$, we introduce and study some types of wavelet packets and the corresponding wavelet transforms. We prove for these transforms the Plancherel formulas, the reconstruction formulas and their scale discrete scaling functions.

Keywords Generalized Weinstein operator · Generalized Weinstein transform · Weinstein Wavelet transform · Wavelet packets · Scale discrete functions

Mathematics Subject Classification 42B10 · 42C40 · 42A38 · 43A32 · 44A05 · 44A35

1 Introduction

In this paper, we consider the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ defined on $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times (0,\ +\infty)$, by:

$$\Delta_W^{\alpha,d,n} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} - \frac{4n\,(\alpha+n)}{x_{d+1}^2} = \Delta_W^{\alpha,d} - \frac{4n\,(\alpha+n)}{x_{d+1}^2}, \tag{1.1}$$

where $n \in \mathbb{N}$, $\alpha > -\frac{1}{2}$ and $\Delta_W^{\alpha,d}$ the classical Weinstein operator given by:

$$\Delta_W^{\alpha,d} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} = \Delta_d + L_\alpha. \tag{1.2}$$

Here L_α is the Bessel operator for the last variable defined on $(0,\ +\infty)$ by:

$$L_\alpha = \frac{\partial^2}{\partial x_{d+1}^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} = \frac{1}{x_{d+1}^{2\alpha+1}} \frac{\partial}{\partial x_{d+1}} \left[x_{d+1}^{2\alpha+1} \frac{\partial}{\partial x_{d+1}} \right].$$

(See [1–3,9,10]).

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The generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ is the function given by:

$$\forall x,\, y\in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d,n}(x,y) = x_{d+1}^{2n} e^{-i\langle x',y'\rangle}\, j_{\alpha+2n}(x_{d+1}y_{d+1}), \tag{1.3}$$

where $x=(x',x_{d+1})$, $x'=(x_1,x_2,\ldots,x_d)$ and j_α is the normalized Bessel function of index α defined by:

$$\forall \xi\in\mathbb{C},\ j_\alpha(\xi) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha+1)}(\frac{\xi}{2})^{2n}. \tag{1.4}$$

Using the Weinstein kernel $\Lambda_{\alpha,d,n}$, we define the Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ by:

$$\forall \lambda\in\mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x)\Lambda_{\alpha,d,n}(x,\lambda)d\mu_{\alpha,d}(x),$$

where $x_{d+1}^{2n} f\in \mathbb{L}^1(\mathbb{R}_+^{d+1},\mu_{\alpha,d}(x))$ and $\mu_{\alpha,d}$ is the measure on $\mathbb{R}_+^{d+1}$ given by:

$$d\mu_{\alpha,d}(x) = x_{d+1}^{2\alpha+1}dx. \tag{1.5}$$

Very recently, many authors have investigated some type of wavelet packets associated with generalized Fourier transforms for instance see [2,8,11–13].

The main objective of this paper is to introduce and studt three types of wavelet packets associated with generalized Weinstein operator $\Delta_W^{d,\alpha,n}$. The contents of the paper are as follows:

In the Sect. 2, we recapitulate some results related to the harmonic analysis associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ given by the relation (1.1).

The Sect. 3, is devoted to introduce the first type of wavelet packets associated with generalized Weinstein operator. We give some harmonic analysis properties for it. We prove for the P-wavelet packet transform $\Phi_\psi^{P,W}$ associated with $\Delta_W^{d,\alpha,n}$ the Plancherel formula, the reconstruction formula.

In the Sect. 4, we introduce the scale discrete scaling function and we give some properties for it.

In the Sect. 5, we study the second type of wavelet packets. We prove for this transform the Plancherel formula and the reconstruction formula.

In the Sect. 6, we introduce the S-wavelet packet, the corresponding dual S-wavelet packet and the S-wavelet pack transforms.

2 Preliminaires

In this section, we shall collect some results and definitions from the theory of the harmonic analysis associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ defined on $\mathbb{R}_+^{d+1}$ by the relation (1.1).

Notations. In what follows, we need the following notations:

- $\mathscr{E}_*(\mathbb{R}^{d+1})$, the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{S}_*(\mathbb{R}^{d+1})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{D}_*(\mathbb{R}^{d+1})$, the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^{d+1}$ which are of compact support, even with respect to the last variable.
- $\mathscr{H}_*(\mathbb{C}^{d+1})$, the space of entire functions on $\mathbb{C}^{d+1}$, even with respect to the last variable, rapidly decreasing and of exponential type.
- $\mathscr{M}_n$, the map defined by:

$$\forall x\in\mathbb{R}_+^{d+1},\ \mathscr{M}_n(f)(x) = x_{d+1}^{2n} f(x), \tag{2.1}$$

where $x=(x',x_{d+1})$ and $x'=(x_1,x_2,\ldots,x_d)$

- $\mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$, $1\le p\le+\infty$, the space of measurable functions on $\mathbb{R}_+^{d+1}$ such that

$$\begin{aligned}\|f\|_{\alpha,n,p} &:= \left[\int_{\mathbb{R}_+^{d+1}} |\mathscr{M}_n^{-1}f(x)|^p d\mu_{\alpha+2n,d}(x)\right]^{\frac{1}{p}} < +\infty,\ \text{if } 1\le p<+\infty,\\ \|f\|_{\alpha,n,\infty} &:= \operatorname*{ess\,sup}_{x\in\mathbb{R}_+^{d+1}} \left|\mathscr{M}_n^{-1}f(x)\right| < +\infty\end{aligned}$$

where $\mu_{\alpha,d}$ is the measure given by the relation (1.5).

• $\mathbb{L}_\alpha^p(\mathbb{R}_+^{d+1}) := L_{\alpha,0}^p(\mathbb{R}_+^{d+1})$, $1 \le p \le +\infty$, and $\|f\|_{\alpha,p} := \|f\|_{\alpha,0,p}$.
• $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$, $\mathscr{D}_{n,*}(\mathbb{R}^{d+1})$ and $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ repespectively stand for the subspace of $\mathscr{E}_*(\mathbb{R}^{d+1})$, $\mathscr{D}_*(\mathbb{R}^{d+1})$ and $\mathscr{S}_*(\mathbb{R}^{d+1})$ consisting of functions f such that

$$\forall k \in \{1, \dots, 2n-1\}, \ \frac{\partial^k f}{\partial x_{d+1}^k}(x', 0) = f(x', 0) = 0.$$

Let us begin by the following result.

Lemma 2.1 (See [4]–[7])

i) The map $\mathscr{M}_n$ is an isomorphism from $\mathscr{E}_*(\mathbb{R}^{d+1})$ $\left(resp.\ \mathscr{S}_*(\mathbb{R}^{d+1})\right)$ onto $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$ $\left(resp.\ \mathscr{S}_{n,*}(\mathbb{R}^{d+1})\right)$.
ii) For all $f \in \mathscr{E}_*(\mathbb{R})$, we have

$$L_{\alpha,n} \circ \mathscr{M}_n(f) = \mathscr{M}_n \circ L_{\alpha+2n}(f), \tag{2.2}$$

where

$$L_{\alpha,n} = \frac{\partial^2}{\partial x_{d+1}^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} - \frac{4n(\alpha+n)}{x_{d+1}^2}. \tag{2.3}$$

iii) For all $f \in \mathscr{E}_*(\mathbb{R}^{d+1})$, we have

$$\Delta_W^{\alpha,d,n} \circ \mathscr{M}_n(f) = \mathscr{M}_n \circ \Delta_W^{\alpha+2n}(f). \tag{2.4}$$

Definition 2.1 i) The generalized Weinstein intertwining operator is the operator $\mathscr{R}_W^{\alpha,d,n}$ defined on $\mathscr{E}_*(\mathbb{R}^{d+1})$ by:

$$\forall x \in \mathbb{R}_+^{d+1}, \ \mathscr{R}_W^{\alpha,d,n} f(x) = a_{\alpha+2n} x_{d+1}^{2n} \int_0^1 \left(1-t^2\right)^{\alpha+2n-\frac{1}{2}} f(x', t x_{d+1}) dt, \tag{2.5}$$

where a_α is the constant given by the relation:

$$a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}. \tag{2.6}$$

ii) The dual of the Weinstein intertwining operator is defined on $\mathscr{D}_*(\mathbb{R}^{d+1})$ by:

$$\forall y \in \mathbb{R}_+^{d+1}, \ {}^t\mathscr{R}_W^{\alpha,d,n}(f)(y) = a_{\alpha+2n} \int_{y_{d+1}}^{+\infty} \left(s^2 - y_{d+1}^2\right)^{\alpha+2n-\frac{1}{2}} f(y', s) s^{1-2n} ds. \tag{2.7}$$

Definition 2.2 The generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ is the function given by:

$$\forall x, y \in \mathbb{C}^{d+1}, \ \Lambda_{\alpha,d,n}(x, y) = x_{d+1}^{2n} e^{-i\langle x', y'\rangle} j_{\alpha+2n}(x_{d+1} y_{d+1}), \tag{2.8}$$

where $x = (x', x_{d+1})$, $x' = (x_1, x_2, \dots, x_d)$ and j_α is given by the relation (1.4).

Remark 2.1 The function $\Lambda_{\alpha,d,n}$ can be written in the form:

$$\forall x, y \in \mathbb{C}^{d+1}, \ \Lambda_{\alpha,d,n}(x, y) = \mathscr{M}_n \Lambda_{\alpha+2n,d}(., y)(x), \tag{2.9}$$

where

$$\forall x, y \in \mathbb{C}^{d+1}, \ \Lambda_{\alpha,d}(x, y) = e^{-i\langle x', y'\rangle} j_\alpha(x_{d+1} y_{d+1}). \tag{2.10}$$

Using the properties of the the classical Weinstein kernel $\Lambda_{\alpha,d} : (\lambda, y) \mapsto \Lambda_{\alpha,d}(x, y)$, it is easy to see that the generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ has a unique extention to $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ and satisifies the following properties.

Proposition 2.1 *i) We have*

$$\forall x, y \in \mathbb{R}_+^{d+1}, \ |\Lambda_{\alpha,d,n}(x, y)| \le x_{d+1}^{2n}. \tag{2.11}$$

ii) The function $x \mapsto \Lambda_{\alpha,d,n}(x, y)$ satisfies the differentiel equation

$$\Delta_W^{\alpha,d,n}\left(\Lambda_{\alpha,d,n}(., y)\right)(x) = -\|y\|^2 \Lambda_{\alpha,d,n}(x, y). \tag{2.12}$$

iii) For all $x,\ y \in \mathbb{C}^{d+1}$, *we have*

$$\Lambda_{\alpha,d,n}(x,y) = a_{\alpha+2n} e^{-i\langle x',y'\rangle} x_{d+1}^{2n} \int_0^1 \left(1-t^2\right)^{\alpha+2n-\frac{1}{2}} \cos(t x_{d+1} y_{d+1}) dt, \tag{2.13}$$

where a_α *is the constant given by the relation* (2.6).

Definition 2.3 The generalized Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ is given for $f \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}_+^{d+1})$ by:

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x) \Lambda_{\alpha,d,n}(x,\lambda) d\mu_{\alpha,d}(x), \tag{2.14}$$

where $\mu_{\alpha,d}$ is the measure on $\mathbb{R}_+^{d+1}$ given by the relation (1.5).

Example 2.1 Let $\psi_{s,n},\ s>0,\ n \in \mathbb{N}$, be the function defined by:

$$\forall x \in \mathbb{R}^{d+1},\ \psi_{s,n}(x) = C_{\alpha+2n,d} \left(2s\,(2\alpha+4n+d+1) - \|x\|^2\right) \frac{x_{d+1}^{2n}}{(2s)^{\alpha+2n+\frac{d}{2}+1}} e^{-\frac{\|x\|^2}{4s}}, \tag{2.15}$$

where $C_{\alpha,d}$ is the constant given by:

$$C_{\alpha,d} = \frac{1}{(2\pi)^{\frac{d}{2}}\, 2^\alpha \Gamma(\alpha+1)}. \tag{2.16}$$

Then the Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ of $\psi_{s,n}$ is given by:

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}(\psi_{s,n})(\lambda) = \|\lambda\|^2 e^{-s\|\lambda\|^2}. \tag{2.17}$$

Remark 2.2 Using the relations (2.9) and (2.14), the generalized Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ can be written in the form:

$$\mathscr{F}_W^{\alpha,d,n} = \mathscr{F}_W^{\alpha+2n,d} \circ \mathscr{M}_n^{-1}, \tag{2.18}$$

where $\mathscr{F}_W^{\alpha,d}$ is the classical Weinstein transform given by:

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x) \Lambda_{\alpha,d}(x,\lambda) d\mu_{\alpha,d}(x). \tag{2.19}$$

Some basic properties of the transform $\mathscr{F}_W^{\alpha,d,n}$ are summarized in the following results. For the proofs, we refer to [4–7].

Proposition 2.2 (See [4]–[7])

i) For all $f \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}_+^{d+1})$, *we have*

$$\|\mathscr{F}_W^{\alpha,d,n}(f)\|_{\alpha,n,\infty} \le \|f\|_{\alpha,n,1}. \tag{2.20}$$

ii) Let $f \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}_+^{d+1})$, *we have*

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left(\overline{f}\right)(\lambda) = \overline{\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{f}\right)(\lambda)}, \tag{2.21}$$

and

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left(\widetilde{f}\right)(\lambda) = \mathscr{F}_W^{\alpha,d,n}(f)(-\lambda), \tag{2.22}$$

where

$$\forall \lambda \in \mathbb{R}^{d+1},\ \widetilde{f}(x) = f(-x). \tag{2.23}$$

iii) Let $f \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}_+^{d+1})$. *If* $\mathscr{F}_W^{\alpha,d,n}(f) \in \mathbb{L}^1_{\alpha+2n}(\mathbb{R}_+^{d+1})$, *then we have*

$$f(x) = C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(f)(y) \Lambda_{\alpha,d,n}(-x,y) d\mu_{\alpha+2n,d}(y),\ a.e\ x \in \mathbb{R}_+^{d+1}, \tag{2.24}$$

where $C_{\alpha,d}$ *is the constant given by the relation* (2.16).

iv) The Weinstein transform $\mathcal{F}_W^{\alpha,d,n}$ is a topological isomorphism from $\mathcal{S}_{n,*}(\mathbb{R}^{d+1})$ onto itself and from $\mathcal{D}_{n,*}(\mathbb{R}^{d+1})$ onto $\mathcal{H}_*(\mathbb{C}^{d+1})$.

Theorem 2.1 (See [4]–[7])

i) For all $f, g \in \mathcal{S}_{n,*}(\mathbb{R}^{d+1})$, we have the following Parseval formula:

$$\int_{\mathbb{R}_+^{d+1}} f(x)\overline{g(x)}d\mu_{\alpha,d}(x) = C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_W^{\alpha,d,n}(f)(\lambda)\overline{\mathcal{F}_W^{\alpha,d,n}(g)(\lambda)}d\mu_{\alpha+2n,d}(\lambda), \tag{2.25}$$

where $C_{\alpha,d}$ is the constant given by the relation (2.16).

ii) (Plancherel formula).
For all $f \in \mathcal{S}_{n,*}(\mathbb{R}^{d+1})$, we have:

$$\int_{\mathbb{R}_+^{d+1}} |f(x)|^2 \, d\mu_{\alpha,d}(x) = C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} \left|\mathcal{F}_W^{\alpha,d,n}(f)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda). \tag{2.26}$$

iii) (Plancherel Theorem):
The transform $\mathcal{F}_W^{\alpha,d,n}$ extends uniquely to an isomorphism from $\mathbb{L}_\alpha^2(\mathbb{R}_+^{d+1})$ onto $\mathbb{L}_{\alpha+2n}^2(\mathbb{R}_+^{d+1})$.

Definition 2.4 The translation operator $T_x^{\alpha,d,n}$, $x \in \mathbb{R}_+^{d+1}$, associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ is defined on $\mathcal{E}_{n,*}(\mathbb{R}_+^{d+1})$ by:

$$\forall y \in \mathbb{R}_+^{d+1}, \; T_x^{\alpha,d,n} f\,(y) = x_{d+1}^{2n} \mathcal{M}_n T_x^{\alpha+2n,d} \mathcal{M}_n^{-1} f\,(y) \tag{2.27}$$

where

$$T_x^{\alpha,d} f\,(y) = \frac{a_\alpha}{2} \int_0^\pi f\left(x' + y', \; \sqrt{x_{d+1}^2 + y_{d+1}^2 + 2x_{d+1}y_{d+1}\cos\theta}\right)(\sin\theta)^{2\alpha}\, d\theta, \tag{2.28}$$

$x' + y' = (x_1 + y_1, \ldots, x_d + y_d)$ and a_α is the constant given by (2.6).

The following proposition summarizes some properties of the generalized Weinstein translation operator.

Proposition 2.3 (See [4]–[7])

i) For all $f \in \mathcal{E}_{n,*}(\mathbb{R}^{d+1})$ and $y \in \mathbb{R}_+^{d+1}$, the function $x \mapsto T_x^{\alpha,d} f\,(y)$ belongs to $\mathcal{E}_{n,*}(\mathbb{R}^{d+1})$.

ii) Let $f \in \mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$, $1 \le p \le +\infty$ and $x \in \mathbb{R}_+^{d+1}$. Then $T_x^{\alpha,d,n} f$ belongs to $\mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$ and we have

$$\|T_x^{\alpha,d,n} f\|_{\alpha,n,p} \le x_{d+1}^{2n} \|f\|_{\alpha,n,p}. \tag{2.29}$$

iii) The function $\Lambda_{\alpha,d,n}(.,\lambda)$, $\lambda \in \mathbb{C}^{d+1}$, satisfies on $\mathbb{R}_+^{d+1}$ the following product formula:

$$\forall x, y \in \mathbb{R}_+^{d+1}, \; \Lambda_{\alpha,d,n}(x,\lambda)\,\Lambda_{\alpha,d,n}(y,\lambda) = T_x^{\alpha,d,n}\left[\Lambda_{\alpha,d,n}(.,\lambda)\right](y). \tag{2.30}$$

iv) Let $f \in \mathcal{S}_{n,*}(\mathbb{R}^{d+1})$ and $x \in \mathbb{R}_+^{d+1}$, we have

$$\forall y \in \mathbb{R}_+^{d+1}, \; \mathcal{F}_W^{\alpha,d,n}\left(T_x^{\alpha,d,n} f\right)(y) = \Lambda_{\alpha,d,n}(-x, y)\,\mathcal{F}_W^{\alpha,d,n}(f)(y). \tag{2.31}$$

Definition 2.5 The generalized Weinstein convolution product of $f, g \in \mathbb{L}_{\alpha,n}^1(\mathbb{R}_+^{d+1})$ is given by :

$$f *_{\alpha,n} g\,(x) = \int_{\mathbb{R}_+^{d+1}} T_x^{\alpha,d,n} f\,(-y)\, g\,(y)\, d\mu_{\alpha,d}(y), \; x \in \mathbb{R}_+^{d+1}. \tag{2.32}$$

Proposition 2.4 (See [4]–[7])

i) Let $p, q, r \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then for all $f \in \mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$ and $g \in L_{\alpha,n}^q(\mathbb{R}_+^{d+1})$, the function $f *_{\alpha,n} g \in \mathbb{L}_{\alpha,n}^r(\mathbb{R}_+^{d+1})$ and we have

$$\|f *_{\alpha,n} g\|_{\alpha,n,r} \le \|f\|_{\alpha,n,p} \|g\|_{\alpha,n,q}. \tag{2.33}$$

ii) For all $f, g \in \mathbb{L}_{\alpha,n}^1(\mathbb{R}_+^{d+1})$, $f *_{\alpha,n} g \in \mathbb{L}_{\alpha,n}^1(\mathbb{R}_+^{d+1})$ and we have

$$\mathcal{F}_W^{\alpha,d,n}(f *_{\alpha,n} g) = \mathcal{F}_W^{\alpha,d,n}(f)\mathcal{F}_W^{\alpha,d,n}(g). \tag{2.34}$$

3 P-wavelet packets associated with generalized Weinstein operator

In this section, we introduce the first type of wavelet packets associated with generalized Weinstein operator. We give some harmonic analysis properties for it. More precisely, we prove for the P-wavelet packet transform $\Phi_{\psi}^{P,W}$ associated with $\Delta_W^{d,\alpha,n}$ the Plancherel formula, Parseval formula and the reconstruction formula.

Definition 3.1 A generalized Weinstein wavelet on $\mathbb{R}_+^{d+1}$ is a function $\psi \in \mathbb{L}_\alpha^2(\mathbb{R}^{d+1})$ satisfying for almost all $x \in \mathbb{R}_+^{d+1}$, the condition:

$$0 < C_\psi^{\alpha,n} = \int_0^{+\infty} |\mathscr{F}_W^{\alpha,d,n}(\psi)(tx)|^2 \frac{dt}{t} < +\infty. \tag{3.1}$$

Example 3.1 The function $\psi_{s,n},\ s > 0,\ n \in \mathbb{N}$, given by the relation (2.15) is a generalized Weinstein wavelet on $\mathbb{R}_+^{d+1}$ and using the relation (2.17), we obtain

$$\forall s > 0, \forall n \in \mathbb{N},,\ C_{\psi_{s,n}}^{\alpha,n} = \frac{1}{8s^2}.$$

Definition 3.2 Let $t > 0$. We define the dilation operator $\mathscr{D}_t$ of a measurable function $\psi : \mathbb{R}_+^{d+1} \to \mathbb{C}$ by:

$$\forall x \in \mathbb{R}^{d+1}, \quad \mathscr{D}_t\psi(x) := \psi_t(x) := \frac{1}{t^{2\alpha+2n+d+2}}\psi(\frac{x}{t}). \tag{3.2}$$

It's obvious that these operators satisfy the following properties.

Proposition 3.1 *i) Let ψ be a measurable function on $\mathbb{R}_+^{d+1}$, for all $s, t > 0$, we have*

$$\forall x \in \mathbb{R}^{d+1},\ \mathscr{D}_t\mathscr{D}_s\psi(x) = \mathscr{D}_{st}\psi(x). \tag{3.3}$$

ii) Let $t > 0$ and ψ be a measurable function on $\mathbb{R}_+^{d+1}$, we have

$$|\mathscr{D}_t\psi|^2 = \frac{1}{t^{2\alpha+2n+d+2}}\mathscr{D}_t\left(|\psi|^2\right). \tag{3.4}$$

iii) Let $t > 0$. For all ψ in $\mathbb{L}_\alpha^p(\mathbb{R}^{d+1})$, the function $\mathscr{D}_t\psi$ belongs also to $\mathbb{L}_\alpha^p(\mathbb{R}^{d+1})$ and we have

$$||\mathscr{D}_t\psi||_{\alpha,p} = t^{\left(\frac{1}{p}-1\right)(2\alpha+d+2)-2n}||\psi||_{\alpha,p}. \tag{3.5}$$

In particular, we have

$$||\mathscr{D}_t\psi||_{\alpha,2} = \frac{1}{t^{\alpha+2n+\frac{d}{2}+1}}||\psi||_{\alpha,2},\ \psi \in \mathbb{L}_\alpha^2(\mathbb{R}^{d+1}). \tag{3.6}$$

iv) Let $t > 0$. For all $\varphi, \psi \in \mathbb{L}_\alpha^2(\mathbb{R}_+^{d+1})$, we have

$$\langle\mathscr{D}_t\psi,\ \varphi\rangle_{\alpha,2} = \frac{1}{t^{2\alpha+4n+d+2}}\langle\psi,\ \mathscr{D}_{\frac{1}{t}}\varphi\rangle_{\alpha,2}, \tag{3.7}$$

where

$$\langle\psi,\ \varphi\rangle_{\alpha,2} = \int_{\mathbb{R}_+^{d+1}} \psi(\xi)\overline{\varphi(\xi)}d\mu_{\alpha,d}(\xi). \tag{3.8}$$

v) Let $t > 0$ and $x \in \mathbb{R}_+^{d+1}$. We have

$$\mathscr{D}_t T_x^{\alpha,d,n} = T_{\frac{x}{t}}^{\alpha,d,n}\mathscr{D}_t. \tag{3.9}$$

vi) Let $t > 0$ and $\psi \in \mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$, $p = 1, 2$, the function $\psi_t := \mathscr{D}_t\psi$ belongs to $\mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$ and we have

$$\forall \lambda \in \mathbb{R}^{d+1},\ \mathscr{F}_W^{\alpha,d,n}(\psi_t)(\lambda) = \mathscr{F}_W^{\alpha,d,n}(\psi)(t\lambda). \tag{3.10}$$

In what follows, we need the following notation:
Notation. We denote by:
• $(\xi_k)_{k\in\mathbb{Z}}$ denotes a scale sequence in $]0,\ +\infty[$ that is strictly decreasing and satisfying:

$$\lim_{k\to-\infty}\xi_k=+\infty \qquad \text{and} \qquad \lim_{k\to+\infty}\xi_k=0. \tag{3.11}$$

Proposition 3.2 *Let $k\in\mathbb{Z}$ and $\psi\in\mathbb{L}^2_\alpha(\mathbb{R}^{d+1}_+)$ be a Weinstein Wavelet. Then*

i) The function $x\mapsto\left(\frac{1}{C^{\alpha,n}_\psi}\int_{\xi_{k+1}}^{\xi_k}|\mathscr{F}^{\alpha,d,n}_W(\psi)(tx)|^2\frac{dt}{t}\right)^{\frac{1}{2}}$ *belongs in* $\mathbb{L}^2_{\alpha+2n}(\mathbb{R}^{d+1}_+)$.

ii) There exists a function $\psi^P_k\in\mathbb{L}^2_\alpha(\mathbb{R}^{d+1}_+)$ *such that*

$$\mathscr{F}^{\alpha,d,n}_W(\psi^P_k)(x)=\left(\frac{1}{C^{\alpha,n}_\psi}\int_{\xi_{k+1}}^{\xi_k}|\mathscr{F}^{\alpha,d,n}_W(\psi)(tx)|^2\frac{dt}{t}\right)^{\frac{1}{2}},\ a.e,\ x\in\mathbb{R}^{d+1}_+. \tag{3.12}$$

Proof i) Using Fubini–Tonelli's theorem and the relations (3.6), (2.26) and (3.10), we get

$$\begin{aligned}
&C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}\left(\int_{\xi_{k+1}}^{\xi_k}|\mathscr{F}^{\alpha,d,n}_W(\psi)(tx)|^2\frac{dt}{t}\right)d\mu_{\alpha+2n,d}(x)\\
&=\int_{\xi_{k+1}}^{\xi_k}\left(C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}|\mathscr{F}^{\alpha,d,n}_W(\psi_t)(x)|^2d\mu_{\alpha+2n,d}(x)\right)\frac{dt}{t}\\
&=\int_{\xi_{k+1}}^{\xi_k}\left(\int_{\mathbb{R}^{d+1}_+}|\psi_t(x)|^2d\mu_{\alpha,d}(x)\right)\frac{dt}{t}\\
&=||\psi||^2_{\alpha,2}\int_{\xi_{k+1}}^{\xi_k}\frac{dt}{t^{2\alpha+4n+d+3}}<+\infty.
\end{aligned}$$

Which gives the desired result.
ii) We obtain the result from the Plancherel Theorem. □

Corollary 3.1 *The function $\psi^P_k,\ k\in\mathbb{Z}$, satisfy the following properties:*

$$\forall k\in\mathbb{Z},\ 0\le\mathscr{F}^{\alpha,d,n}_W\left(\psi^P_k\right)(\lambda)\le 1,\ a.e,\ \lambda\in\mathbb{R}^{d+1}_+, \tag{3.13}$$

and

$$\sum_{k=-\infty}^{+\infty}\left(\mathscr{F}^{\alpha,d,n}_W\left(\psi^P_k\right)(\lambda)\right)^2=1,\ a.e,\ \lambda\in\mathbb{R}^{d+1}_+. \tag{3.14}$$

Definition 3.3 i) The function $\psi^P_k,\ k\in\mathbb{Z}$, is called P-wavelet packet member of step k and the sequence $\left(\psi^P_k\right)_{k\in\mathbb{Z}}$ is called P- wavelet packet.

ii) The P-wavelet packet transform $\Phi^{P,W}_\psi$ associated with the generalized Weinstein operator on $\mathbb{R}^{d+1}_+$ is defined on $\mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$ by:

$$\forall k\in\mathbb{Z},\forall x\in\mathbb{R}^{d+1}_+,\ \Phi^{P,W}_\psi(f)(k,x)=\int_{\mathbb{R}^{d+1}_+}f(y)\overline{\psi^P_{k,x}(y)}d\mu_{\alpha,d}(y), \tag{3.15}$$

where $\psi^P_{k,x},\ k\in\mathbb{Z},\ ,x\in\mathbb{R}^{d+1}_+$ are the family of the generalized Weinstein wavelets on $\mathbb{R}^{d+1}_+$ in $\mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$ given by:

$$\forall y\in\mathbb{R}^{d+1}_+,\ \psi^P_{k,x}(y)=T^{\alpha,d,n}_x\psi^P_k(y). \tag{3.16}$$

Here $T^{\alpha,d,n}_x,\ x\in\mathbb{R}^{d+1}_+$, are the generalized Weinstein translation operators given by the relation (2.27).

Proposition 3.3 *i) For all $k \in \mathbb{Z}$ and $x \in \mathbb{R}_+^{d+1}$, the function $\psi_{k,x}^P$ belongs to $\mathbb{L}_{\alpha,n}^2(\mathbb{R}_+^{d+1})$ and we have*

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left(\psi_{k,x}^P\right)(\lambda) = \Lambda_{\alpha,d,n}(-x,\lambda)\,\mathscr{F}_W^{\alpha,d,n}\left(\psi_k^P\right)(\lambda). \tag{3.17}$$

ii) For all $k \in \mathbb{Z}$ and $x \in \mathbb{R}_+^{d+1}$, we have

$$\left\|\psi_{k,x}^P\right\|_{\alpha,n,2} \leq x_{d+1}^{2n}\left\|\psi_k^P\right\|_{\alpha,n,2}. \tag{3.18}$$

Remark 3.1 The transform $\Phi_\psi^{P,W}$ can also be written in the form

$$\Phi_\psi^{P,W}(f)(k,x) = \widetilde{f} *_{\alpha,n} \overline{\psi_k^P}(x), \tag{3.19}$$

where $*_{\alpha,n}$ is the Weinstein convolution product given by the relation (2.32).

Theorem 3.1 *(*Parseval formula for $\Phi_\psi^{P,W}$*)*

For all $f, g \in \mathbb{L}_{\alpha,n}^2(\mathbb{R}_+^{d+1})$, we have:

$$\int_{\mathbb{R}_+^{d+1}} f(x)\overline{g(x)}d\mu_{\alpha,d}(x) = \sum_{k=-\infty}^{+\infty}\int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{P,W}(f)(k,x)\overline{\Phi_\psi^{P,W}(g)(k,x)}d\mu_{\alpha,d}(x). \tag{3.20}$$

Proof According to the relations (2.25) and (3.19), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{P,W}(f)(k,x)\overline{\Phi_\psi^{P,W}(g)(k,x)}d\mu_{\alpha,d}(x)\\
&= C_{\alpha+2n,d}^2\int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}\left(\Phi_\psi^{P,W}(f)(k,.)\right)(\lambda)\overline{\mathscr{F}_W^{\alpha,d,n}\left(\Phi_\psi^{P,W}(g)(k,.)\right)(\lambda)}d\mu_{\alpha+2n,d}(\lambda)\\
&= C_{\alpha+2n,d}^2\int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}\left(\widetilde{f} *_{\alpha,n} \overline{\psi_k}\right)(\lambda)\overline{\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{g} *_{\alpha,n} \overline{\psi_k}\right)(\lambda)}d\mu_{\alpha+2n,d}(\lambda)\\
&= C_{\alpha+2n,d}^2\int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(f)(-\lambda)\overline{\mathscr{F}_W^{\alpha,d,n}(g)(-\lambda)}\left|\mathscr{F}_W^{\alpha,d,n}(\psi_k)(-\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda).
\end{aligned}$$

Then from the relation (3.14), we get

$$\begin{aligned}
&\sum_{k=-\infty}^{+\infty}\int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{P,W}(f)(t,x)\overline{\Phi_\psi^{P,W}(g)(k,x)}d\mu_{\alpha,d}(x)\\
&= C_{\alpha+2n,d}^2\int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\overline{\mathscr{F}_W^{\alpha,d,n}(g)(\lambda)}\sum_{k=-\infty}^{+\infty}\left|\mathscr{F}_W^{\alpha,d,n}(\psi_k)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda)\\
&= C_{\alpha+2n,d}^2\int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\overline{\mathscr{F}_W^{\alpha,d,n}(g)(\lambda)}d\mu_{\alpha+2n,d}(\lambda)\\
&= \int_{\mathbb{R}_+^{d+1}} f(x)\overline{g(x)}d\mu_{\alpha,d}(x).
\end{aligned}$$

□

Corollary 3.2 *(*Plancherel formula for $\Phi_\psi^{P,W}$*) For all $f \in \mathbb{L}_{\alpha,n}^2(\mathbb{R}_+^{d+1})$, we have:*

$$\int_{\mathbb{R}_+^{d+1}} |f(x)|^2 d\mu_{\alpha,d}(x) = \sum_{k=-\infty}^{+\infty}\int_{\mathbb{R}_+^{d+1}} \left|\Phi_\psi^{P,W}(f)(k,x)\right|^2 d\mu_{\alpha,d}(x). \tag{3.21}$$

Proof The result follows immediatly from the relation (3.20) with $g = f$. □

The following theorem gives an inversion formula for the transform $\Phi_\psi^{P,W}$.

Theorem 3.2 *Let $\left(\psi_k^P\right)_{k\in\mathbb{Z}}$ be a P-wavelet packet. For all $f \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}_+^{d+1})$ such that $\mathscr{F}_W^{\alpha,d,n}(f) \in \mathbb{L}^1_{\alpha+2n}(\mathbb{R}_+^{d+1})$, we have*

$$f(x) = \sum_{k=-\infty}^{+\infty} \int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{P,W}(f)(k,y)\psi_{k,x}^P(y)d\mu_{\alpha,d}(y), \ \ a.e, \ \ x \in \mathbb{R}_+^{d+1}. \tag{3.22}$$

Proof Let $\left(\psi_k^P\right)_{k\in\mathbb{Z}}$ be a P- wavelet packet and $f \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}_+^{d+1})$ such that $\mathscr{F}_W^{\alpha,d,n}(f) \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}_+^{d+1})$. We put

$$\forall k \in \mathbb{Z}, \forall x \in \mathbb{R}_+^{d+1}, \ \mathscr{L}_k(x) = \int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{P,W}(f)(k,y)\psi_{k,x}^P(y)d\mu_{\alpha,d}(y).$$

By invoking the relations (3.16), (3.19), (2.21), (2.22) and Parseval's formula (2.25), we can write

$$\begin{aligned}
\mathscr{L}_k(x) &= \int_{\mathbb{R}_+^{d+1}} \widetilde{f} *_{\alpha,n} \overline{\psi_k^P}(y)\overline{T_x^{\alpha,d,n}\overline{\psi_k^P}(y)}d\mu_{\alpha,d}(y) \\
&= C^2_{\alpha+2n,d} \int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}\left(\widetilde{f}\right)(\lambda)\, \mathscr{F}_W^{\alpha,d,n}\left(\overline{\psi_k^P}\right)(\lambda)\, \overline{\mathscr{F}_W^{\alpha,d,n}\left(T_x^{\alpha,d,n}\overline{\psi_k^P}\right)(\lambda)}d\mu_{\alpha+2n,d}(\lambda) \\
&= C^2_{\alpha+2n,d} \int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(f)(-\lambda)\, \overline{\mathscr{F}_W^{\alpha,d,n}\left(\psi_k^P\right)}(-\lambda)\, \Lambda_{\alpha,d,n}(x,\lambda)\, \mathscr{F}_W^{\alpha,d,n}\left(\psi_k^P\right)(-\lambda)\, d\mu_{\alpha+2n,d}(\lambda).
\end{aligned}$$

Thus, we obtain

$$\mathscr{L}_k(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\, \Lambda_{\alpha,d,n}(-x,\lambda) \left|\mathscr{F}_W^{\alpha,d,n}\left(\psi_k^P\right)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda). \tag{3.23}$$

Hence from the relation (3.14) and the inversion formula (2.24), for all $x \in \mathbb{R}_+^{d+1}$, we get

$$\begin{aligned}
\sum_{k=-\infty}^{+\infty} \mathscr{L}_k(x) &= C^2_{\alpha+2n,d} \int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\, \Lambda_{\alpha,d,n}(-x,\lambda) \sum_{k=-\infty}^{+\infty} \left|\mathscr{F}_W^{\alpha,d,n}\left(\psi_k^P\right)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda) \\
&= C^2_{\alpha+2n,d} \int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\, \Lambda_{\alpha,d,n}(-x,\lambda)\, d\mu_{\alpha+2n,d}(\lambda) \\
&= f(x).
\end{aligned}$$

□

Theorem 3.3 *Let $\left(\psi_k^P\right)_{k\in\mathbb{Z}}$ be a P-wavelet packet. For all $f \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$ such that $\mathscr{F}_W^{\alpha,d,n}(f) \in \mathbb{L}^1_{\alpha+2n}(\mathbb{R}_+^{d+1}) \cap \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$, we have*

$$f(x) = \sum_{k=-\infty}^{+\infty} \int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{P,W}(f)(k,y)\psi_{k,x}^P(y)d\mu_{\alpha,d}(x), \ \ a.e, \ \ x \in \mathbb{R}_+^{d+1}.$$

Proof The result can be proved in the same way of Theorem 3.2. □

4 Scale discrete scaling function on $\mathbb{R}_+^{d+1}$

Proposition 4.1 *Let $\left(\psi_k^P\right)_{k\in\mathbb{Z}}$ be a P- wavelet packet.*

i) For all $j \in \mathbb{Z}$ and $x \in \mathbb{R}_+^{d+1}$, we have

$$\sum_{k=-\infty}^{j-1} \left|\mathscr{F}_W^{\alpha,d,n}\left(\psi_k^P\right)(x)\right|^2 = \frac{1}{C_\psi^{\alpha,n}} \int_{\xi_j}^{+\infty} |\mathscr{F}_W^{\alpha,d,n}(\psi)(tx)|^2 \frac{dt}{t}. \tag{4.1}$$

ii) For all $j \in \mathbb{Z}$, there exists only one function $\phi_j^P \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$ such that

$$\forall x \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left(\phi_j^P\right)(x) = \left(\sum_{k=-\infty}^{j-1}\left|\mathscr{F}_W^{\alpha,d,n}\left(\psi_k^P\right)(x)\right|^2\right)^{\frac{1}{2}}. \tag{4.2}$$

Proof i) From the relations (3.12) and (3.11), we get

$$\begin{aligned}\sum_{k=-\infty}^{j-1}\left|\mathscr{F}_W^{\alpha,d,n}\left(\psi_k^P\right)(x)\right|^2 &= \frac{1}{C_\psi^{\alpha,n}}\sum_{k=-\infty}^{j-1}\int_{\xi_{k+1}}^{\xi_k}|\mathscr{F}_W^{\alpha,d,n}(\psi)(tx)|^2\frac{dt}{t}\\ &= \frac{1}{C_\psi^{\alpha,n}}\int_{\xi_j}^{+\infty}|\mathscr{F}_W^{\alpha,d,n}(\psi)(tx)|^2\frac{dt}{t}.\end{aligned}$$

ii) We obtain the result from the Plancherel Theorem. □

Definition 4.1 The sequence $\left(\phi_j^P\right)_{j\in\mathbb{Z}}$ is called scale discrete scaling function.
We need the following lemma which we obtain by a simple calculation.

Lemma 4.1 *The function ϕ_j^P, $j \in \mathbb{Z}$, satisfy the following properties:*

i) We have

$$\forall j \in \mathbb{Z},\ 0 \le \mathscr{F}_W^{\alpha,d,n}\left(\phi_j^P\right)(\lambda) \le 1,\ a.e,\ \lambda \in \mathbb{R}_+^{d+1}, \tag{4.3}$$

and

$$\lim_{j\to+\infty}\mathscr{F}_W^{\alpha,d,n}\left(\phi_j^P\right)(\lambda) = 1. \tag{4.4}$$

ii) We have

$$\forall j \in \mathbb{Z},\ \left|\mathscr{F}_W^{\alpha,d,n}\left(\psi_j^P\right)(\lambda)\right|^2 = \left|\mathscr{F}_W^{\alpha,d,n}\left(\phi_{j+1}^P\right)(\lambda)\right|^2 - \left|\mathscr{F}_W^{\alpha,d,n}\left(\phi_j^P\right)(\lambda)\right|^2, \tag{4.5}$$

and

$$\sum_{j=-\infty}^{+\infty}\left(\left|\mathscr{F}_W^{\alpha,d,n}\left(\phi_{j+1}^P\right)(\lambda)\right|^2 - \left|\mathscr{F}_W^{\alpha,d,n}\left(\phi_j^P\right)(\lambda)\right|^2\right) = 1,\ a.e,\ \lambda \in \mathbb{R}_+^{d+1}. \tag{4.6}$$

Proposition 4.2 *Let $j \in \mathbb{Z}$ and $x \in \mathbb{R}_+^{d+1}$, we consider the function $\phi_{j,x}^P$ defined on $\mathbb{R}_+^{d+1}$ by:*

$$\forall y \in \mathbb{R}_+^{d+1},\ \phi_{j,x}^P(y) = T_x^{\alpha,d,n}\phi_j^P(y). \tag{4.7}$$

Then the function $\phi_{j,x}^P \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$ and we have

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left(\phi_{j,x}^P\right)(\lambda) = \Lambda_{\alpha,d,n}(-x,\lambda)\mathscr{F}_W^{\alpha,d,n}\left(\phi_j^P\right)(\lambda). \tag{4.8}$$

Moreover, we have

$$\forall j \in \mathbb{Z},\ \forall x \in \mathbb{R}_+^{d+1},\ \left\|\phi_{j,x}^P\right\|_{\alpha,n,2} \le x_{d+1}^{2n}\left\|\phi_j^P\right\|_{\alpha,n,2}. \tag{4.9}$$

Proof The relation (4.7) together with (2.31) gives the relation (4.8).
An easily combination of the relations (4.7) and (2.29), we obtain the relation (4.9). □

Now, we give a Parseval formula and Plancherel formula involving both the scale discrete scaling function $\left(\phi_j^P\right)_{j\in\mathbb{Z}}$ and The P-wavelet packet transform $\Phi_\psi^{P,W}$ associated with the generalized Weinstein operator.

Theorem 4.1 *i) (Parseval formula)*
For all $f, g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$ *and* $j \in \mathbb{Z}$, *we have:*

$$\int_{\mathbb{R}^{d+1}_+} f(x)\overline{g(x)}d\mu_{\alpha,d}(x) = \int_{\mathbb{R}^{d+1}_+} \langle f,\ \phi^P_{j,x}\rangle_{\alpha,2}\overline{\langle g,\ \phi^P_{j,x}\rangle_{\alpha,2}}d\mu_{\alpha,d}(x) + \sum_{k=j}^{+\infty}\int_{\mathbb{R}^{d+1}_+} \Phi^{P,W}_{\psi}(f)(k,x)\,\overline{\Phi^{P,W}_{\psi}(g)(k,x)}d\mu_{\alpha,d}(x). \tag{4.10}$$

ii) (Plancherel formula)
For all $f \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$ *and* $j \in \mathbb{Z}$, *we have:*

$$\int_{\mathbb{R}^{d+1}_+} |f(x)|^2\, d\mu_{\alpha,d}(x) = \int_{\mathbb{R}^{d+1}_+} \left|\langle f,\ \phi^P_{j,x}\rangle_{\alpha,2})\right|^2 d\mu_{\alpha,d}(x) + \sum_{k=j}^{+\infty}\int_{\mathbb{R}^{d+1}_+} \left|\Phi^{P,W}_{\psi}(f)(k,x)\right|^2 d\mu_{\alpha,d}(x). \tag{4.11}$$

Proof i) Let $f, g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. From, the relation (4.7), we have

$$\langle f,\ \phi^P_{j,x}\rangle_{\alpha,2} = \int_{\mathbb{R}^{d+1}_+} f(y)\,\overline{T^{\alpha,d,n}_x \phi^P_j(y)}d\mu_{\alpha,d}(y) = \widetilde{f} * \overline{\phi^P_j}(x). \tag{4.12}$$

Then according to the relations (4.12), (2.25), (2.34) and (4.2), we get

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}_+} \langle f,\ \phi^P_{j,x}\rangle_{\alpha,2}\overline{\langle g,\ \phi^P_{j,x}\rangle_{\alpha,2}}d\mu_{\alpha,d}(x)\\
&= \int_{\mathbb{R}^{d+1}_+} \widetilde{f} * \overline{\phi^P_j}(x)\,\overline{\widetilde{g} * \overline{\phi^P_j}(x)}d\mu_{\alpha,d}(x)\\
&= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W\left(\widetilde{f} * \overline{\phi^P_j}\right)(\lambda)\overline{\mathscr{F}^{\alpha,d,n}_W\left(\widetilde{g} * \overline{\phi^P_j}\right)(\lambda)}d\mu_{\alpha+2n,d}(\lambda)\\
&= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W\left(\widetilde{f}\right)(\lambda)\overline{\mathscr{F}^{\alpha,d,n}_W(\widetilde{g})(\lambda)}\left|\mathscr{F}^{\alpha,d,n}_W\left(\phi^P_j\right)(-\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda)\\
&= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(-\lambda)\overline{\mathscr{F}^{\alpha,d,n}_W(g)(-\lambda)}\sum_{k=-\infty}^{j-1}\left|\mathscr{F}^{\alpha,d,n}_W\left(\psi^P_k\right)(-\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{k=j}^{+\infty}\int_{\mathbb{R}^{d+1}_+} \Phi^{P,W}_{\psi}(f)(k,x)\,\overline{\Phi^{P,W}_{\psi}(g)(k,x)}d\mu_{\alpha,d}(x)\\
&= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(-\lambda)\overline{\mathscr{F}^{\alpha,d,n}_W(g)(-\lambda)}\sum_{k=j}^{+\infty}\left|\mathscr{F}^{\alpha,d,n}_W\left(\psi^P_k\right)(-\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda).
\end{aligned}$$

Then using the relations (2.25) and (3.14), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+^{d+1}} \langle f,\ \phi_{j,x}^P\rangle_{\alpha,2}\overline{\langle g,\ \phi_{j,x}^P\rangle_{\alpha,2}} d\mu_{\alpha,d}(x) + \sum_{k=j}^{+\infty}\int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{P,W}(f)(k,x)\,\overline{\Phi_\psi^{P,W}(g)(k,x)} d\mu_{\alpha,d}(x)\\
&= C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\overline{\mathscr{F}_W^{\alpha,d,n}(g)(\lambda)} \sum_{k=-\infty}^{+\infty} \left|\mathscr{F}_W^{\alpha,d,n}\left(\psi_k^P\right)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda)\\
&= C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\overline{\mathscr{F}_W^{\alpha,d,n}(g)(\lambda)} d\mu_{\alpha+2n,d}(\lambda)\\
&= \int_{\mathbb{R}_+^{d+1}} f(x)\overline{g(x)} d\mu_{\alpha,d}(x).
\end{aligned}$$

ii) The result follows immediatly from the relation (4.10) with $g = f$. □

Corollary 4.1 *i)* $\left(\textit{Parseval formula for } \left(\phi_j^P\right)_{j\in\mathbb{Z}}\right)$

For all $f, g \in \mathbb{L}_{\alpha,n}^2(\mathbb{R}_+^{d+1})$*, we have:*

$$\int_{\mathbb{R}_+^{d+1}} f(x)\overline{g(x)} d\mu_{\alpha,d}(x) = \lim_{j\to+\infty}\int_{\mathbb{R}_+^{d+1}} \langle f,\ \phi_{j,x}^P\rangle_{\alpha,2}\overline{\langle g,\ \phi_{j,x}^P\rangle_{\alpha,2}} d\mu_{\alpha,d}(x). \tag{4.13}$$

ii) $\left(\textit{Plancherel formula for } \left(\phi_j^P\right)_{j\in\mathbb{Z}}\right)$

For all $f \in \mathbb{L}_{\alpha,n}^2(\mathbb{R}_+^{d+1})$*, we have:*

$$\int_{\mathbb{R}_+^{d+1}} |f(x)|^2 d\mu_{\alpha,d}(x) = \lim_{j\to+\infty}\int_{\mathbb{R}_+^{d+1}} \left|\langle f,\ \phi_{j,x}^P\rangle_{\alpha,2})\right|^2 d\mu_{\alpha,d}(x). \tag{4.14}$$

The following theorem gives two reconstruction formulas.

Theorem 4.2 *Let* $\left(\psi_k^P\right)_{k\in\mathbb{Z}}$ *be a P- wavelet packet and* $f \in \mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$, $p = 1, 2$ *such that* $\mathscr{F}_W^{\alpha,d,n}(f) \in \mathbb{L}_{\alpha+2n}^1(\mathbb{R}_+^{d+1})$.

i) We have

$$f(x) = \lim_{j\to+\infty}\int_{\mathbb{R}_+^{d+1}} \langle f,\ \phi_{j,y}^P\rangle_{\alpha,2}\phi_{j,y}^P(x)\, d\mu_{\alpha,d}(y),\ \textit{a.e},\ \ x \in \mathbb{R}_+^{d+1}. \tag{4.15}$$

ii) For all $j \in \mathbb{Z}$*, we have*

$$f(x) = \int_{\mathbb{R}_+^{d+1}} \langle f,\ \phi_{j,y}^P\rangle_{\alpha,2}\phi_{j,y}^P(x)\, d\mu_{\alpha,d}(y) + \sum_{k=j}^{+\infty}\int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{P,W}(f)(k,y)\psi_{k,y}^P(x)\, d\mu_{\alpha,d}(y),\ \textit{a.e}, x \in \mathbb{R}_+^{d+1}. \tag{4.16}$$

Proof i) Let $\left(\psi_k^P\right)_{k\in\mathbb{Z}}$ be a P- wavelet packet and $f \in \mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$, $p = 1, 2$ such that $\mathscr{F}_W^{\alpha,d,n}(f) \in \mathbb{L}_{\alpha+2n}^1(\mathbb{R}_+^{d+1})$, we put

$$\forall x \in \mathbb{R}_+^{d+1},\ \forall j \in \mathbb{Z},\ \mathscr{K}_j(x) = \int_{\mathbb{R}_+^{d+1}} \langle f,\ \phi_{j,y}^P\rangle_{\alpha,2}\phi_{j,y}^P(x)\, d\mu_{\alpha,d}(y).$$

Using the relation (4.12), we can write

$$\forall x \in \mathbb{R}_+^{d+1},\ \forall j \in \mathbb{Z},\ \mathscr{K}_j(x) = \int_{\mathbb{R}_+^{d+1}} \widetilde{f} * \overline{\phi_j^P}(y)\,\overline{T_x^{\alpha,d,n}\overline{\phi_j^P}(y)} d\mu_{\alpha,d}(y).$$

Now, from the relation (2.34) and the Parseval's formula, we obtain

$$\mathscr{K}_j(x) = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W\left(\widetilde{f}\right)(\lambda)\,\mathscr{F}^{\alpha,d,n}_W\left(\overline{\phi^P_j}\right)(\lambda)\,\overline{\mathscr{F}^{\alpha,d,n}_W\left(T^{\alpha,d,n}_x\overline{\phi^P_j}\right)(\lambda)}d\mu_{\alpha+2n,d}(\lambda)$$
$$= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(-\lambda)\,\overline{\mathscr{F}^{\alpha,d,n}_W\left(\phi^P_j\right)}(-\lambda)\,\Lambda_{\alpha,d,n}(x,\lambda)\,\mathscr{F}^{\alpha,d,n}_W\left(\phi^P_j\right)(-\lambda)\,d\mu_{\alpha+2n,d}(\lambda).$$

Then, we get

$$\mathscr{K}_j(x) = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\,\Lambda_{\alpha,d,n}(-x,\lambda)\left|\mathscr{F}^{\alpha,d,n}_W\left(\phi^P_j\right)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda). \tag{4.17}$$

Hence from the dominated convergence theorem, the relation (4.4) and the inversion formula (2.24), for $j\in\mathbb{Z}$ and for almost every $x\in\mathbb{R}^{d+1}_+$, we obtain

$$\begin{aligned}
&\lim_{j\to+\infty}\int_{\mathbb{R}^{d+1}_+}\langle f,\ \phi^P_{j,y}\rangle_{\alpha,2}\phi^P_{j,y}(x)\,d\mu_{\alpha,d}(y)\\
&= \lim_{j\to+\infty} C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\,\Lambda_{\alpha,d,n}(-x,\lambda)\left|\mathscr{F}^{\alpha,d,n}_W\left(\phi^P_j\right)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda)\\
&= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\,\Lambda_{\alpha,d,n}(-x,\lambda)\,d\mu_{\alpha+2n,d}(\lambda)\\
&= f(x).
\end{aligned}$$

ii) An easily combination of the relations (3.23), (4.17), (3.14) and (2.24), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}_+}\langle f,\phi^P_{j,y}\rangle_{\alpha,2}\phi^P_{j,y}(x)\,d\mu_{\alpha,d}(y) + \sum_{k=j}^{+\infty}\int_{\mathbb{R}^{d+1}_+}\Phi^{P,W}_\psi(f)(k,y)\psi^P_{k,y}(x)\,d\mu_{\alpha,d}(y)\\
&= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\,\Lambda_{\alpha,d,n}(-x,\lambda)\sum_{k=-\infty}^{j-1}\left|\mathscr{F}^{\alpha,d,n}_W\left(\psi^P_k\right)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda)\\
&\quad + C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\,\Lambda_{\alpha,d,n}(-x,\lambda)\sum_{k=j}^{+\infty}\left|\mathscr{F}^{\alpha,d,n}_W\left(\psi^P_k\right)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda)\\
&= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\,\Lambda_{\alpha,d,n}(-x,\lambda)\sum_{k=-\infty}^{+\infty}\left|\mathscr{F}^{\alpha,d,n}_W\left(\psi^P_k\right)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda)\\
&= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\,\Lambda_{\alpha,d,n}(-x,\lambda)\,d\mu_{\alpha+2n,d}(\lambda)\\
&= f(x).
\end{aligned}$$

Which gives the desired result. □

Theorem 4.3 *For all $f\in\mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$ such that $\mathscr{F}^{\alpha,d,n}_W(f)\in\mathbb{L}^p_{\alpha,n}(\mathbb{R}^{d+1}_+)$, $p=1,2$, for all $x\in\mathbb{R}^{d+1}_+$ and $k\in\mathbb{Z}$, we have*

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}_+}\Phi^{P,W}_\psi(f)(k,y)\psi^P_{k,x}(y)d\mu_{\alpha,d}(y)\\
&= \int_{\mathbb{R}^{d+1}_+}\langle f,\phi^P_{k+1,y}\rangle_{\alpha,2}\phi^P_{k+1,y}(x)\,d\mu_{\alpha,d}(y) - \int_{\mathbb{R}^{d+1}_+}\langle f,\phi^P_{k,y}\rangle_{\alpha,2}\phi^P_{k,y}(x)\,d\mu_{\alpha,d}(y).
\end{aligned}\tag{4.18}$$

Proof From the relations (3.23) and (4.5), we obtain the result. □

5 Modified wavelet packets associated with the generalized Weinstein operator

Definition 5.1 We consider the functions χ_j^M and $\widetilde{\chi}_j^M$ given by:

$$\chi_j^M = \phi_{j+1}^P + \phi_j^P \text{ and } \widetilde{\chi}_j^M = \phi_{j+1}^P - \phi_j^P \tag{5.1}$$

where $\left(\phi_j^P\right)_{j\in\mathbb{Z}}$ is the scale discrete scaling function given by (4.2).

i) The sequences $\left(\chi_j^M\right)_{j\in\mathbb{Z}}$ and $\left(\widetilde{\chi}_j^M\right)_{j\in\mathbb{Z}}$ are called repectively modifeted wavelet packet and the corresponding dual modifeted wavelet packet.

ii) The M-wavelet packet transform $\left(\Phi_\psi^{M,W}\right)_{j\in\mathbb{Z}}$ defined for regular function on $\mathbb{R}_+^{d+1}$ is given by:

$$\forall j\in\mathbb{Z},\ \forall x\in\mathbb{R}_+^{d+1},\ \Phi_\psi^{M,W}(f)(j,x) = \int_{\mathbb{R}_+^{d+1}} f(y)\,\overline{\chi_{j,x}^M(y)}d\mu_{\alpha,d}(y), \tag{5.2}$$

where

$$\forall j\in\mathbb{Z},\ \forall x,y\in\mathbb{R}_+^{d+1},\ \chi_{j,x}^M(y) = T_x^{\alpha,d,n}\chi_j^M(y). \tag{5.3}$$

iii) The dual M-wavelet packet transform $\left(\widetilde{\Phi}_\psi^{M,W}\right)_{j\in\mathbb{Z}}$ defined for regular function on $\mathbb{R}_+^{d+1}$ is given by:

$$\forall j\in\mathbb{Z},\ \forall x\in\mathbb{R}_+^{d+1},\ \widetilde{\Phi}_\psi^{M,W}(f)(j,x) = \int_{\mathbb{R}_+^{d+1}} f(y)\,\overline{\widetilde{\chi}_{j,x}^M(y)}d\mu_{\alpha,d}(y), \tag{5.4}$$

where

$$\forall j\in\mathbb{Z},\ \forall x,y\in\mathbb{R}_+^{d+1},\ \widetilde{\chi}_{j,x}^M(y) = T_x^{\alpha,d,n}\widetilde{\chi}_j^M(y). \tag{5.5}$$

Remark 5.1 We can write

$$\forall j\in\mathbb{Z},\ \forall x\in\mathbb{R}_+^{d+1},\ \Phi_\psi^{M,W}(f)(j,x) = \widetilde{f} * \overline{\chi_j^M}(x), \tag{5.6}$$

and

$$\forall j\in\mathbb{Z},\ \forall x\in\mathbb{R}_+^{d+1},\ \widetilde{\Phi}_\psi^{M,W}(f)(j,x) = \widetilde{f} * \overline{\widetilde{\chi}_j^M}(x). \tag{5.7}$$

Proposition 5.1 *i) The functions χ_j^M and $\widetilde{\chi}_j^M$ belong to $\mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$.*

ii) The functions $\mathscr{F}_W^{\alpha,d,n}\left(\chi_j^M\right)$ and $\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\chi}_j^M\right)$ belong to $\mathbb{L}^2_{\alpha+2n}(\mathbb{R}_+^{d+1})\cap\mathbb{L}^\infty_{\alpha,n}(\mathbb{R}_+^{d+1})$. Moreover, we have

$$\left\|\mathscr{F}_W^{\alpha,d,n}\left(\chi_j^M\right)\right\|_{\alpha,n,\infty} \le 2 \text{ and } \left\|\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\chi}_j^M\right)\right\|_{\alpha,n,\infty} \le 2. \tag{5.8}$$

iii) For almost all $\lambda\in\mathbb{R}_+^{d+1}$, we have

$$\sum_{j=-\infty}^{+\infty} \mathscr{F}_W^{\alpha,d,n}\left(\chi_j^M\right)(\lambda)\,\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\chi}_j^M\right)(\lambda) = 1. \tag{5.9}$$

Proof An easily combination of the relations (4.2), (4.3) and (4.6), we obtain the results. □

Theorem 5.1 *(Plancherel formula) For all $f\in\mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$, we have:*

$$\int_{\mathbb{R}_+^{d+1}} |f(x)|^2\,d\mu_{\alpha,d}(x) = \sum_{j=-\infty}^{+\infty}\int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{M,W}(f)(j,x)\,\overline{\widetilde{\Phi}_\psi^{M,W}(f)(j,x)}d\mu_{\alpha,d}(x). \tag{5.10}$$

Proof By invoking the relations (5.6), (5.7) and (2.25), we obtain

$$\int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{M,W}(f)(j,x)\,\overline{\widetilde{\Phi}_\psi^{M,W}(ff)(j,x)}d\mu_{\alpha,d}(x)$$
$$= C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} \left|\mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\right|^2 \mathscr{F}_W^{\alpha,d,n}\left(\chi_j^M\right)(\lambda)\,\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\chi}_j^M\right)(\lambda)\,d\mu_{\alpha+2n,d}(\lambda).$$

Then from the relation (5.9), we get

$$\sum_{j=-\infty}^{+\infty}\int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{M,W}(f)(j,x)\,\overline{\widetilde{\Phi}_\psi^{M,W}(f)(j,x)}d\mu_{\alpha,d}(x)$$
$$= C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} \left|\mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\right|^2 \sum_{j=-\infty}^{+\infty} \mathscr{F}_W^{\alpha,d,n}\left(\chi_j^M\right)(\lambda)\,\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\chi}_j^M\right)(\lambda)\,d\mu_{\alpha+2n,d}(\lambda)$$
$$= C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} \left|\mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda)$$
$$= \int_{\mathbb{R}_+^{d+1}} |f(x)|^2\,d\mu_{\alpha,d}(x).$$

Which achieves the proof. □

Proposition 5.2 *Let $f \in \mathbb{L}_\alpha^p(\mathbb{R}_+^{d+1})$, $p = 1, 2$ such that $\mathscr{F}_W^{\alpha,d,n}(f)$ belongs to $\mathbb{L}_{\alpha+2n}^1(\mathbb{R}_+^{d+1})$. Then for all $j \in \mathbb{Z}$ and $x \in \mathbb{R}_+^{d+1}$, we have*

$$\int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{M,W}(f)(j,x)\,\widetilde{\chi}_{j,y}^M(x)\,d\mu_{\alpha,d}(y) = \int_{\mathbb{R}_+^{d+1}} \widetilde{\Phi}_\psi^{M,W}(f)(j,x)\,\chi_{j,y}^M(x)\,d\mu_{\alpha,d}(y)$$
$$= \int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{P,W}(f)(j,x)\,\psi_{j,y}^P(x)\,d\mu_{\alpha,d}(y).$$

Proof Using the relations (2.25), (2.31) and (3.23), we obtain

$$\int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{M,W}(f)(j,x)\,\widetilde{\chi}_{j,y}^M(x)\,d\mu_{\alpha,d}(y)$$
$$= \int_{\mathbb{R}_+^{d+1}} \widetilde{\Phi}_\psi^{M,W}(f)(j,x)\,\chi_{j,y}^M(x)\,d\mu_{\alpha,d}(y)$$
$$= C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\,\mathscr{F}_W^{\alpha,d,n}\left(\chi_j^M\right)(\lambda)\,\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\chi}_j^M\right)(\lambda)\,\Lambda_{\alpha,d,n}(-x,\lambda)\,d\mu_{\alpha+2n,d}(\lambda)$$
$$= \int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\left|\mathscr{F}_W^{\alpha,d,n}\left(\phi_j^P\right)(\lambda)\right|^2 \Lambda_{\alpha,d,n}(-x,\lambda)\,d\mu_{\alpha+2n,d}(\lambda)$$
$$= \int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{P,W}(f)(j,x)\,\psi_{j,y}^P(x)\,d\mu_{\alpha,d}(y).$$

□

Theorem 5.2 *Let $f \in \mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$, $p = 1, 2$ such that $\mathscr{F}_W^{\alpha,d,n}(f) \in \mathbb{L}_{\alpha,n}^1(\mathbb{R}_+^{d+1})$.*

i) We have

$$f(x) = \sum_{j=-\infty}^{+\infty} \int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{M,W}(f)(j,x)\,\chi_{j,y}^M(x)\,d\mu_{\alpha,d}(y),\ a.e, x \in \mathbb{R}_+^{d+1}.$$

ii) We have

$$f(x)=\sum_{j=-\infty}^{+\infty}\int_{\mathbb{R}_+^{d+1}}\widetilde{\Phi}_\psi^{M,W}(f)(j,x)\,\widetilde{\chi}_{j,y}^{M}(x)\,d\mu_{\alpha,d}(y),\ a.e,x\in\mathbb{R}_+^{d+1}.$$

Proof The results can be proved in the same way of Theorem 3.2. □

The following theorem is as an immediate consequence of Theorem 4.2 and Proposition 5.2.

Theorem 5.3 *Let $f\in\mathbb{L}_{\alpha,n}^{p}(\mathbb{R}_+^{d+1}),\ p=1,2$ such that $\mathscr{F}_W^{\alpha,d,n}(f)\in\mathbb{L}_{\alpha+2n}^{1}(\mathbb{R}_+^{d+1})$.*

i) For all $j\in\mathbb{Z}$ and almost every $x\in\mathbb{R}_+^{d+1}$, we have

$$f(x)=\int_{\mathbb{R}_+^{d+1}}\langle f,\ \phi_{j,y}^{P}\rangle_{\alpha,2}\phi_{j,y}^{P}(x)\,d\mu_{\alpha,d}(y)+\sum_{k=j}^{+\infty}\int_{\mathbb{R}_+^{d+1}}\Phi_\psi^{M,W}(f)(k,y)\,\widetilde{\chi}_{k,x}^{M}(y)\,d\mu_{\alpha,d}(y).$$

ii) For all $j\in\mathbb{Z}$ and almost every $x\in\mathbb{R}_+^{d+1}$, we have

$$f(x)=\int_{\mathbb{R}_+^{d+1}}\langle f,\ \phi_{j,y}^{P}\rangle_{\alpha,2}\phi_{j,y}^{P}(x)\,d\mu_{\alpha,d}(y)+\sum_{k=j}^{+\infty}\int_{\mathbb{R}_+^{d+1}}\widetilde{\Phi}_\psi^{M,W}(f)(k,y)\,\chi_{k,x}^{M}(y)\,d\mu_{\alpha,d}(y).$$

6 S-wavelet packet associated with the generalized Weinstein operator

Definition 6.1 A sequence $\left(\varkappa_j^S\right)_{j\in\mathbb{Z}}$ in $\mathbb{L}_{\alpha,n}^{2}(\mathbb{R}_+^{d+1})$ is called a S-wavelet packet associated with the generalized Weinstein operator if it satisfies the following conditions:

i) For all $j\in\mathbb{Z}$, $\mathscr{F}_W^{\alpha,d,n}\left(\varkappa_j^S\right)$ is real-valued.

ii) For all $\lambda\in\mathbb{R}_+^{d+1}$, we have

$$\forall j\in\mathbb{Z},\ a\le\mathscr{F}_W^{\alpha,d,n}\left(\varkappa_j^S\right)(\lambda)\le b,\tag{6.1}$$

where $a,b\in\mathbb{R}$ are the stability constants with $0<a<b<+\infty$.

Definition 6.2 Let $\left(\varkappa_j^S\right)_{j\in\mathbb{Z}}$ be a S-wavelet packet associated with the generalized Weinstein operator.

i) The corresponding dual S-wavelet packet $\left(\widetilde{\varkappa}_j^S\right)_{j\in\mathbb{Z}}$ is given by:

$$\forall\lambda\in\mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\varkappa}_j^S\right)(\lambda)=\frac{\mathscr{F}_W^{\alpha,d,n}\left(\varkappa_j^S\right)(\lambda)}{\sum_{j=-\infty}^{+\infty}\left(\mathscr{F}_W^{\alpha,d,n}\left(\varkappa_j^S\right)(\lambda)\right)^2}.\tag{6.2}$$

ii) The S-wavelet packet transform $\Phi_\psi^{S,W}$ is defined for a regular function f on $\mathbb{R}_+^{d+1}$ by:

$$\forall j\in\mathbb{Z},\ \forall x\in\mathbb{R}_+^{d+1},\ \Phi_\psi^{S,W}(f)(j,x)=\int_{\mathbb{R}_+^{d+1}}f(y)\overline{\varkappa_{j,x}^S(y)}d\mu_{\alpha,d}(y),\tag{6.3}$$

where

$$\forall j\in\mathbb{Z},\ \forall x,y\in\mathbb{R}_+^{d+1},\ \varkappa_{j,x}^S(y)=T_x^{\alpha,d,n}\varkappa_j^S(y).\tag{6.4}$$

iii) The S-wavelet packet transform $\widetilde{\Phi}_\psi^{S,W}$ is defined for a regular function f on $\mathbb{R}_+^{d+1}$ by:

$$\forall j\in\mathbb{Z},\ \forall x\in\mathbb{R}_+^{d+1},\ \widetilde{\Phi}_\psi^{S,W}(f)(j,x)=\int_{\mathbb{R}_+^{d+1}}f(y)\overline{\widetilde{\varkappa}_{j,x}^S(y)}d\mu_{\alpha,d}(y),\tag{6.5}$$

where

$$\forall j\in\mathbb{Z},\ \forall x,y\in\mathbb{R}_+^{d+1},\ \widetilde{\varkappa}_{j,x}^S(y)=T_x^{\alpha,d,n}\widetilde{\varkappa}_j^S(y).\tag{6.6}$$

Remark 6.1 We can write

$$\forall j \in \mathbb{Z},\ \forall x \in \mathbb{R}_+^{d+1},\ \ \Phi_\psi^{S,W}(f)(j,x) = \widetilde{f} * \overline{\varkappa_j^S}(x), \tag{6.7}$$

and

$$\forall j \in \mathbb{Z},\ \forall x \in \mathbb{R}_+^{d+1},\ \ \widetilde{\Phi}_\psi^{S,W}(f)(j,x) = \widetilde{f} * \overline{\widetilde{\varkappa}_j^S}(x). \tag{6.8}$$

We need the following lemma which we obtain by a simple calculation.

Lemma 6.1 *Let $\left(\varkappa_j^S\right)_{j\in\mathbb{Z}}$ be a S-wavelet packet and $\left(\widetilde{\varkappa}_j^S\right)_{j\in\mathbb{Z}}$ the corresponding dual S-wavelet packet.*

i) For all $\lambda \in \mathbb{R}_+^{d+1}$, we have

$$\sum_{j=-\infty}^{+\infty} \mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\varkappa}_j^S\right)(\lambda)\, \mathscr{F}_W^{\alpha,d,n}\left(\varkappa_j^S\right)(\lambda) = 1, \tag{6.9}$$

and

$$\sum_{j=-\infty}^{+\infty} \left(\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\varkappa}_j^S\right)(\lambda)\right)^2 = \left(\sum_{j=-\infty}^{+\infty} \left(\mathscr{F}_W^{\alpha,d,n}\left(\varkappa_j^S\right)(\lambda)\right)^2\right)^{-1}. \tag{6.10}$$

ii) For all $j \in \mathbb{Z}$ and $\lambda \in \mathbb{R}_+^{d+1}$, we have

$$\frac{\sum_{k=-\infty}^{j-1}\left(\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\varkappa}_k^S\right)(\lambda)\right)^2}{\sum_{k=-\infty}^{+\infty}\left(\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\varkappa}_j^S\right)(\lambda)\right)^2} = \frac{\sum_{k=-\infty}^{j-1}\left(\mathscr{F}_W^{\alpha,d,n}\left(\varkappa_k^S\right)(\lambda)\right)^2}{\sum_{k=-\infty}^{+\infty}\left(\mathscr{F}_W^{\alpha,d,n}\left(\varkappa_k^S\right)(\lambda)\right)^2}. \tag{6.11}$$

iii) For all $j \in \mathbb{Z}$ and $\lambda \in \mathbb{R}_+^{d+1}$, we have

$$\sum_{k=-\infty}^{j-1} \mathscr{F}_W^{\alpha,d,n}\left(\varkappa_k^S\right)(\lambda)\, \mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\varkappa}_k^S\right)(\lambda) = \frac{\sum_{k=-\infty}^{j-1}\left(\mathscr{F}_W^{\alpha,d,n}\left(\varkappa_k^S\right)(\lambda)\right)^2}{\sum_{k=-\infty}^{+\infty}\left(\mathscr{F}_W^{\alpha,d,n}\left(\varkappa_k^S\right)(\lambda)\right)^2}. \tag{6.12}$$

iv) For all $\lambda \in \mathbb{R}_+^{d+1}$, we have

$$\frac{1}{b} \le \sum_{k=-\infty}^{+\infty}\left(\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\varkappa}_k^S\right)(\lambda)\right)^2 \le \frac{1}{a}, \tag{6.13}$$

where $a, b \in \mathbb{R}_+^*$ are the stability constants given by the relation (6.1).

Remark 6.2 If $a = b$, then we obtain:

$$\forall j \in \mathbb{Z},\ \varkappa_j^S = a\widetilde{\varkappa}_j^S.$$

Theorem 6.1 (Plancherel formula)
For all $f \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$, we have:

$$\int_{\mathbb{R}_+^{d+1}} |f(x)|^2\, d\mu_{\alpha,d}(x) = \sum_{j=-\infty}^{+\infty} \int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{S,W}(f)(j,x)\, \overline{\widetilde{\Phi}_\psi^{S,W}(f)(j,x)} d\mu_{\alpha,d}(x). \tag{6.14}$$

Proof From the relations (6.7), (6.8) and (2.25), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{S,W}(f)(j,x)\, \overline{\widetilde{\Phi}_\psi^{S,W}(f)(j,x)} d\mu_{\alpha,d}(x) \\
&= C^2_{\alpha+2n,d} \int_{\mathbb{R}_+^{d+1}} \left|\mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\right|^2 \mathscr{F}_W^{\alpha,d,n}\left(\varkappa_j^S\right)(\lambda)\, \mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\varkappa}_j^S\right)(\lambda)\, d\mu_{\alpha+2n,d}(\lambda).
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\sum_{j=-\infty}^{+\infty}\int_{\mathbb{R}_+^{d+1}}\Phi_\psi^{S,W}(f)(j,x)\,\overline{\widetilde{\Phi}_\psi^{S,W}(f)(j,x)}d\mu_{\alpha,d}(x)\\
&=C_{\alpha+2n,d}^2\int_{\mathbb{R}_+^{d+1}}\left|\mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\right|^2\left[\sum_{j=-\infty}^{+\infty}\mathscr{F}_W^{\alpha,d,n}\left(\varkappa_j^S\right)(\lambda)\,\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\varkappa}_j^S\right)(\lambda)\right]d\mu_{\alpha+2n,d}(\lambda)\\
&=C_{\alpha+2n,d}^2\int_{\mathbb{R}_+^{d+1}}\left|\mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda)\\
&=\int_{\mathbb{R}_+^{d+1}}|f(x)|^2\,d\mu_{\alpha,d}(x).
\end{aligned}$$

Which achieves the proof. □

Theorem 6.2 *Let $f\in\mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$, $p=1,2$ such that $\mathscr{F}_W^{\alpha,d,n}(f)\in\mathbb{L}_{\alpha+2n}^1(\mathbb{R}_+^{d+1})$.*

i) We have

$$f(x)=\sum_{j=-\infty}^{+\infty}\int_{\mathbb{R}_+^{d+1}}\Phi_\psi^{S,W}(f)(j,y)\,\widetilde{\varkappa}_{j,x}^S(y)\,d\mu_{\alpha,d}(y),\ a.e,\ \ x\in\mathbb{R}_+^{d+1}.$$

ii) We have

$$f(x)=\sum_{j=-\infty}^{+\infty}\int_{\mathbb{R}_+^{d+1}}\widetilde{\Phi}_\psi^{S,W}(f)(j,y)\,\varkappa_{j,x}^S(y)\,d\mu_{\alpha,d}(y),\ a.e,\ \ x\in\mathbb{R}_+^{d+1}.$$

Proof The results can be proved in the same way of Theorem 3.2. □

Definition 6.3 Let $\left(\varkappa_j^S\right)_{j\in\mathbb{Z}}$ be a S-wavelet packet and $\left(\widetilde{\varkappa}_j^S\right)_{j\in\mathbb{Z}}$ the corresponding dual S-wavelet packet. The scale discrete scaling function $\left(\chi_j^S\right)_{j\in\mathbb{Z}}$ corresponding to $\left(\varkappa_j^S\right)_{j\in\mathbb{Z}}$ is defined by:

$$\forall\lambda\in\mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left(\chi_j^S\right)(\lambda)=\left(\sum_{k=-\infty}^{j-1}\mathscr{F}_W^{\alpha,d,n}\left(\varkappa_k^S\right)(\lambda)\,\mathscr{F}_W^{\alpha,d,n}\left(\widetilde{\varkappa}_k^S\right)(\lambda)\right)^{\frac{1}{2}}.$$

Proposition 6.1 *The scale discrete scaling function $\left(\chi_j^S\right)_{j\in\mathbb{Z}}$ corresponding to $\left(\varkappa_j^S\right)_{j\in\mathbb{Z}}$ satisfy the following properties:*

$$\forall j\in\mathbb{Z},\ 0\le\mathscr{F}_W^{\alpha,d,n}\left(\chi_j^S\right)(\lambda)\le 1,\ \ a.e,\ \ \lambda\in\mathbb{R}_+^{d+1},$$

and

$$\lim_{j\to+\infty}\mathscr{F}_W^{\alpha,d,n}\left(\chi_j^S\right)(\lambda)=1,\ a.e,\ \ \lambda\in\mathbb{R}_+^{d+1}.$$

Proof The result follows from the relation (6.12). □

Theorem 6.3 (Plancherel formula)
For all $f\in\mathbb{L}_{\alpha,n}^2(\mathbb{R}_+^{d+1})$ and $j\in\mathbb{Z}$, we have

$$\int_{\mathbb{R}_+^{d+1}}|f(x)|^2\,d\mu_{\alpha,d}(x)=\lim_{j\to+\infty}\int_{\mathbb{R}_+^{d+1}}\left|\langle f,\ \chi_{j,x}^S\rangle_{\alpha,2}\right|^2 d\mu_{\alpha,d}(x),$$

and

$$\int_{\mathbb{R}_+^{d+1}} |f(x)|^2 \, d\mu_{\alpha,d}(x) = \int_{\mathbb{R}_+^{d+1}} \left| \langle f,\ \chi_{j,x}^S \rangle_{\alpha,2} \right|^2 d\mu_{\alpha,d}(x) + \sum_{k=j}^{+\infty} \int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{S,W}(f)(k,x) \, \overline{\widetilde{\Phi}_\psi^{S,W}(f)(k,x)} d\mu_{\alpha,d}(x),$$

where

$$\forall j \in \mathbb{Z},\ \forall x, y \in \mathbb{R}_+^{d+1},\ \chi_{j,x}^S(y) = T_x^{\alpha,d,n} \chi_j^S(y).$$

Proof The results can be proved in the same way of Theorem 4.1 and Corollary 4.1. □

Theorem 6.4 *Let* $f \in \mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1}),\ p = 1, 2$ *such that* $\mathscr{F}_W^{\alpha,d,n}(f) \in \mathbb{L}_{\alpha+2n}^1(\mathbb{R}_+^{d+1})$.

i) We have

$$f(x) = \lim_{j \to +\infty} \int_{\mathbb{R}_+^{d+1}} \langle f,\ \chi_{j,y}^S \rangle_{\alpha,2} \chi_{j,y}^S(x) \, d\mu_{\alpha,d}(y),\ a.e,\ \ x \in \mathbb{R}_+^{d+1}.$$

ii) For all $j \in \mathbb{Z}$ *and for almost all* $x \in \mathbb{R}_+^{d+1}$, *we have*

$$f(x) = \int_{\mathbb{R}_+^{d+1}} \langle f,\ \chi_{j,y}^S \rangle_{\alpha,2} \chi_{j,y}^S(x) \, d\mu_{\alpha,d}(y) + \sum_{k=j}^{+\infty} \int_{\mathbb{R}_+^{d+1}} \Phi_\psi^{S,W}(f)(k,x) \, \widetilde{\varkappa}_{k,x}^S(y) \, d\mu_{\alpha,d}(y).$$

iii) For all $j \in \mathbb{Z}$ *and for almost all* $x \in \mathbb{R}_+^{d+1}$, *we have*

$$f(x) = \int_{\mathbb{R}_+^{d+1}} \langle f,\ \chi_{j,y}^S \rangle_{\alpha,2} \chi_{j,y}^S(x) \, d\mu_{\alpha,d}(y) + \sum_{k=j}^{+\infty} \int_{\mathbb{R}_+^{d+1}} \widetilde{\Phi}_\psi^{S,W}(f)(k,x) \, \varkappa_{k,x}^S(y) \, d\mu_{\alpha,d}(y).$$

Proof The proof of these results is in the same sprite with of Theorem 4.2. □

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