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ORIGINAL RESEARCH

Existence of positive weak solutions for a class of fractional Laplacian systems with sign-changing weight functions

Rafik Guefaifia · S. H. Rasouli

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Abstract This paper deals with the existence of positive weak solutions for a class of fractional Laplacian systems with sign-changing weight functions satisfying the homogeneous Dirichlet condition on the exterior of a bounded domain. In particular, we employ the comparison principle and the sub and supersolutions method to prove sufficient conditions for the existence of positive solutions. Our results extend and complement previously known results in the literature.

Keywords Fractional Laplacian systems · Positive solutions · Sub-supersolution · Sign-changing weight functions

1 Introduction

Consider the fractional Laplacian system

$$\begin{cases} (-\Delta)^s u = \lambda\, F(x,u,v), & \text{in } \Omega, \\ (-\Delta)^s v = \lambda G(x,u,v), & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $F(x,u,v) = g(x)\,a(u) + f(v)$, $G(x,u,v) = g(x)\,b(v) + h(u)$, λ is a positives parameter and $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with $C^{1,1}$ boundary $\partial\Omega$ for $N \geq 2$. We assume that f, g, a, b are nontrivial $C^1([0,+\infty))$ functions satisfying the following assumptions:

H1 $f, h \in C^1([0,+\infty))$ are nondecreasing, $f(t) \to \infty, h(t) \to \infty$ as $t \to \infty$, and $\lim_{t\to+\infty} \frac{f(Lg(t))}{t} = 0$, for all $L > 0$ (a combined sublinear effect at ∞).

H2 $a, b \in C^1([0,+\infty))$ are non-negative, nondecreasing, and $\lim_{t\to+\infty} \frac{a(t)}{t} = \lim_{t\to+\infty} \frac{b(t)}{t} = 0$ (sublinear functions).

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R. Guefaifia
Department of Mathematics, College of Sciences and Arts, Qassim University, Buraydah 51452, Saudi Arabia
E-mail: r.guefaifia@qu.edu.sa

S. H. Rasouli (✉)
Department of Mathematics, Faculty of Basic Sciences, Babol Noshirvani University of Technology, Babol, Iran
E-mail: s.h.rasouli@nit.ac.ir

For a fixed $s \in (0, 1)$, $(-\Delta)^s$ is the fractional Laplacian operator defined by

$$(-\Delta)^s u(x) := C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

where $C_{N,s} := s2^{2s}\pi^{\frac{-N}{2}}\Gamma\left(\frac{N+2s}{2}\right)/\Gamma(1-s)$ is a positive normalizing constant with Γ as the usual gamma function and $P.V.$ stands for the Cauchy principal value of the singular integral.

Here we focus on sign-changing weight functions g and also do not assume any sign condition on $f(0)$ or $h(0)$. Hence in our system (1.1) $F(x, 0, 0)$ or $G(x, 0, 0)$ could be negative for some $x \in \Omega$. Such systems are referred to in the literature as semipositone problems and it is well known from the literature that the study of positive solutions is mathematically challenging see ([1,2,13–15,19]).

We prove an existence for positive weak solutions for a certain range of parameter λ. We use the method of sub-super solutions to establish our results. Several authors have used this method to prove that there is a weak solution to Dirichlet-type boundary problems, for instance, ([4,7,9,11,17]).

On the other hand, many authors have been interested in fractional equations see [8,10,12,18], for exemple in [10] the existence of at least three solutions for a fractional system involving the p-Laplacian with Lipschitz nonlinearity is demonstrated by means of the three critical points theorems. In [6], by using the method of sub- and supersolutions, the authors discussed the existence of positive weak solutions for a class of nonlinear fractional Laplacian problems satisfying the homogeneous Dirichlet condition on the exterior of a bounded domain. They also provided numerical bifurcation diagrams and the profile of positive solutions, corresponding to the theoretical results using the finite elements method in one dimension.

The motivation for this study is the case when $s = 1$ (local case) [3], where the authors established the existence of positive solutions for the nonlinear system

$$\begin{cases} -\Delta u = \lambda F(x, u, v), & x \in \Omega, \\ -\Delta u = \lambda H(x, u, v), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases}$$

where $F(x, u, v) = g(x)\, a(u) + f(v)$, $H(x, u, v) = g(x)\, b(v) + h(u)$, λ is positives parameter and $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain in $\mathbb{R}^N$, $N \geq 1$ with smooth boundary $\partial\Omega$ and Δ is the Laplacian operator, g is a C^1 sign-changing function that may be negative near the boundary. the authors discussed the existence of positive solutions when f, h, a, b and g satisfy certain additional conditions.

The aim of this work is to extend the results of [3] to the fractional Laplacian system (1.1). The fractional Laplacian operator $(-\Delta)^s$, is associated with super diffusion driven by Lévy flights and appears to be of interest from the application point of view, see [5,16], and references therein. More specifically, such operators are associated with the effcient foraging strategy of living organisms, see e.g.[16] and references therein.

This paper is organized as follows. In section 2, we recall some properties of fractional Sobolev spaces and definitions. In section 3, we will state and prove our existence results.

2 Functional spaces and definitions

We first discuss some properties of fractional Sobolev spaces, see[5,8] for more details. Let

$$H^s\left(\mathbb{R}^N\right) = \left\{u \in L^2\left(\mathbb{R}^N\right) : \|u\|_{H^s(\mathbb{R}^N)} < +\infty\right\},$$

where

$$\|u\|_{H^s(\mathbb{R}^N)} := \left(\|u\|^2_{L^s(\mathbb{R}^N)} + [u]^2_{H^s(\mathbb{R}^N)}\right),$$

and

$$[u]^2_{H^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|}{|x - y|^{N+2s}} dxdy\right)^{\frac{1}{2}},$$

is the Gagliardo seminorm of u. The fractional Sobolev space $H^s\left(\mathbb{R}^N\right)$ is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} uvdx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy. \tag{2.1}$$

Further, the fractional Sobolev space

$$H_0^s(\Omega) := \left\{u \in H^s\left(\mathbb{R}^N\right) : u = 0 \text{ a.e } \mathbb{R}^N\backslash\Omega\right\}$$

is also a Hilbert space with respect to the inner product

$$\langle u, v\rangle_{H_0^s(\Omega)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy. \tag{2.2}$$

The norms generated by (2.1) and (2.2) are equivalent in $H_0^s(\Omega)$. This fact follows from [[5], Lemmas 1.28 and 1.29] for $N \geq 2$ and Lemma 8.1 for $N = 1$ with $\Omega = (0,1) \subset \mathbb{R}$. We utilize this important fact in our analysis. To simplify notation, for

$$\mathcal{L}(\psi,\phi) := \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\psi(x)-\psi(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}dxdy. \tag{2.3}$$

Let σ_1 be the first eigenvalue of $(-\Delta)^s$ in Ω and $\varphi_1 > 0$ be the corresponding eigenfunction [5] i.e.

$$\begin{cases} (-\Delta)^s\varphi_1 = \sigma_1\varphi_1 \text{ in } \Omega, \\ \varphi_1 > 0 \text{in } \Omega, \\ \varphi_1 = 0 \text{in } \mathbb{R}^N\diagdown\Omega. \end{cases} \tag{2.4}$$

The variational characterization of σ_1 is given by

$$\sigma_1 := \inf_{u\in H_0^s(\Omega)\backslash\{0\}} \frac{\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx}{\int_\Omega u^2 dx}.$$

Moreover, it follows from [18] that there exist positive constants c_1, c_2 such that

$$0 < c_1\delta^s < \varphi_1(x) < c_2\delta^s. \tag{2.5}$$

Also, it follows from [12] that there exist positive constants β such that $\beta < l(x) < +\infty$ for all $x \in \Omega$, where

$$l(x) := \int_{\mathbb{R}^N} \frac{|\varphi_1(x)-\varphi_1(y)|^2}{|x-y|^{N+2s}}dy.$$

By simple computation [12], we obtain

$$(-\Delta)^s\varphi_1^2(x) = 2\sigma_1\varphi_1^2(x) - l(x) \text{ in } \Omega, \tag{2.6}$$

and there exist $m, \mu > 0$ such that

$$m < l(x) - 2\sigma_1\varphi_1^2(x) \text{ in } \overline{\Omega}_\delta, \tag{2.7}$$

and

$$\mu \leq \varphi_1 \leq 1 \text{ in } \Omega\backslash\overline{\Omega}_\delta, \tag{2.8}$$

where

$$\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}.$$

Let e be the solution of the following problem

$$\begin{cases} (-\Delta)^s e = 1 \text{ in } \Omega, \\ \\ e = 0 \text{ on } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{2.9}$$

Moreover, it follows from [5] that there exist positive constants $\widehat{c}_1, \widehat{c}_2$ such that

$$0 < \widehat{c}_1 \delta^s < e(x) < \widehat{c}_2 \delta^s \text{ a.e.in } \Omega. \tag{2.10}$$

Here we assume that the weight g takes negative values in Ω_δ but require g to be strictly positive in $\Omega \backslash \overline{\Omega}_\delta$. To be precise we assume that there exist positive constants β and η such that $g(x) \geq -\beta$ on $\overline{\Omega}_\delta$, and $g(x) \geq \eta$ on $\Omega \backslash \overline{\Omega}_\delta$. Let $\tau_0 \geq 0$ be such that $\eta a(\tau) + f(\tau) > 0$ and $\eta b(\tau) + h(\tau) > 0$ for every $\tau > \tau_0$, and $f_0 = \max\{0, -f(0)\}$, $h_0 = \max\{0, -h(0)\}$. For $\gamma > \frac{\tau_0}{\alpha}$ we define

$$Q_1(\gamma) = \min\left\{\frac{m\gamma}{\beta a(\gamma) + f_0}, \frac{m\gamma}{\beta b(\gamma) + h_0}\right\},$$
$$Q_2(\gamma) = \max\left\{\frac{2\gamma\sigma_1}{\eta a(\gamma\alpha) + f(\gamma\alpha)}, \frac{2\gamma\sigma_1}{\eta b(\gamma\alpha) + h(\gamma\alpha)}\right\},$$

where $\alpha = \mu^2$.

Definition 2.1 $(u, v) \in \left(H_0^s(\Omega) \times H_0^s(\Omega)\right)$, is said a weak solution of (1.1) if it satisfies

$$\begin{aligned} \mathcal{L}(u, \phi) &= \lambda \int_\Omega [g(x) a(u) + f(v)] \phi dx \text{ in } \Omega, \\ \mathcal{L}(v, \psi) &= \lambda \int_\Omega [g(x) b(v) + h(u)] \psi dx \text{ in } \Omega, \end{aligned} \tag{2.11}$$

for all $(\phi, \psi) \in H_0^s(\Omega) \times H_0^s(\Omega)$.

Definition 2.2 A pair of nonnegative functions $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$ in $H^s\left(\mathbb{R}^N\right) \times H^s\left(\mathbb{R}^N\right)$ are called a weak subsolution and supersolution of (1.1) if they satisfy $(\underline{u}, \underline{v}) \leq (0, 0)$, $(\overline{u}, \overline{v}) \geq (0, 0)$ *a.e.* in $\mathbb{R}^N \backslash \Omega$.

$$\begin{aligned} \mathcal{L}(\underline{u}, \phi) &\leq \lambda \int_\Omega \left[g(x) a\left(\underline{u}\right) + f\left(\underline{v}\right)\right] \phi dx \text{ in } \Omega, \\ \mathcal{L}(\underline{v}, \psi) &\leq \lambda \int_\Omega \left[g(x) b\left(\underline{v}\right) + h\left(\underline{u}\right)\right] \psi dx \text{ in } \Omega, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \mathcal{L}(\overline{u}, \phi) &\geq \lambda \int_\Omega [g(x) a(\overline{u}) + f(\overline{v})] \phi dx \text{ in } \Omega, \\ \mathcal{L}(\overline{v}, \psi) &\geq \lambda \int_\Omega [g(x) b(\overline{v}) + h(\overline{u})] \psi dx \text{ in } \Omega. \end{aligned} \tag{2.13}$$

for all $(\phi, \psi) \in H_0^s(\Omega) \times H_0^s(\Omega)$.

3 Existence results

In this section, we shall state and prove the main result of this paper

Now we state a general sub- and supersolution result without monotonicity assumption for the problem

$$\begin{cases} (-\Delta)^s u = g(x, u) \text{ in } \Omega, \\ u = 0, \text{ in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.1}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying the following assumptions:

$(h1)$ for all $r > 0$, there is $a_r \in L^\infty(\Omega)$ such that $|g(x, t)| \leq a_r$ for all $|t| \leq r$ a.e. $x \in \Omega$,

$(h2)$ for all $r > 0$, there is a continuous nondecreasing function b_r with $b_r(0) = 0$ such that $|g(x, t_1) - g(x, t_2)| \leq b_r |t_1 - t_2|$ for all $|t_1|, |t_2| \leq r$ a.e. $x \in \Omega$.

Theorem 3.1 [6] Assume (h1) and (h2) hold. Let $\underline{u}$ and $\overline{u} \in H^s(\mathbb{R}^N) \cap L^\infty(\Omega)$ be weak subsolution and weak supersolution, respectively of (3.1) satisfying $\underline{u} \leq \overline{u}$ a.e. $x \in \Omega$. Then, there exists a weak solution u to (3.1) satisfying $\underline{u} \leq u \leq \overline{u}$ a.e. $x \in \Omega$.

For systems (1.1), if there exist subsolutions $(\underline{u}, \underline{v})$ and supersolutions $(\overline{u}, \overline{v})$ such that $(\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v})$ on Ω, then these systems have at least one solution (u, v) satisfying $(\underline{u}, \underline{v}) \leq (u, v) \leq (\overline{u}, \overline{v})$ on Ω. This follows by the natural extension of the Theorem 3.1 to systems (1.1).

Theorem 3.2 *Let* $H := \left\{\tau > \frac{\tau_0}{\alpha} : Q_2(\tau) \leq Q_1(\tau)\right\}$ *and* $S = \cup_{\tau \in H} [Q_2(\tau), .Q_1(\tau)]$. *Assume that the conditions* $(H1) - (H2)$ *be satisfied and* $H \neq \emptyset$. *Then, for* $\lambda \in S$, *systems (1.1) has at least one positive solution.*

Proof Let $\lambda \in S$ and $\gamma > \frac{\tau_0}{\alpha}$ be such that $\lambda \in [Q_2(\tau), .Q_1(\tau)]$. Let $\underline{u} = \underline{v} = \gamma\varphi_1^2$, where φ_1 is the eigenfunction as described in Section 2. We will prove that $\left(\underline{u}, \underline{v}\right)$ is a sub-solution.

Now, by (2.6)

$$\begin{aligned}\mathcal{L}(\underline{u}, \phi) &= \mathcal{L}\left(\gamma\varphi_1^2, \phi\right) = \gamma\mathcal{L}\left(\varphi_1^2, \phi\right) \\ &= \gamma\int_\Omega \left(2\sigma_1\varphi_1^2(x) - l(x)\right)\phi(x)\,dx, \text{ for all } 0 < \phi \in H_0^S(\Omega).\end{aligned}$$

First we consider the case when $x \in \overline{\Omega}_\delta$. Since $\lambda \leq Q_1(\gamma)$, we have $\lambda \leq \frac{m\gamma}{\beta a(\gamma) + f_0}$. Thus

$$-m\gamma \leq \lambda\left[-\beta a(\gamma) - f_0\right].$$

Hence

$$\begin{aligned}\mathcal{L}(\underline{u}, \phi) &= \gamma\int_{\overline{\Omega}_\delta} \left(2\sigma_1\varphi_1^2(x) - l(x)\right)\phi(x)\,dx \\ &\leq -\gamma m\int_{\overline{\Omega}_\delta} \phi(x)\,dx \\ &\leq \lambda\left[-\beta a(\gamma) - f_0\right]\int_{\overline{\Omega}_\delta} \phi(x)\,dx \\ &\leq \lambda\int_{\overline{\Omega}_\delta} \left[g(x)\,a\left(\underline{u}\right) + f\left(\underline{v}\right)\right]\phi(x)\,dx.\end{aligned} \tag{3.2}$$

A similar argument shows that

$$\mathcal{L}(\underline{v}, \phi) \leq \lambda\int_{\overline{\Omega}_\delta} \left[g(x)\,b\left(\underline{v}\right) + h\left(\underline{u}\right)\right]\psi(x)\,dx.$$

Next consider the case when $x \in \Omega\backslash\overline{\Omega}_\delta$. Since $\underline{u} = \gamma\varphi_1^2 \geq \gamma\mu^2 = \gamma\alpha$, and $\lambda \geq Q_2(\gamma)$, we have

$$\lambda \geq \frac{2\gamma\sigma_1}{\eta a(\gamma\alpha) + f(\gamma\alpha)}.$$

Hence

$$\begin{aligned}\mathcal{L}(\underline{u}, \phi) &= \gamma\int_{\Omega\backslash\overline{\Omega}_\delta} \left(2\sigma_1\varphi_1^2(x) - l(x)\right)\phi(x)\,dx \\ &\leq 2\gamma\sigma_1\int_{\Omega\backslash\overline{\Omega}_\delta} \phi(x)\,dx \\ &\leq \lambda\left[\eta a(\gamma\alpha) + f(\gamma\alpha)\right]\int_{\Omega\backslash\overline{\Omega}_\delta} \phi(x)\,dx \\ &\leq \lambda\int_{\Omega\backslash\overline{\Omega}_\delta} \left[g(x)\,a\left(\underline{u}\right) + f\left(\underline{v}\right)\right]\phi(x)\,dx.\end{aligned}$$

A similar argument shows that

$$\mathcal{L}(\underline{v}, \psi) \leq \lambda \int_{\Omega} \left[g(x) b\left(\underline{v}\right) + h\left(\underline{u}\right) \right] \psi(x) dx.$$

Thus we have shown that $\left(\underline{u}, \underline{v}\right)$ is a sub-solution.

Now we will prove that for given λ there exists a C large enough so that

$$(\overline{u}, \overline{v}) = \left(\frac{C}{\nu} \lambda e, 2h(C\lambda) \lambda e \right)$$

is a supersolution of (1.1), where e is the unique positive solution of (2.9). Since a is sublinear and f, h satisfy $(H2)$, choosing C large we have

$$\frac{\nu \left[\|g\|_{\infty} a(C\lambda) \right] + f(2h(C\lambda)\lambda\nu)}{C} \leq 1,$$

where $\nu = \|e\|_{\infty}$ .Thus

$$\frac{C}{\nu} \geq \left[\|g\|_{\infty} a(C\lambda) + f(2h(C\lambda)\lambda\nu) \right].$$

Hence

$$\begin{aligned}
\mathcal{L}(\overline{u}, \phi) &= \frac{C}{\nu} \lambda \mathcal{L}(e, \phi) \\
&\geq \lambda \left[\|g\|_{\infty} a(C\lambda) + f(2h(C\lambda)\lambda\nu) \right] \int_{\Omega} \phi(x) dx \\
&\geq \lambda \int_{\Omega} \left[g(x) a\left(\frac{C}{\nu} \lambda e \right) + f(2h(C\lambda)\lambda e) \right] \phi(x) dx \\
&= \lambda \int_{\Omega} \left[g(x) a(\overline{u}) + f(\overline{v}) \right] \phi(x) dx.
\end{aligned}$$

Similarly, choosing C large (since $h(t) \to \infty$ as $t \to \infty$) so that

$$\frac{\|g\|_{\infty} (2h(C\lambda)\lambda\nu)}{h(C\lambda)} \leq 1,$$

we have

$$\begin{aligned}
\mathcal{L}(\overline{v}, \psi) &= 2h(C\lambda) \lambda \mathcal{L}(e, \psi) \\
&\geq \lambda \left[g(x) b(2h(C\lambda)\lambda e) + h\left(C\lambda \frac{e}{\nu} \right) \right] \int_{\Omega} \psi(x) dx \\
&= \lambda \int_{\Omega} \left[g(x) b(\overline{v}) + h(\overline{u}) \right] \psi dx.
\end{aligned}$$

Hence $(\overline{u}, \overline{v})$ is a super-solution of (1.1).

Finally, using the right estimate in (2.5), the left estimate in (2.10), and taking C larger, if necessary, we get $\left(\underline{u}, \underline{v}\right) \leq (\overline{u}, \overline{v})$ a.e. in Ω. Hence, (1.1) has a positive weak solution $(u, v) \in \left(H_0^s(\Omega) \times H_0^s(\Omega) \right)$ such that $\underline{u} \leq u \leq \overline{u}$, $\underline{v} \leq v \leq \overline{v}$ a.e. in Ω for $\lambda \in S$. This completes the proof of Theorem 3.2 □

Corollary 3.3 *Let f, h, a, b satisfy $(H1) - (H2)$ and*

$$\frac{a(\tau)}{f(\tau)}, \frac{b(\tau)}{f(\tau)}, \frac{a(\tau)}{h(\tau)}, \frac{b(\tau)}{h(\tau)} \to 0 \text{ as } \tau \to \infty. \tag{3.3}$$

Then there exists λ_0 such that for every $\lambda \geq \lambda_0$, systems (1.1) has at least one positive solution.

Corollary 3.4 *Assume there exists* $k > 0$ *such that* $a(\tau) < k, b(\tau) < k \ \forall \tau$, *and*

$$\min\{\eta a(0) + f(0), \eta b(0) + h(0)\} \geq \frac{\sigma_1(\beta k + \zeta_0)}{m}, \tag{3.4}$$

where $\zeta_0 = \max\{0, f_0 h_0\}$. Then for any $\lambda > 0$, systems (1.1) has at least one positive solution.
The proof of the two results is similar to [3].

Example 3.5 Consider the problem

$$\begin{cases} (-\Delta)^s u = \lambda \left[g(x)(u+1)^{\alpha_1} + (v+1)^p - 2 \right] \text{ in } \Omega, \\ (-\Delta)^s v = \lambda \left[g(x)(v+1)^{\alpha_2} + (u+1)^q - 2 \right] \text{ in } \Omega, \\ u = v = 0 \text{ in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.5}$$

where $0 < \alpha_j < 1$, and $p, q > 0$ are such that $pq < 1$. Clearly $a(\tau) = (\tau+1)^{\alpha_1}$, $b(\tau) = (\tau+1)^{\alpha_2}$, $f(\tau) = (\tau+1)^p - 2$, and $h(\tau) = (\tau+1)^q - 2$ satisfy $(H1)$, $(H2)$ and

$$\frac{a(\tau)}{f(\tau)}, \frac{b(\tau)}{f(\tau)}, \frac{a(\tau)}{h(\tau)}, \frac{b(\tau)}{h(\tau)} \to 0 \text{ as } \tau \to \infty.$$

Hence by Corollary 3.3, (3.5) has a positive solution for $\lambda \geq \lambda_0$ for some $\lambda_0 > 0$.

Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors state no conflict of interest.

Ethical approval Hereby, We consciously assure that the following is fulfilled: 1) This material is the authors' own original work, which has not been previously published elsewhere. 2) The paper is not currently being considered for publication elsewhere. 3) The paper reflects the authors' own research and analysis in a truthful and complete manner.

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