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ORIGINAL RESEARCH

Results on approximate solutions of second order differential equations with non-instantaneous impulses in a separable Hilbert space

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Abstract This paper is concerned with the approximation of solutions to a class of second order non linear non-instantaneous impulsive abstract differential equations. The finite-dimensional approximate solutions of the given system are built with the aid of the projection operators. We investigate the connection between the approximate solutions and exact solution, and the question of convergence. Moreover, we define the Faedo-Galerkin (F-G) approximations and prove the existence and convergence results. The results are obtained by using the theory of cosine functions, Banach fixed point theorem and fractional power of closed linear operators. At last, some examples of abstract formulation are provided.

Keywords Second order differential equations · Fixed point theorem · Cosine family of operators · Non-instantaneous impulses

Mathematics Subject Classification 34A45 · 47D09 · 34G20 · 34A37

1 Introduction

The Cauchy problem for first-order differential equations with unbounded operator coefficients, as studied by Hille and Yosida in 1948, has led to an increased interest in abstract differential equations. The theory of strongly continuous cosine family (SCCF) of bounded linear operators can be applied to study second-order differential equations in Banach spaces. Cosine functions are important since the relationship between cosine functions and second-order Cauchy problems is analogous to that between semigroups and first-order Cauchy problems. For the fundamental concepts of cosine functions, we refer the readers to Goldstein [1], Kisyński [2], Travis and Webb [3] and the cited references therein. The transverse motion of an extensive beam and the oscillation of a bar fixed at one end are some examples of physical phenomena that can be expressed as partial differential equations. These equations can be formulated as a second-order abstract differential equation.

The modeling of dynamic processes involving sudden and discontinuous jumps can be naturally accomplished through impulsive differential equations. There are two categories of impulsive differential equations, distinguished by the length of their impulsive actions. Instantaneous impulsive differential equations belong to the first type, in which the abrupt changes occur in a negligible duration relative to the overall time of the

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phenomenon. The second type comprises non-instantaneous impulsive differential equations(NIIDEs), where the impulsive action commences at an arbitrary fixed point and persists for a specific length of time. Impulsive differential equations have a wide range of applications in various fields, including physics, engineering, economics, population dynamics and so on. The administration of insulin to control glucose levels in diabetic patients is a well known example of a non-instantaneous impulsive action in biology. The introduction of insulin into the bloodstream and subsequent absorption by the body occurs gradually and continuously, rather than instantaneously. Therefore, the above situation can be model as a non-instantaneous impulsive action. To obtain further information about impulsive differential equations, we refer [4–6].

Solvability of differential equations is often limited, and obtaining their analytical solutions are challenging. As a consequence of these difficulties, the significance of approximate solutions increases. The methods utilized to attain approximate solutions do not require linearization of the nonlinear equations or discretization of space time variables, for example, Adomian decomposition method, Euler method, Picard method, and the variational iteration method. The Adomian decomposition method offers an analytical approximation for a broad spectrum of nonlinear equations, presenting a significant advantage by eliminating the need for linearization or relying on weak nonlinearity assumptions, this is the main advantage of this method.

The F-G method is among the most powerful technique for finding solutions to equations that are more regular, even when the data is weak and the problem is presented in a variational formulation [7]. The key feature of F-G method is the finite dimensional approximation. In this method, we employ a projection scheme $\{\mathcal{H}_n, \mathcal{P}^n\}$, which consists of projection operators $\mathcal{P}^n$ and increasing subspaces $\mathcal{H}_n$. The utilization of this method is that it reduces the differential equation problem to that of finite dimensional linear algebra problem (system of differential equations), which is often more computationally tractable.

In our knowledge, Bazley [8,9] was the first person who studied the approximation of solutions to a wave equation. Then Goethel [10] demonstrated the similar outcome for a functional Cauchy problem by using the premise that the nonlinear function must be defined on the entire Hilbert space. Subsequently, Miletta [11] extended the results of Goethel [10] by relaxing the assumptions previously imposed by Goethel. Recently, Malik [12] extended the F-G approximation results to second order differential equations. The literature survey indicates that numerous studies [13–17] have been conducted within this particular field but there is not a single research work that addresses the utilization of the F-G method for deriving approximate solutions to abstract second-order nonlinear NIIDEs. Thus, motivated by the above observation, we investigate a finite dimensional approximation to a class of second order non-instantaneous impulsive differential system in the infinite dimensional space and study the existence and convergence of approximate solutions.

We consider the following class of second order NIIDEs in a separable Hilbert space $\mathcal{H}$:

$$\begin{aligned}
\frac{d^2\nu(\varrho)}{d\varrho^2} &= -\mathcal{A}\nu(\varrho) + \mathcal{F}(\varrho, \nu(\varrho)), \quad \varrho \in (\vartheta_i, \varrho_{i+1}],\ i = 0, 1, 2, \cdots, q,\\
\nu(\varrho) &= h_i^1(\varrho, \nu(\varrho_i^-)), \quad \varrho \in (\varrho_i, \vartheta_i],\ i = 1, 2, \cdots, q,\\
\frac{d\nu(\varrho)}{d\varrho} &= h_i^2(\varrho, \nu(\varrho_i^-)), \quad \varrho \in (\varrho_i, \vartheta_i],\ i = 1, 2, \cdots, q,\\
\nu(0) &= y_0, \quad \nu'(0) = z_0
\end{aligned} \tag{1}$$

where $\nu(\cdot)$ be a state function takes value in a separable Hilbert space $\mathcal{H}$, $-\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a SCCF of bounded linear operators $(C(\varrho))_{\varrho\in\mathbb{R}}$, initial values $z_0, y_0 \in \mathcal{H}$ and ϑ_i, ϱ_i are some pre-fixed numbers satisfying the relation $0 = \vartheta_0 = \varrho_0 < \varrho_1 < \vartheta_1 < \varrho_2 < \cdots < \vartheta_i < \varrho_{i+1} = T$, $\nu(\varrho_i^-) = \lim_{k\to 0^+} \nu(\varrho_i - k)$ denotes the left limit of $\nu(\varrho)$ at $\varrho = \varrho_i$ and the maps $\mathcal{F} \in C([0, T] \times \mathcal{H}; \mathcal{H})$, $h_i^j \in C((\varrho_i, \vartheta_i] \times \mathcal{H}; \mathcal{H})$ for all $i = 1, 2, \cdots, q$, $j = 1, 2$ satisfying few conditions to be specified later.

The following are the major contributions of our manuscript:

- This is the first work that presents F-G approach to find approximate solutions for the second order NIIDEs.
- We consider a class of second order NIIDEs in an arbitrary infinite dimensional space and obtain its approximate solutions in finite dimensional subspace.
- We constrained the problem (1) to a finite dimensional subspace by using the projection operators.
- Conversion of a class of partial differential equations with impulsive conditions to a class of abstract impulsive second order differential systems in infinite dimensional space is demonstrated through an example.
- This work can be extended for finding approximate solutions to higher order differential equations.

The paper is divided into the following sections: Sect. 2 concerns definitions and some useful facts regarding the cosine family. In Sect. 3, we define the approximate solutions and prove their existence. Section 4 deals with the convergence of the approximate solutions. Section 5 introduces the F-G approximations and their convergence results. At last, in Sect. 6, the applications of these obtained abstract results are demonstrated by examples.

2 Preliminaries and assumptions

This sections presents assumptions, propositions and some definitions of the theory of the cosine family.

Definition 1 [18] A one parameter family $(C(\varrho))_{\varrho\in\mathbb{R}}$ of bounded linear operators from a Banach space $\mathcal{H}$ into $\mathcal{H}$ is referred to as a SCCF if the following conditions hold:

(i) $C(0) = I$.
(ii) $C(\vartheta+\varrho) + C(\vartheta-\varrho) = 2C(\vartheta)C(\varrho)$, for all $\varrho, \vartheta \in \mathbb{R}$.
(iii) $\varrho \to C(\varrho)z$ is continuous on $\mathbb{R}$ for each $z \in \mathcal{H}$.

Definition 2 $(S(\varrho))_{\varrho\in\mathbb{R}}$ denotes the sine family associated with the cosine family is given by

$$S(\varrho)z = \int_0^\varrho C(\vartheta)z\,d\vartheta, \quad z \in \mathcal{H}.$$

Let $M, \tilde{M} > 0$ such that $\|S(\varrho)\| \le \tilde{M}$ and $\|C(\varrho)\| \le M\ \forall \varrho \in [0,T]$. The operator $\mathcal{A}$ is defined by

$$\mathcal{A}z = \frac{d^2}{d\varrho^2}C(0)z, \quad z \in D(\mathcal{A}),$$

where the domain of the operator $\mathcal{A}$ is defined by

$$D(\mathcal{A}) = \{z \in \mathcal{H} : C(\varrho)z \text{ is a twice continuously differentiable function of } \varrho\}.$$

$D(\mathcal{A})$ is the Banach space equipped with the graph norm $\|z\|_{\mathcal{A}} = \|z\| + \|\mathcal{A}z\|$. Furthermore, E refers to the space comprising of vectors $z \in \mathcal{H}$ for which the function $C(\cdot)z$ belongs to the class C^1. Kisyński [2] demonstrated that the space E is a Banach space endowed with the norm $\|z\|_E = \|z\| + \sup_{0\le\varrho\le1} \|\mathcal{A}S(\varrho)z\|$, $z \in E$.

The operator valued function $G(\varrho) = \begin{bmatrix} C(\varrho) & S(\varrho) \\ \mathcal{A}S(\varrho) & C(\varrho) \end{bmatrix}$ on the space $E \times \mathcal{H}$, is a strongly continuous group of linear operators and generated by the operator $D = \begin{bmatrix} 0 & I \\ \mathcal{A} & 0 \end{bmatrix}$ defined on $D(\mathcal{A}) \times E$. As a consequence, $\mathcal{A}S(\varrho) : E \to \mathcal{H}$ is a bounded linear operator and for each $z \in E$, $\mathcal{A}S(\varrho)z \to 0$ as $\varrho \to 0$. Moreover, if $x : [0,\infty) \to \mathcal{H}$ is a locally integrable function then $y(\varrho) = \int_0^\varrho S(\varrho-\vartheta)x(\vartheta)d\vartheta$, defines an $E-$valued continuous function which follows from the fact that

$$\int_0^\varrho G(\varrho-\vartheta)\begin{bmatrix} 0 \\ x(\vartheta) \end{bmatrix} d\vartheta = \left[\int_0^\varrho S(\varrho-\vartheta)x(\vartheta)d\vartheta, \int_0^\varrho C(\varrho-\vartheta)x(\vartheta)d\vartheta\right]^T$$

defines an $E \times \mathcal{H}$ valued continuous function.

Consider the second order abstract Cauchy problem

$$\begin{aligned} \frac{d^2z(\varrho)}{d\varrho^2} &= \mathcal{A}z(\varrho) + h(\varrho), \quad \varrho \in [0,T], \\ z(0) &= z_0, \quad z'(0) = z_1 \end{aligned} \tag{2}$$

where $h : [0,T] \to \mathcal{H}$ is an integrable function. The function

$$z(\varrho) = C(\varrho)z_0 + S(\varrho)z_1 + \int_0^\varrho S(\varrho-\vartheta)h(\vartheta)d\vartheta$$

is said to be mild solution $z(\cdot)$ of problem (2) and that when $z_0 \in E$, $z(\cdot)$ is continuously differentiable and

$$z'(\varrho) = \mathcal{A}S(\varrho)z_0 + C(\varrho)z_1 + \int_0^\varrho C(\varrho-\vartheta)h(\vartheta)d\vartheta.$$

For deeper knowledge of the theory of the cosine function, please see references [18–23].

Proposition 1 [18] *Let $\mathcal{A}$ be the generator of a SCCF $(C(\varrho))_{\varrho\in\mathbb{R}}$. Then*

(i) *$\mathcal{A}$ is closed operator in $\mathcal{H}$ and $D(\mathcal{A})$ is dense in $\mathcal{H}$.*
(ii) *If $z \in E$, then $S(\varrho)z \in D(\mathcal{A})$ and $\frac{d}{d\varrho}C(\varrho)z = \mathcal{A}S(\varrho)z$.*
(iii) *If $z \in D(\mathcal{A})$, then $S(\varrho)z \in D(\mathcal{A})$ and $\mathcal{A}S(\varrho)z = S(\varrho)\mathcal{A}z$.*
(iv) *If $z \in D(\mathcal{A})$, then $C(\varrho)z \in D(\mathcal{A})$ and $\frac{d^2}{d\varrho^2}C(\varrho)z = \mathcal{A}C(\varrho)z = C(\varrho)\mathcal{A}z$.*
(v) *If $a, b \in \mathbb{R}$ and $z \in \mathcal{H}$, then $\int_a^b S(u)zdu \in D(\mathcal{A})$ and $\mathcal{A}z = C(b)z - C(a)z$.*

Lemma 2 [21] *If $-\mathcal{A}$ is the infinitesimal generator of a SCCF of $(C(\varrho))_{\varrho\in\mathbb{R}}$ on $\mathcal{H}$. Then $\mathcal{A}$ generates an analytic semigroup $T(\varrho)$ with angle $\frac{\pi}{2}$ defined by*

$$T(\varrho)z = \frac{1}{\sqrt{\pi\varrho}}\int_0^\infty e^{-\frac{\vartheta^2}{4\varrho}}C(\vartheta)zd\vartheta, \quad \varrho > 0, z \in \mathcal{H}.$$

Since $-\mathcal{A}$ generates an analytic semigroup $T(\varrho)$ in $\mathcal{H}$ and $0 \in \rho(-\mathcal{A})$. Hence, for every $\alpha > 0$, we can define the negative fractional power of $\mathcal{A}$ as

$$\mathcal{A}^{-\alpha} = \frac{1}{\Gamma\alpha}\int_0^\infty \varrho^{\alpha-1}T(\varrho)d\varrho.$$

Since $\mathcal{A}^{-\alpha}$ is uniformly bounded and one-one. Thus inverse of $\mathcal{A}^{-\alpha}$ is $\mathcal{A}^{\alpha} : D(\mathcal{A}^{\alpha}) \subset \mathcal{H} \to \mathcal{H}$ is well defined. We denote $D(\mathcal{A}^{\alpha})$ by $\mathcal{H}_\alpha$ and it forms a Banach space with the norm $\|\nu\|_\alpha = \|\mathcal{A}^{\alpha}\nu\|$.

When dealing with impulsive conditions, we consider the space of piecewise continuous functions $\mathcal{PC} = \mathcal{PC}([0, T]; \mathcal{H}_\alpha)$ formed by all the functions $\nu : [0, T] \to \mathcal{H}_\alpha$ such that $\nu(\cdot)$ is continuous at $\varrho \neq \varrho_i$, $\lim\limits_{\varrho\downarrow\varrho_i}\nu(\varrho)$ exists for all $i = 1, 2, \cdots, q$ and $\nu(\cdot)$ is left continuous on [0,T]. $\mathcal{PC}$ is a Banach space endowed with the norm $\|\nu\|_{\mathcal{PC},\alpha} = \sup\limits_{\varrho\in[0,T]}\|\mathcal{A}^{\alpha}\nu(\varrho)\|$.

We introduce the following assumptions in order to establish results:

(A1): Operator $\mathcal{A}$ is self adjoint, positive definite and $\mathcal{A}$ has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

where $\lambda_n \to \infty$ as $n \to \infty$ and corresponding orthonormal system of eigenfunctions ψ_j is complete such that $\mathcal{A}\psi_j = \lambda_j\psi_j$ and $\langle\psi_i, \psi_j\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta function.

(A2): The map $\mathcal{F} : [0, T] \times \mathcal{H}_\alpha \to \mathcal{H}$ is continuous and satisfying

$$\|\mathcal{F}(\varrho, x) - \mathcal{F}(\varrho, \nu)\| \leq K_1\|x - \nu\|_\alpha$$

and

$$\max_{\varrho\in[0,T]}\|\mathcal{F}(\varrho, x)\| \leq K_2$$

where K_1, K_2 are positive constants.

(A3): The maps $h_i^j : (\varrho_i, \vartheta_i] \times \mathcal{H}_\alpha \to \mathcal{H}_\alpha, i = 1, 2, \cdots, q,\ j = 1, 2$ are continuous and uniformly bounded. There exist constants $C_{h_i^j}, D_{h_i^j} > 0$, $j = 1, 2, i = 1, 2, \cdots, q$, such that

$$\|h_i^j(\varrho, \nu) - h_i^j(\varrho, x)\|_\alpha \leq C_{h_i^j}\|\nu - x\|_\alpha$$

and

$$\|h_i^j(\varrho, x)\|_\alpha \leq D_{h_i^j}.$$

Now onwards, throughout the manuscript, we set

$\rho = \sup\limits_{\varrho\in[0,T]}\|\mathcal{A}S(\varrho)\|$; $D = \max\limits_{1\leq i\leq q}\big(\max\limits_{0\leq i\leq q} Q_i, C_{h_i^1}\big)$;

$N_0 = M\|y_0\|_\alpha + \tilde{M}\|z_0\|_\alpha + \|\mathcal{A}^{\alpha-1}\|\rho K_2\varrho_1$; $N_i = MD_{h_i^1} + \tilde{M}D_{h_i^2} + \|\mathcal{A}^{\alpha-1}\|\rho K_2\varrho_{i+1},\ i = 1, 2, \cdots, q$;

$Q_0 = \|\mathcal{A}^{\alpha-1}\|\rho K_1\varrho_1$; $Q_i = MC_{h_i^1} + \tilde{M}C_{h_i^2} + \|\mathcal{A}^{\alpha-1}\|\rho K_1\varrho_{i+1},\ i = 1, 2, \cdots, q$.

Definition 3 A function $\nu \in \mathcal{PC}([0,T];\mathcal{H})$ is considered to be a mild solution of problem (1) if $\nu(\varrho)$ fulfils $\nu(0) = y_0,\ \nu'(0) = z_0$ and

$$\begin{cases} \nu(\varrho) = h_i^1(\varrho, \nu(\varrho_i^-)),\ \varrho \in (\varrho_i, \vartheta_i], i = 1, 2, \cdots, q; \\ \frac{d\nu(\varrho)}{d\varrho} = h_i^2(\varrho, \nu(\varrho_i^-)),\ \varrho \in (\varrho_i, \vartheta_i], i = 1, 2, \cdots, q; \\ \nu(\varrho) = C(\varrho)y_0 + S(\varrho)z_0 + \int_0^\varrho S(\varrho - \vartheta)\mathcal{F}(\vartheta, \nu(\vartheta))d\vartheta,\ \varrho \in (0, \varrho_1]; \\ \nu(\varrho) = C(\varrho - \vartheta_i)h_i^1(\vartheta_i, \nu(\varrho_i^-)) + S(\varrho - \vartheta_i)h_i^2(\vartheta_i, \nu(\varrho_i^-)) \\ \qquad + \int_{\vartheta_i}^\varrho S(\varrho - \vartheta)\mathcal{F}(\vartheta, \nu(\vartheta))d\vartheta, \quad \varrho \in (\vartheta_i, \varrho_{i+1}],\ i = 1, 2, \cdots, q. \end{cases} \tag{3}$$

3 Existence of approximate solutions

This section focus on finding approximate solutions for problem (1), and demonstrate that such solutions exist and are unique.

Moving forward, we discretize the separable Hilbert space $\mathcal{H}$ by projecting it onto $\mathcal{H}_n = \text{span}\{\psi_1, \psi_2, \cdots, \psi_n\}$. The orthogonal projection operators $\mathcal{P}^n : \mathcal{H} \to \mathcal{H}_n$ are defined by

$$\mathcal{P}^n \nu = \sum_{i=1}^{n} \langle \nu, \psi_i \rangle \psi_i, \quad \nu \in \mathcal{H}.$$

For each $n \in \mathbb{N}$, we define

$$\mathcal{F}_n : [0,T] \times \mathcal{H}_\alpha \to \mathcal{H}$$

such that

$$\mathcal{F}_n(\varrho, \nu(\varrho)) = \mathcal{F}(\varrho, \mathcal{P}^n \nu(\varrho)).$$

Similarly, we define

$$h_{i,n}^j : (\varrho_i, \vartheta_i] \times \mathcal{H}_\alpha \to \mathcal{H}_\alpha, \qquad i = 1, 2, \cdots, q, \quad j = 1, 2$$

such that

$$h_{i,n}^j(\varrho, \nu(\varrho)) = h_i^j(\varrho, \mathcal{P}^n \nu(\varrho)).$$

Throughout this paper $B_R(\mathcal{H})$ denotes the closed ball with center zero and radius $R > 0$ in a Banach space $\mathcal{H}$. We define

$$B_R(\mathcal{PC}([0,T];\mathcal{H}_\alpha)) = \{z \in \mathcal{PC}([0,T];\mathcal{H}_\alpha) : \|z\|_{\mathcal{PC},\alpha} \le R\}$$

where

$$R = \max_{1 \le i \le q} \left(\max_{0 \le i \le q} N_i, D_{h_i^1} \right).$$

Theorem 3 *Let $z_0 \in E$, $y_0 \in D(\mathcal{A})$ and assumptions* **(A1)** $-$ **(A3)** *fulfilled, then there exists a unique $\nu_n \in B_R(\mathcal{PC}([0,T];\mathcal{H}_\alpha))$, provided*

$$D < 1. \tag{4}$$

Proof We define the map $F_n : B_R(\mathcal{PC}([0,T];\mathcal{H}_\alpha)) \to B_R(\mathcal{PC}([0,T];\mathcal{H}_\alpha))$, $n \in \mathbb{N}$, as follows,

$$(F_n\nu)(\varrho) = \begin{cases} C(\varrho)y_0 + S(\varrho)z_0 + \int_0^\varrho S(\varrho-\vartheta)\mathcal{F}_n(\vartheta,\nu(\vartheta))d\vartheta, \quad \varrho \in (0,\varrho_1], \\ h^1_{i,n}(\varrho,\nu(\varrho_i^-)), \quad \varrho \in (\varrho_i,\vartheta_i], i = 1,,2,\cdots,q, \\ C(\varrho-\vartheta_i)h^1_{i,n}(\vartheta_i,\nu(\varrho_i^-)) + S(\varrho-\vartheta_i)h^2_{i,n}(\vartheta_i,\nu(\varrho_i^-)) \\ \quad + \int_{\vartheta_i}^\varrho S(\varrho-\vartheta)\mathcal{F}_n(\vartheta,\nu(\vartheta))d\vartheta, \ \varrho \in (\vartheta_i,\varrho_{i+1}], \ i = 1,2,\cdots,q. \end{cases}$$

The proof is split up into the following two steps.

Step 1: Before applying the Banach fixed point theorem, first we need to establish that F_n is well defined. Now, for any $\varrho \in (0,\varrho_1]$ and $\nu \in B_R(\mathcal{PC}([0,T];\mathcal{H}_\alpha))$, we have

$$\begin{aligned} \|(F_n\nu)(\varrho)\|_\alpha &\le \|C(\varrho)\|\|y_0\|_\alpha + \|S(\varrho)\|\|z_0\|_\alpha + \int_0^\varrho \|\mathcal{A}^{\alpha-1}\|\|\mathcal{A}S(\varrho-\vartheta)\|\|\mathcal{F}_n(\vartheta,\nu(\vartheta))\|d\vartheta \\ &\le M\|y_0\|_\alpha + \tilde{M}\|z_0\|_\alpha + \|\mathcal{A}^{\alpha-1}\|\rho K_2\varrho \\ &\le N_0. \end{aligned} \tag{5}$$

Similarly, for any $\nu \in B_R(\mathcal{PC}([0,T];\mathcal{H}_\alpha))$ and $\varrho \in (\varrho_i,\vartheta_i]$, $i = 1,2,\cdots,q$

$$\|(F_n\nu)(\varrho)\|_\alpha = \|h^1_{i,n}(\varrho,\nu(\varrho_i^-))\|_\alpha \le D_{h_i^1}. \tag{6}$$

Now, for any $\nu \in B_R(\mathcal{PC}([0,T];\mathcal{H}_\alpha))$ and $\varrho \in (\vartheta_i,\varrho_{i+1}]$, $i = 1,2,\cdots,q$, we have

$$\begin{aligned} \|(F_n\nu)(\varrho)\|_\alpha &\le \|C(\varrho-\vartheta_i)\|\|h^1_{i,n}(\vartheta_i,\nu(\varrho_i^-))\|_\alpha + \|S(\varrho-\vartheta_i)\|\|h^2_{i,n}(\vartheta_i,\nu(\varrho_i^-))\|_\alpha \\ &\quad + \int_{\vartheta_i}^\varrho \|\mathcal{A}^{\alpha-1}\|\|\mathcal{A}S(\varrho-\vartheta)\|\|\mathcal{F}_n(\vartheta,\nu(\vartheta))\|d\vartheta \\ &\le MD_{h_i^1} + \tilde{M}D_{h_i^2} + \|\mathcal{A}^{\alpha-1}\|\rho K_2(\varrho-\vartheta_i) \\ &\le N_i. \end{aligned} \tag{7}$$

Hence, from the inequalities (5), (6) and (7), for any $\varrho \in [0,T]$, we get

$$\|F_n\nu\|_{\mathcal{PC},\alpha} \le R.$$

Step 2: Here, we demonstrate that F_n is a strict contracting operator. For any $\nu, x \in B_R(\mathcal{PC}([0,T];\mathcal{H}_\alpha))$ and $\varrho \in (0,\varrho_1]$

$$\begin{aligned} \|(F_n\nu)(\varrho) - (F_nx)(\varrho)\|_\alpha &\le \int_0^\varrho \|\mathcal{A}^{\alpha-1}\|\|\mathcal{A}S(\varrho-\vartheta)\|\|\mathcal{F}_n(\vartheta,\nu(\vartheta)) - \mathcal{F}_n(\vartheta,x(\vartheta))\|d\vartheta \\ &\le \|\mathcal{A}^{\alpha-1}\|\rho K_1\varrho\|\nu(\vartheta) - x(\vartheta)\|_\alpha \\ &\le Q_0\|\nu - x\|_{\mathcal{PC},\alpha}. \end{aligned} \tag{8}$$

Also, for any $x, \nu \in B_R(\mathcal{PC}([0,T];\mathcal{H}_\alpha))$ and $\varrho \in (\varrho_i,\vartheta_i]$, $i = 1,2,\cdots,q$

$$\begin{aligned} \|(F_n\nu)(\varrho) - (F_nx)(\varrho)\|_\alpha &= \|h^1_{i,n}(\varrho,\nu(\varrho_i^-)) - h^1_{i,n}(\varrho,x(\varrho_i^-))\|_\alpha \\ &\le C_{h_i^1}\|\nu - x\|_{\mathcal{PC},\alpha}. \end{aligned} \tag{9}$$

Similarly, for any $x, \nu \in B_R(\mathcal{PC}([0,T];\mathcal{H}_\alpha))$ and $\varrho \in (\vartheta_i,\varrho_{i+1}]$, $i = 1,2,\cdots,q$, we have

$$\begin{aligned} \|(F_n\nu)(\varrho) - (F_nx)(\varrho)\|_\alpha &\le \|C(\varrho-\vartheta_i)\|\|h^1_{i,n}(\vartheta_i,\nu(\varrho_i^-)) - h^1_{i,n}(\vartheta_i,x(\varrho_i^-))\|_\alpha \\ &\quad + \|S(\varrho-\vartheta_i)\|\|h^2_{i,n}(\vartheta_i,\nu(\varrho_i^-)) - h^2_{i,n}(\vartheta_i,x(\varrho_i^-))\|_\alpha \\ &\quad + \int_{\vartheta_i}^\varrho \|\mathcal{A}^{\alpha-1}\|\|\mathcal{A}S(\varrho-\vartheta)\|\|\mathcal{F}_n(\vartheta,\nu(\vartheta)) - \mathcal{F}_n(\vartheta,x(\vartheta))\|d\vartheta \\ &\le [MC_{h_i^1} + \tilde{M}C_{h_i^2} + \|\mathcal{A}^{\alpha-1}\|\rho K_1(\varrho-\vartheta_i)]\|\nu - x\|_{\mathcal{PC},\alpha} \\ &\le Q_i\|\nu - x\|_{\mathcal{PC},\alpha}. \end{aligned} \tag{10}$$

Therefore, from the inequalities (8), (9) and (10), for any $\varrho \in [0, T]$, we have

$$\|F_n v - F_n x\|_{\mathcal{PC},\alpha} \leq D\|v - x\|_{\mathcal{PC},\alpha}. \tag{11}$$

Hence, from the inequality (4), F_n is a contracting operator. Consequently, by considering both step 1 and step 2, it can be concluded that the operator F_n fulfills all the requirements of Banach fixed point theorem. Hence, F_n has a unique fixed point v_n in $B_R(\mathcal{PC}([0, T]; \mathcal{H}_\alpha))$ which is given by

$$v_n(\varrho) = \begin{cases} C(\varrho)y_0 + S(\varrho)z_0 + \int_0^\varrho S(\varrho - \vartheta)\mathcal{F}_n(\vartheta, v_n(\vartheta))d\vartheta, \ \varrho \in (0, \varrho_1]; \\ h^1_{i,n}(\varrho, v_n(\varrho_i^-)), \ \varrho \in (\varrho_i, \vartheta_i], \ i = 1, 2, \cdots, q; \\ C(\varrho - \vartheta_i)h^1_{i,n}(\vartheta_i, v_n(\varrho_i^-)) + S(\varrho - \vartheta_i)h^2_{i,n}(\vartheta_i, v_n(\varrho_i^-)) \\ \quad + \int_{\vartheta_i}^\varrho S(\varrho - \vartheta)\mathcal{F}_n(\vartheta, v_n(\vartheta))d\vartheta, \ \varrho \in (\vartheta_i, \varrho_{i+1}], \quad i = 1, 2, \cdots, q. \end{cases} \tag{12}$$

Corollay 4 *Let* **(A1)** $-$ **(A3)** *fulfilled. If* $z_0 \in E$ *and* $y_0, h^j_{i,n}(\varrho, v_n(\varrho_i^-)) \in D(\mathcal{A})$ $\forall$ $\varrho \in (\varrho_i, \vartheta_i]$, $i = 1, 2, \cdots, q$, $j = 1, 2$ *then* $v_n(\varrho) \in D(\mathcal{A}^\eta)$ $\forall$ $\varrho \in [0, T]$, *where* $0 \leq \eta \leq 1$.

Proof This result is true for $\varrho \in (\varrho_i, \vartheta_i]$. Since $z_0 \in E$ and $y_0, h^j_{i,n}(\varrho, v_n(\varrho_i^-)) \in D(\mathcal{A})$ thus, in view of Proposition 1, $C(\varrho)y_0$, $S(\varrho)z_0$, $C(\varrho - \vartheta_i)h^1_{i,n}(\vartheta_i, v_n(\varrho_i^-))$, $S(\varrho - \vartheta_i)h^2_{i,n}(\vartheta_i, v_n(\varrho_i^-)) \in D(\mathcal{A})$. Therefore their sum is also in $D(\mathcal{A})$. Also $\mathcal{F}_n(\vartheta, v_n(\vartheta)) \in \mathcal{H}$ therefore by Proposition 1, $\int_{\vartheta_i}^\varrho S(\varrho - \vartheta)\mathcal{F}_n(\vartheta, v_n(\vartheta))d\vartheta \in D(\mathcal{A})$. Hence, the require outcome can be inferred from the fact that $D(\mathcal{A}) \subseteq D(\mathcal{A}^\eta)$ for $\eta \in [0, 1]$. □

Lemma 5 *Let* **(A1)** $-$ **(A3)** *fulfilled and* $y_0, z_0 \in D(\mathcal{A})$ *then there exists a constant* U', *independent of n, such that*

$$\|\mathcal{A}^\eta v_n(\varrho)\| \leq U', \quad \forall\, \varrho \in [0, T] \ \text{ and } 0 < \eta < 1, \quad \text{where}$$

$$U' = \max_{1 \leq i \leq q} \Big\{ \max_{0 \leq i \leq q} U_i, D_{h_i^1} \Big\}.$$

Proof For any $\varrho \in (0, \varrho_1]$, applying $\mathcal{A}^\eta$ on both sides of Eq. (12) and taking norm

$$\begin{aligned} \|v_n(\varrho)\|_\eta &\leq \|C(\varrho)\|\|y_0\|_\eta + \|S(\varrho)\|\|z_0\|_\eta + \int_0^\varrho \|\mathcal{A}^{\eta-1}\|\|\mathcal{A}S(\varrho - \vartheta)\|\|\mathcal{F}_n(\vartheta, v_n(\vartheta))\|d\vartheta \\ &\leq M\|y_0\|_\eta + \tilde{M}\|z_0\|_\eta + \|\mathcal{A}^{\eta-1}\|\rho K_2\varrho \\ &\leq U_0, \end{aligned} \tag{13}$$

where

$$U_0 = M\|y_0\|_\eta + \tilde{M}\|z_0\|_\eta + \|\mathcal{A}^{\eta-1}\|\rho K_2\varrho_1.$$

Now, for any $\varrho \in (\varrho_i, \vartheta_i]$, $i = 1, 2, \cdots, q$, after applying $\mathcal{A}^\eta$ on both sides of Eq. (12), we have

$$\|v_n(\varrho)\|_\eta = \|h^1_{i,n}(\varrho, v_n(\varrho_i^-))\|_\eta \leq D_{h_i^1}. \tag{14}$$

Similarly, for any $\varrho \in (\vartheta_i, \varrho_{i+1}]$, $i = 1, 2, \cdots, q$, after applying $\mathcal{A}^\eta$ on both sides of Eq. (12), we have

$$\begin{aligned} \|v_n(\varrho)\|_\eta &\leq \|C(\varrho - \vartheta_i)\|\|h^1_{i,n}(\vartheta_i, v_n(\varrho_i^-))\|_\eta + \|S(\varrho - \vartheta_i)\|\|h^2_{i,n}(\vartheta_i, v_n(\varrho_i^-))\|_\eta \\ &\quad + \int_{\vartheta_i}^\varrho \|\mathcal{A}^{\eta-1}\|\|\mathcal{A}S(\varrho - \vartheta)\|\|\mathcal{F}_n(\vartheta, v_n(\vartheta))d\vartheta \\ &\leq MD_{h_i^1} + \tilde{M}D_{h_i^2} + \|\mathcal{A}^{\eta-1}\|\rho K_2(\varrho - \vartheta_i) \\ &\leq U_i, \end{aligned} \tag{15}$$

where

$$U_i = MD^1_{h_i} + \tilde{M}D^2_{h_i} + \|\mathcal{A}^{\eta-1}\|\rho K_2\varrho_{i+1}. \tag{16}$$

Thus, we get the desired result. □

4 Convergence of approximate solutions

This section shows that the approximate solutions (12) form a Cauchy sequence and converges to a unique mild solution $\nu(\cdot)$ of the Eq. (3).

Theorem 6 *Let* (**A1**) − (**A3**) *fulfilled and* $y_0, z_0 \in D(\mathcal{A})$ *then*
$\lim_{m\to\infty} \sup_{\{n\geq m,\ 0\leq\varrho\leq T\}} \|\nu_n(\varrho) - \nu_m(\varrho)\|_\alpha = 0.$

Proof Let $0 < \alpha < \eta < 1$ and $\varrho \in [0, T]$. From Lemma 5, it follows that

$$\begin{aligned}\|(\mathcal{P}^n - \mathcal{P}^m)\nu_m(\varrho)\|_\alpha &\leq \lambda_m^{\alpha-\eta}\|\mathcal{A}^\eta \nu_m(\varrho)\| \\ &\leq \frac{1}{\lambda_m^{\eta-\alpha}} U'. \end{aligned} \tag{17}$$

Using (**A2**) and inequality (17), we get

$$\begin{aligned}&\|\mathcal{F}_n(\vartheta, \nu_n(\vartheta)) - \mathcal{F}_m(\vartheta, \nu_m(\vartheta))\| \\ &\leq \|\mathcal{F}_n(\vartheta, \nu_n(\vartheta)) - \mathcal{F}_n(\vartheta, \nu_m(\vartheta))\| + \|\mathcal{F}_n(\vartheta, \nu_m(\vartheta)) - \mathcal{F}_m(\vartheta, \nu_m(\vartheta))\| \\ &\leq K_1\left[\|\nu_n(\vartheta) - \nu_m(\vartheta)\|_\alpha + \|(\mathcal{P}^n - \mathcal{P}^m)\nu_m(\vartheta)\|_\alpha\right] \\ &\leq K_1\left[\|\nu_n - \nu_m\|_{\mathcal{PC},\alpha} + \frac{U'}{\lambda_m^{(\eta-\alpha)}}\right]. \end{aligned} \tag{18}$$

Similarly, (**A3**) and inequality (17) imply that

$$\begin{aligned}&\|h_{i,n}^j(\varrho, \nu_n(\varrho_i^-)) - h_{i,n}^j(\varrho, \nu_m(\varrho_i^-))\|_\alpha \\ &\leq \left[\|h_{i,n}^j(\varrho, \nu_n(\varrho_i^-)) - h_{i,n}^j(\varrho, \nu_m(\varrho_i^-))\|_\alpha + \|h_{i,n}^j(\varrho, \nu_m(\varrho_i^-)) - h_{i,m}^j(\varrho, \nu_m(\varrho_i^-))\|_\alpha\right] \\ &\leq C_{h_i^j}\left[\|\nu_n(\varrho_i^-) - \nu_m(\varrho_i^-)\|_\alpha + \|(\mathcal{P}^n - \mathcal{P}^m)\nu_m(\varrho_i^-)\|_\alpha\right] \\ &\leq C_{h_i^j}\left[\|\nu_n - \nu_m\|_{\mathcal{PC},\alpha} + \frac{U'}{\lambda_m^{(\eta-\alpha)}}\right]. \end{aligned} \tag{19}$$

Now, for any $\varrho \in (0, \varrho_1]$, by using inequality (18), we have

$$\begin{aligned}\|\nu_n(\varrho) - \nu_m(\varrho)\|_\alpha &\leq \int_0^\varrho \|\mathcal{A}^{\alpha-1}\|\|\mathcal{A}S(\varrho - \vartheta)\|\|\mathcal{F}_n(\vartheta, \nu_n(\vartheta)) - \mathcal{F}_m(\vartheta, \nu_m(\vartheta))\|d\vartheta \\ &\leq \|\mathcal{A}^{\alpha-1}\|\rho K_1\varrho_1\left[\|\nu_n - \nu_m\|_{\mathcal{PC},\alpha} + \frac{U'}{\lambda_m^{(\eta-\alpha)}}\right]. \end{aligned}$$

Hence

$$\|\nu_n - \nu_m\|_{\mathcal{PC},\alpha} \leq \frac{Q_0}{(1 - Q_0)}\frac{1}{\lambda_m^{(\eta-\alpha)}} U'. \tag{20}$$

Similarly, for any $\varrho \in (\varrho_i, \vartheta_i]$, $i = 1, 2, \cdots, q$, we utilize inequality (19) and obtain

$$\begin{aligned}\|\nu_n(\varrho) - \nu_m(\varrho)\|_\alpha &= \|h_{i,n}^1(\varrho, \nu_n(\varrho_i^-)) - h_{i,m}^1(\varrho, \nu_m(\varrho_i^-))\|_\alpha \\ &\leq C_{h_i^1}\left[\|\nu_n - \nu_m\|_{\mathcal{PC},\alpha} + \frac{U'}{\lambda_m^{(\eta-\alpha)}}\right]. \end{aligned}$$

Hence

$$\|\nu_n - \nu_m\|_{\mathcal{PC},\alpha} \leq \frac{C_{h_i^1}}{(1 - C_{h_i^1})}\frac{1}{\lambda_m^{(\eta-\alpha)}} U'. \tag{21}$$

Also, for any $\varrho \in (\vartheta_i, \varrho_{i+1}]$, $i = 1, 2, \cdots, q$, we use inequalities (18), (19) and compute

$$\begin{aligned}\|\nu_n(\varrho) - \nu_m(\varrho)\|_\alpha &\le \|C(\varrho - \vartheta_i)\|\|h^1_{i,n}(\vartheta_i, \nu_n(\varrho_i^-)) - h^1_{i,m}(\vartheta_i, \nu_m(\varrho_i^-))\|_\alpha \\ &\quad + \|S(\varrho - \vartheta_i)\|\|h^2_{i,n}(\vartheta_i, \nu_n(\varrho_i^-)) - h^2_{i,m}(\vartheta_i, \nu_m(\varrho_i^-))]\|_\alpha \\ &\quad + \int_{\vartheta_i}^{\varrho} \|\mathcal{A}^{\alpha-1}\|\|\mathcal{A}S(\varrho - \vartheta)\|\|\mathcal{F}_n(\vartheta, \nu_n(\vartheta)) - \mathcal{F}_m(\vartheta, \nu_m(\vartheta))\|d\vartheta \\ &\le \left(MC_{h_i^1} + \tilde{M}C_{h_i^2} + \|\mathcal{A}^{\alpha-1}\|\rho K_1 \varrho_{i+1}\right)\left[\|\nu_n - \nu_m\|_{\mathcal{PC},\alpha} + \frac{U'}{\lambda_m^{(\eta-\alpha)}}\right].\end{aligned}$$

Hence

$$\|\nu_n - \nu_m\|_{\mathcal{PC},\alpha} \le \frac{Q_i}{(1 - Q_i)} \frac{1}{\lambda_m^{(\eta-\alpha)}} U'. \tag{22}$$

By the assumption **(A1)**, $\frac{1}{\lambda_m^{\eta-\alpha}} \to 0$ as $m \to \infty$. Therefore, from the inequalities (20), (21) and (22), we get the desired result. □

Theorem 7 *If $y_0, z_0 \in D(\mathcal{A})$ and* **(A1)** $-$ **(A3)** *fulfilled. Then there exists a function $\nu \in B_R(\mathcal{PC}(0, T]; \mathcal{H}_\alpha))$ satisfying Eq.* (3) *such that $\nu_n \to \nu$ as $n \to \infty$ in $B_R(\mathcal{PC}(0, T]; \mathcal{H}_\alpha))$.*

Proof Theorems 3 and 6 have already verified the existence and convergence of ν_n. Now, it suffices to show that the limit ν of ν_n is given by Eq. (3). We have

$$\begin{aligned}\|\mathcal{F}_n(\varrho, \nu_n(\varrho)) - \mathcal{F}(\varrho, \nu(\varrho))\| &\le K_1\big[\|(\mathcal{P}^n - I)\nu_n(\varrho)\|_\alpha + \|\nu_n(\varrho) - \nu(\varrho)\|_\alpha\big] \\ &\le K_1\big[\|\mathcal{P}^n(\nu_n - \nu)\|_{\mathcal{PC},\alpha} + \|(\mathcal{P}^n - I)\nu\|_{\mathcal{PC},\alpha}\big].\end{aligned} \tag{23}$$

Also

$$\begin{aligned}\|h^j_{i,n}(\varrho, \nu_n(\varrho_i^-)) - h^j_i(\varrho, \nu(\varrho_i^-))\|_\alpha &\le C_{h_i^j}\big[\|(\mathcal{P}^n - I)\nu_n(\varrho_i^-)\|_\alpha + \|\nu_n(\varrho_i^-) - \nu(\varrho_i^-)\|_\alpha\big] \\ &\le C_{h_i^j}\big[\|\mathcal{P}^n(\nu_n - \nu)\|_{\mathcal{PC},\alpha} + \|(\mathcal{P}^n - I)\nu\|_{\mathcal{PC},\alpha}\big].\end{aligned} \tag{24}$$

Since $\nu_n \to \nu$ and $\mathcal{P}^n\nu \to \nu$ as $n \to \infty$ therefore the right hand side of inequalities (23) and (24) tends to zero.

Thus, for any $\varrho \in (\vartheta_i, \varrho_{i+1}]$, $i = 1, 2, \cdots, q$

$$\begin{aligned}\|\nu_n(\varrho) - \nu(\varrho)\|_\alpha &\le \|C(\varrho - \vartheta_i)\|\|h^1_{i,n}(\vartheta_i, \nu_n(\varrho_i^-)) - h^1_{i,n}(\vartheta_i, \nu_n(\varrho_i^-))\|_\alpha \\ &\quad + \|S(\varrho - \vartheta_i)\|\|h^2_{i,n}(\vartheta_i, \nu_n(\varrho_i^-)) - h^2_{i,n}(\vartheta_i, \nu_n(\varrho_i^-))\|_\alpha \\ &\quad + \int_{\vartheta_i}^{\varrho} \|\mathcal{A}S(\varrho - \vartheta)\|\|\mathcal{F}_n(\vartheta, \nu_n(\vartheta)) - \mathcal{F}(\vartheta, \nu(\vartheta))\|d\vartheta \\ &\quad \to 0 \quad \text{as} \quad n \to \infty.\end{aligned} \tag{25}$$

As a result of the preceding inequalities, one can easily verified the same argument for any ϱ lies in $(0, \varrho_1]$ and $(\varrho_i, \vartheta_i]$, $i = 1, 2, \cdots, q$. Hence, the limit ν is given by Eq. (3). □

5 Faedo-Galerkin approximations

This section deals with the existence and convergence analysis of F-G approximate solutions of system (1).

By projecting the approximate solutions (12) onto $\mathcal{H}_n$, we obtain finite dimensional approximations i.e. $\bar{\nu}_n(\varrho) = \mathcal{P}^n\nu_n(\varrho)$, referred to as F-G approximations satisfying

$$\bar{\nu}_n(\varrho) = \begin{cases} C(\varrho)\mathcal{P}^n y_0 + S(\varrho)\mathcal{P}^n z_0 + \int_0^\varrho S(\varrho - \vartheta)\mathcal{P}^n\mathcal{F}_n(\vartheta, \mathcal{P}^n\nu_n(\vartheta))d\vartheta, \quad \varrho \in (0, \varrho_1], \\ \mathcal{P}^n h^1_{i,n}(\varrho, \mathcal{P}^n\nu_n(\varrho_i^-)), \quad \varrho \in (\varrho_i, \vartheta_i], \quad i = 1, 2, \cdots, q, \\ C(\varrho - \vartheta_i)\mathcal{P}^n h^1_{i,n}(\vartheta_i, \nu_n(\varrho_i^-)) + S(\varrho - \vartheta_i)\mathcal{P}^n h^2_{i,n}(\vartheta_i, \mathcal{P}^n\nu_n(\varrho_i^-)) \\ \quad + \int_{\vartheta_i}^\varrho S(\varrho - \vartheta)\mathcal{P}^n\mathcal{F}_n(\vartheta, \mathcal{P}^n\nu_n(\vartheta))d\vartheta, \ \varrho \in (\vartheta_i, \varrho_{i+1}], \ i = 1, 2, \cdots, q. \end{cases} \tag{26}$$

Solutions ν and $\bar{\nu}_n$ of Eqs. (3) and (26) respectively, have the following representation

$$\nu(\varrho) = \sum_{l=1}^{\infty} \alpha_l(\varrho)\psi_l, \quad \alpha_l = \langle \nu(\varrho), \psi_l \rangle, \ l \in \mathbb{N}; \tag{27}$$

$$\bar{\nu}_n(\varrho) = \sum_{l=1}^{n} \alpha_l^n(\varrho)\psi_l, \quad \alpha_l^n(\varrho) = \langle \bar{\nu}_n(\varrho), \psi_l \rangle, \ l = 1, 2, \cdots, n. \tag{28}$$

Now, we define

$$\mathcal{P}^n \nu(\varrho) = \sum_{l=1}^{n} \alpha_l^n(\varrho)\psi_l, \quad \alpha_l^n(\varrho) = \langle \nu(\varrho), \psi_l \rangle, \ l = 1, 2, \cdots, n. \tag{29}$$

The F-G method approximates problem (1) as

$$\begin{cases} \frac{d^2\mathcal{P}^n\nu(\varrho)}{d\varrho^2} + \mathcal{P}^n\mathcal{A}\mathcal{P}^n\nu(\varrho) = \mathcal{P}^n\mathcal{F}(\varrho, \mathcal{P}^n\nu(\varrho)), \ \varrho \in (\vartheta_i, \varrho_{i+1}], i = 0, 1, 2, \cdots, q, \\ \mathcal{P}^n\nu(\varrho) = \mathcal{P}^n h_i^1(\varrho, \mathcal{P}^n\nu(\varrho_i^-)), \ \varrho \in (\varrho_i, \vartheta_i], \ i = 1, 2, \cdots, q, \\ \frac{d\mathcal{P}^n\nu(\varrho)}{d\varrho} = \mathcal{P}^n h_i^2(\varrho, \mathcal{P}^n\nu(\varrho_i^-)), \ \varrho \in (\varrho_i, \vartheta_i], \ i = 1, 2, \cdots, q, \\ \mathcal{P}^n\nu(0) = \mathcal{P}^n y_0, \quad \mathcal{P}^n\nu'(0) = \mathcal{P}^n z_0. \end{cases} \tag{30}$$

Therefore (26) is equivalent to the system of differential equations

$$\begin{cases} \frac{d^2\alpha_l^n(\varrho)}{d\varrho^2} + \lambda_l\alpha_l^n(\varrho) = \mathcal{F}_l^n(\varrho, \alpha_1^n(\varrho), \cdots, \alpha_n^n(\varrho)), \ \varrho \in (\vartheta_i, \varrho_{i+1}], i = 0, 1, 2, \cdots, q, \\ \alpha_l^n(\varrho) = h_{i,l}^{1,n}(\varrho, \alpha_1^n(\varrho), \cdots, \alpha_n^n(\varrho)), \ \varrho \in (\varrho_i, \vartheta_i], \ i = 1, 2, \cdots, q, \\ \frac{d\alpha_l^n(\varrho)}{d\varrho} = h_{i,l}^{2,n}(\varrho, \alpha_1^n(\varrho), \cdots, \alpha_n^n(\varrho)), \ \varrho \in (\varrho_i, \vartheta_i], \ i = 1, 2, \cdots, q, \\ \alpha_l^n(0) = \langle y_0, \psi_l \rangle, \quad \dot{\alpha_l}^n(0) = \langle z_0, \psi_l \rangle, \ i = 1, 2, \cdots, q \end{cases} \tag{31}$$

where

$$\mathcal{F}_l^n(\varrho, \alpha_1^n(\varrho), \cdots, \alpha_n^n(\varrho)) = \langle \mathcal{F}(\varrho, \sum_{j=1}^{n} \alpha_j^n(\varrho)\psi_j), \psi_l \rangle,$$

$$h_{i,l}^{1,n}(\varrho, \alpha_1^n(\varrho), \cdots, \alpha_n^n(\varrho)) = \langle h_i^1(\varrho, \sum_{j=1}^{n} \alpha_j^n(\varrho_i^-)\psi_j), \psi_l \rangle,$$

$$h_{i,l}^{2,n}(\varrho, \alpha_1^n(\varrho), \cdots, \alpha_n^n(\varrho)) = \langle h_i^2(\varrho, \sum_{j=1}^{n} \alpha_j^n(\varrho_i^-)\psi_j), \psi_l \rangle.$$

There exists a unique solution $(\alpha_1^n(\varrho), \cdots, \alpha_n^n(\varrho))$ of this system for $\varrho \in [0, T]$, and for each value of ϱ we can be evaluate it by either analytical or numerical techniques.

Theorem 8 *Let $y_0, z_0 \in D(\mathcal{A})$ and the assumptions* (**A1**) − (**A3**) *satisfied. Then,*

$$\lim_{m\to\infty} \sup_{\{n\geq m, 0\leq\varrho\leq T\}} \|\bar{\nu}_n(\varrho) - \bar{\nu}_m(\varrho)\|_\alpha = 0.$$

Proof For $n \geq m$, we have

$$\begin{aligned} \|\bar{\nu}_n(\varrho) - \bar{\nu}_m(\varrho)\|_\alpha &= \|\mathcal{P}^n\nu_n(\varrho) - \mathcal{P}^m\nu_m(\varrho)\|_\alpha \\ &\leq \|\mathcal{P}^n(\nu_n(\varrho) - \nu_m(\varrho))\|_\alpha + \|(\mathcal{P}^n - \mathcal{P}^m)\nu_m(\varrho)\|_\alpha \\ &\leq \|\nu_n(\varrho) - \nu_m(\varrho)\|_\alpha + \frac{1}{\lambda_m^{\eta-\alpha}}U'. \end{aligned}$$

Thus, we attain the intended result through the utilization of Theorem 6. □

Next, by the following theorem the convergence of F-G approximate solutions of (1) can be demonstrated

Theorem 9 *Let $y_0, z_0 \in D(\mathcal{A})$ and* (**A1**) − (**A3**) *hold. Then there exist functions $\nu, \bar{\nu}_n \in B_R(\mathcal{PC}([0,T];\mathcal{H}_\alpha))$ such that $\bar{\nu}_n \to \nu$ as $n \to \infty$ in $B_R(\mathcal{PC}([0,T];\mathcal{H}_\alpha))$.*

Proof This theorem can be proved by considering the outcomes of Theorems 7 and 8. □

Now, we state the following theorem for $\{\alpha_l^n(s)\}$.

Theorem 10 *Let $y_0, z_0 \in D(\mathcal{A})$ and assumptions* (**A1**) − (**A3**) *fulfilled. Then*

$$\lim_{n\to\infty} \sup_{\{0\le\varrho\le T\}} \left(\sum_{l=1}^{n} \lambda_l^{2\alpha} |\alpha_l^n(\vartheta) - \alpha_l(\vartheta)|^2 \right) = 0.$$

Proof

$$\begin{aligned} \|\bar{\nu}_n(\vartheta) - \nu(\vartheta)\|_\alpha^2 &= \|\mathcal{A}^\alpha \big[\bar{\nu}_n(\vartheta) - \nu(\vartheta)\big]\|^2 \\ &= \|\mathcal{A}^\alpha \Big[\sum_{l=1}^{\infty} \big(\alpha_l^n(\vartheta) - \alpha_l(\vartheta)\big)\psi_l \Big]\|^2 \\ &= \sum_{l=1}^{\infty} \lambda_l^{2\alpha} |\alpha_l^n(\vartheta) - \alpha_l(\vartheta)|^2. \end{aligned}$$

Therefore,

$$\sum_{l=1}^{n} \lambda_l^{2\alpha} |\alpha_l^n(\vartheta) - \alpha_l(\vartheta)|^2 \le \|\bar{\nu}_n(\vartheta) - \nu(\vartheta)\|_\alpha^2.$$

The required result is directly inferred from the findings presented in Theorem 9. □

6 Application

We employ abstract results of this manuscript on a concrete impulsive second order partial differential equations.

Example 1 In this application, let $\mathcal{H} = L^2(0,\pi)$:

$$\begin{aligned} \frac{\partial^2 V(\varrho,\xi)}{\partial \varrho^2} + \frac{\partial^2 V(\varrho,\xi)}{\partial \xi^2} &= \xi \sin\varrho + \frac{e^{-\varrho}}{\sqrt{72} + |V(\varrho,\xi)|}, \ (\varrho,\xi) \in (\vartheta_i, \varrho_{i+1}] \times (0,\pi), \\ & \qquad\qquad i = 1, 2, \cdots, q \\ V(\varrho,0) = V(\varrho,\pi) &= 0, \quad \varrho \in [0,T], \ 0 < T < \infty \end{aligned} \tag{32}$$

$$\begin{aligned} V(\varrho,\xi) &= \frac{1}{(2i+1)} \sin i\varrho \frac{|V(\varrho_i^-,\xi)|}{1+|V(\varrho_i^-,\xi)|}, \quad (\varrho,\xi) \in (\varrho_i, \vartheta_i] \times (0,\pi), \\ & \qquad\qquad i = 1, 2, \cdots, q \\ \frac{\partial}{\partial\varrho} V(\varrho,\xi) &= \frac{i}{(2i+1)} \cos i\varrho \frac{|V(\varrho_i^-,\xi)|}{1+|V(\varrho_i^-,\xi)|}, \quad (\varrho,\xi) \in (\varrho_i, \vartheta_i] \times (0,\pi), \\ & \qquad\qquad i = 1, 2, \cdots, q \\ V(0,\xi) = y_0(\xi), &\quad \frac{\partial}{\partial\varrho} V(0,\xi) = z_0(\xi), \quad \xi \in (0,\pi). \end{aligned}$$

The operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined as follows:

$$\mathcal{A}\nu = -\frac{d^2}{d\xi^2}\nu, \quad \nu \in D(\mathcal{A}), \tag{33}$$

where

$$D(\mathcal{A}) = \{\nu \in \mathcal{H}_0^1(0,\pi) \cap H^2(0,\pi) : \nu'' \in \mathcal{H} \text{ and } \nu(0) = \nu(\pi) = 0\}.$$

Let $\lambda \in \mathbb{R}$ and $\nu \in D(\mathcal{A})$ such that $\mathcal{A}\nu = -\nu''$ also $\langle A\nu, \nu\rangle = \langle \lambda\nu, \nu\rangle$. Hence

$$\langle -\nu'', \nu\rangle = |\nu'|_{L^2}^2 = \lambda|\nu|_{L^2}^2.$$

Therefore, $\lambda > 0$. Furthermore, $\mathcal{A}$ has discrete spectrum $\lambda = \lambda_n = n^2$ with corresponding normalized eigenfunctions $\nu_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi),\ n \in \mathbb{N}$.
For $\nu \in D(\mathcal{A})$, operator $\mathcal{A}$ has the following series representation

$$\mathcal{A}\nu = \sum_{n=1}^{\infty} n^2 < \nu, \nu_n > \nu_n.$$

A SCCF $(C(\varrho))_{\varrho\in\mathbb{R}}$ on $\mathcal{H}$ generated by $-\mathcal{A}$ is defined as

$$C(\varrho)\nu = \sum_{n=1}^{\infty} \cos(n\varrho) < \nu, \nu_n > \nu_n, \quad \nu \in \mathcal{H},$$

and the associated sine family $(S(\varrho))_{\varrho\in\mathbb{R}}$ on $\mathcal{H}$, given by

$$S(\varrho)\nu = \sum_{n=1}^{\infty} \frac{1}{n}\sin(n\varrho) < \nu, \nu_n > \nu_n, \quad \nu \in \mathcal{H}. \tag{34}$$

Therefore, the operator $S(\varrho)$ is compact and $\|C(\varrho)\| = \|S(\varrho)\| = 1$, for all $\varrho \in \mathbb{R}$.
Also, $-\mathcal{A}$ generates an analytic semigroup $(T(\varrho))_{\varrho\geq 0}$ given by

$$T(\varrho)\nu = \sum_{n=1}^{\infty} e^{-n^2\varrho} < \nu, \nu_n > \nu_n, \quad \nu \in \mathcal{H}.$$

To learn more about cosine theory, we refer the readers to [21,22].

Now, let $\alpha = \frac{1}{2}$, $D(\mathcal{A}^{\frac{1}{2}})$ is a Banach space equipped with the norm

$$\|\nu\|_{\frac{1}{2}} = \|\mathcal{A}^{\frac{1}{2}}\nu\|, \quad \nu \in D(\mathcal{A}^{\frac{1}{2}}),$$

we denote this space by $\mathcal{H}_{\frac{1}{2}} = \mathcal{H}_0^1(0,1)$. For $\nu \in D(\mathcal{A}^{\frac{1}{2}})$, we have

$$\mathcal{A}^{\frac{1}{2}}\nu(\xi) = \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}}\gamma_n\nu_n(\xi),$$

where $\gamma_n \in \mathbb{R}$ is an increasing sequence and $\sum_{n=1}^{\infty} \lambda_n\gamma_n^2 < \infty$.
We can transform the system (32) into the following abstract second order impulsive differential equation in $\mathcal{H} = L^2(0,\pi)$:

$$\begin{aligned}
\frac{d^2\nu(\varrho)}{d\varrho^2} &= -\mathcal{A}\nu(\varrho) + \mathcal{F}(\varrho, \nu(\varrho)), \quad \varrho \in (\vartheta_i, \varrho_{i+1}],\ i = 0, 1, 2, \cdots, q,\\
\nu(\varrho) &= h_i^1(\varrho, \nu(\varrho_i^-)), \quad \varrho \in (\varrho_i, \vartheta_i],\ i = 1, 2, \cdots, q,\\
\frac{d\nu(\varrho)}{d\varrho} &= h_i^2(\varrho, \nu(\varrho_i^-)), \quad \varrho \in (\varrho_i, \vartheta_i],\ i = 1, 2, \cdots, q,\\
\nu(0) &= y_0, \quad \nu'(0) = z_0
\end{aligned} \tag{35}$$

where $\nu(\varrho) = V(\varrho, \cdot)$, i.e. $\nu(\varrho)(\xi) = V(\varrho, \xi)$, $(\varrho, \xi) \in [0, T] \times (0, \pi)$ and the operator $\mathcal{A}$ is same as in Eq. (33). We define the functions by

(i) $\mathcal{F} : [0, T] \times \mathcal{H}_{\frac{1}{2}} \to \mathcal{H}$ is given by

$$\mathcal{F}(\varrho, \nu)(\xi) = \xi \sin \varrho + \frac{e^{-\varrho}}{\sqrt{72} + |\nu(\varrho, \xi)|}$$

(ii) $h_i^1 : (\varrho_i, \vartheta_i] \times \mathcal{H}_{\frac{1}{2}} \to \mathcal{H}_{\frac{1}{2}}, \quad i = 1, 2, \cdots, q$

$$h_i^1(\varrho, \nu(\varrho_i^-))(\xi) = \frac{1}{2i+1}(\sin i\varrho)\frac{|\nu(\varrho_i^-, \xi)|}{|1 + |\nu(\varrho_i^-, \xi)|}$$

(iii) $h_i^2 : (\varrho_i, \vartheta_i] \times \mathcal{H}_{\frac{1}{2}} \to \mathcal{H}_{\frac{1}{2}}, \quad i = 1, 2, \cdots, q$

$$h_i^2(\varrho, \nu(\varrho_i^-))(\xi) = \frac{i}{2i+1}(\cos i\varrho)\frac{|\nu(\varrho_i^-, \xi)|}{|1 + |\nu(\varrho_i^-, \xi)|}$$

For each $\varrho \in [0, T]$ and $\nu_1, \nu_2 \in \mathcal{H}$, we compute

$$\|\mathcal{F}(\varrho, \nu_1) - \mathcal{F}(\varrho, \nu_2)\| \leq \frac{\sqrt{\pi}}{72}\|\mathcal{A}^{-\frac{1}{2}}\|\|\nu_1 - \nu_2\|_{\frac{1}{2}}$$

and

$$\|\mathcal{F}(\varrho, \nu)\|^2 = \int_0^\pi \left|\xi \sin \varrho + \frac{e^{-\varrho}}{\sqrt{72} + |\nu(\varrho, \xi)|}\right|^2 d\xi \leq \pi\left(\pi + \frac{1}{\sqrt{72}}\right)^2.$$

Thus the assumption **(A2)** is verified. Also, one can easily verified the assumptions **(A3)** and **(A4)** with Lipschitz constants $C_{h_i^1} = D_{h_i^1} = \frac{1}{2i+1}$ and $C_{h_i^2} = D_{h_i^2} = \frac{i}{2i+1}$.

Next, we consider a particular case of Example 1.

Example 2 Consider the following partial differential equation given by:

$$\begin{aligned}
\frac{\partial^2 V(\varrho, \xi)}{\partial \varrho^2} + \frac{\partial^2 V(\varrho, \xi)}{\partial \xi^2} &= \xi \sin \varrho + \frac{e^{-\varrho}}{\sqrt{72} + |V(\varrho, \xi)|}, \ (\varrho, \xi) \in (0, 1) \cup (2, \pi] \times (0, \pi), \\
V(\varrho, 0) = V(\varrho, \pi) &= 0, \quad \varrho \in [0, T], \ 0 < T < \infty \\
V(\varrho, \xi) &= \frac{1}{3}\sin \varrho \frac{|V(1^-, \xi)|}{1 + |V(1^-, \xi)|}, \quad (\varrho, \xi) \in (1, 2] \times (0, \pi), \\
\frac{\partial}{\partial \varrho} V(\varrho, \xi) &= \frac{1}{3}\cos \varrho \frac{|V(1^-, \xi)|}{1 + |V(1^-, \xi)|}, \quad (\varrho, \xi) \in (1, 2] \times (0, \pi), \\
V(0, \xi) = \frac{1}{20}e^{-\xi}, &\quad \frac{\partial}{\partial \varrho} V(0, \xi) = \frac{-1}{400}e^{-\xi}, \quad \xi \in (0, \pi),
\end{aligned} \tag{36}$$

where $0 = \vartheta_0 = \varrho_0 < \varrho_1 < \vartheta_1 < \varrho_2 = T < \infty$ with $\vartheta_0 = \varrho_0 = 0, \varrho_1 = 1, \vartheta_1 = 2, \varrho_2 = \pi$; i = 0,1. Where $y_0(\xi) = \frac{1}{20}e^{-\xi}$, $z_0(\xi) = \frac{-1}{400}e^{-\xi}$ and $C_{h_1^1} = D_{h_1^1} = C_{h_1^2} = \frac{1}{3}$.

Example 3 Consider the folllowing undamped forced string system in the space $\mathcal{H} = \mathbb{R}$:

$$\begin{aligned}
\frac{d^2\nu(\varrho)}{d\varrho^2} + a\nu(\varrho) + m \sin \nu(\varrho) = \cos \varrho \quad \varrho \in (\vartheta_i, \varrho_{i+1}], \ i = 0, 1, 2, \cdots, q, \\
\nu(\varrho) = a_i \tanh(\nu(\varrho_i^-))e(\varrho) \quad \varrho \in (\varrho_i, \vartheta_i], \ i = 1, 2, \cdots, q, \\
\frac{d\nu(\varrho)}{d\varrho} = a_i \tanh(\nu(\varrho_i^-))e'(\varrho), \quad \varrho \in (\varrho_i, \vartheta_i], \ i = 1, 2, \cdots, q, \\
\nu(0) = y_0, \quad \nu'(0) = z_0,
\end{aligned}$$

where $a \in \mathbb{R}^+$, $m, a_i \in \mathbb{R}$ and $e \in C([0, T]; \mathbb{R})$. We define $\mathcal{A}$ by $\mathcal{A}\nu = -a\nu$ with $D(\mathcal{A}) = \mathbb{R}$. The functions $a_i \tanh(\nu(\varrho_i^-))e(\varrho)$ and $a_i \tanh(\nu(\varrho_i^-))e'(\varrho)$ represent non-instantaneous impulses during the interval $(\varrho_i, \vartheta_i]$, $i = 1, 2, \cdots, q$. $C(\varrho) = \cos(\sqrt{a}\varrho)$ is the SCCF and the associated sine family is given by $S(\varrho) = \frac{1}{\sqrt{a}}\sin(\sqrt{a}\varrho)$.

7 Conclusion

In this manuscript, a class of second order nonlinear NIIDEs in an arbitrary separable Hilbert space is taken into consideration. An exact solution of system (1) is additionally suggested. By projecting the exact solution onto a finite dimensional space, we constructed a sequence of approximate solutions. We have successfully established the existence, uniqueness and convergence of approximate solutions. After that, we provide a rigorous proof demonstrating that approximate solutions constitute a Cauchy sequence, with the mild solution of the original problem (3) emerging as the sequence's limit. Moreover, we proved the similar existence and convergence results for the F-G approximations. Finally, some concrete examples involving partial differential equation are provided to validate the main abstract results. Our research endeavours to present significant outcomes that can be of immense value for future investigations.

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Conflict of interest The authors declare that they have no conflict of interest.

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