Indian J Pure Appl Math (2026) 57:1798–1807
https://doi.org/10.1007/s13226-024-00679-0

ORIGINAL RESEARCH

Counting triangles in smooth cubic hypersurfaces

Mulong Xu

Received: 30 November 2023 / Accepted: 15 July 2024 / Published online: 30 August 2024
© The Indian National Science Academy 2024

Abstract We propose and study the notion of triangles in smooth cubic hypersurfaces. We prove that for a generic cubic n-fold X ($n \geq 2$), the variety of triangles in X is of dimension $3n-6$. We show that on a generic cubic n-fold, the triangles with a given edge can be parametrized by an open subset of a quintic hypersurface in $\mathbb{P}^{n-1}$. In the case of a generic cubic threefold, we show that the locus of the opposite vertices for triangles with a given edge form a curve of degree 10. As a corollary, we get an interesting enumerative result on the number of triangles satisfying some restrictions.

Keywords Cubic hypersurfaces · Varieties of triangles · Varieties of lines

1 Introduction

We work over $\mathbb{C}$, the field of complex numbers, throughout this article. A smooth cubic hypersurface of dimension n, or more simply, a cubic n-fold, is a smooth hypersurface in $\mathbb{P}^{n+1}$ defined by a homogeneous polynomial of degree 3. The study of the lines in smooth cubic hypersurfaces has a long history and occupies a large part of mathematical research [1,4,5]. One of the most famous results in algebraic geometry is the following theorem due to Cayley and Salmon. (The textbook [9] has a nice presentation on this result.)

Every smooth cubic surface has exactly 27 *lines on it.*

When the dimension n of the cubic hypersurface X is strictly larger than 2, the number of lines on X is no longer finite. The set $F = F(X)$ of all lines on X is a projective variety, called the Fano variety of lines of X (whence the letter F). The variety F is relatively quite well-understood [5]. It has very nice geometric properties, and the understanding of F will in turn give some geometric information about the cubic hypersurface X itself. For example, in [4], the Fano variety of lines in the cubic threefold is studied to give rationality results of the cubic threefold. In [3], the Fano variety of lines in the cubic fourfold is proven to be a holomorphic symplectic manifold. In this article, we propose and study the notion *triangles* in smooth cubic hypersurfaces, which is defined as follows.

Definition 1.1 Let X be a smooth cubic hypersurface. A *triangle* on X is a set of three distinct coplanar lines $\{L_1, L_2, L_3\}$ lying on X. Each line L_i of a triangle is called an *edge* of this triangle. The intersection point of L_2 and L_3 is called the *opposite vertex* of L_1.

The notion of triangles has been already proposed and studied for cubic fourfolds in [10], but there does not seem to be systematic research on triangles in cubic hypersurfaces of all dimensions in the literature. This article studies the configurations of triangles and the geometry of the variety of triangles.

Communicated by V Srinivas.

M. Xu (✉)
Nanjing Foreign Language School, Nanjing, China
E-mail: xumulong109@outlook.com

Let W be the variety of triangles in X. In the cubic surface case (i.e., $n = 2$), since there are 27 lines on X, the number of triangles is also finite, hence $\dim W = 0$ in this case. In fact, we can even conclude that there are exactly 45 triangles in any smooth cubic surface by studying the configurations of the 27 lines. See Corollary 2.4 for a discussion on this counting result. In the higher dimensional cases, W is an algebraic variety. About this variety, we have the following result.

Theorem 1.2 *For a generic cubic hypersurface, the dimension of W is $3n - 6$.*

This dimension result coincides with the observation in [10] that the variety of triangles of a cubic *fourfold* has dimension 6.

Since the dimension of W is strictly larger than 0, we can try to impose more restrictions on the triangles and study the geometry of the triangles under these conditions. In this project, we are mostly motivated by the following guiding problem.

Problem 1.3 *Let $X \subset \mathbb{P}^{n+1}$ be a generic cubic n-fold and L a generic line on X. Let S be a generic surface of degree d on X. What is the number of triangles on X with L an edge such that the opposite vertex lies on S?*

By a standard dimension counting argument, it is not hard to see that the number of triangles satisfying the conditions in Problem 1.3 is indeed *finite*. See Proposition 4.1 for a detailed argument. Our first step to attack Problem 1.3 is to study the variety of triangles with a given edge L. Notice that the planes (2-dimensional linear subspaces) containing L in $\mathbb{P}^{n+1}$ can be parametrized by $\mathbb{P}^{n-1}$. We have the following result.

Theorem 1.4 *Let L be a generic line on a generic cubic n-fold X. The triangles in X with L as an edge are parametrized by a non-empty open subset of a quintic hypersurface in $\mathbb{P}^{n-1}$. More precisely, there exists a quintic hypersurface $V \subset \mathbb{P}^{n-1}$ and a codimension 2 subvariety $Y \subset V$ such that the set of triangles in X with L as an edge can be parametrized by $V - Y$.*

Remark 1.5 When $n = 2$, Theorem 1.4 states that there are exactly 5 triangles with a given edge in a cubic surface. This result is classical (see Proposition 2.3). When $n = 3$, V is a quintic curve and the codimension 2 subvariety $Y \subset V$ is empty by dimension reason. Hence, Theorem 1.4 states that the triangles with a given edge in a generic cubic threefold can be parametrized by a plane quintic curve.

Theorem 1.4 helps us to answer Problem 1.3, at least in the case of cubic *threefolds*. Our approach to counting the triangles with a given edge and restricted opposite vertex is to determine the geometry of the locus of the opposite vertices, and then use intersection theory to determine the intersection of this locus and the given surface S. For cubic threefolds, we get the following theorem.

Theorem 1.6 *Let L be a generic line in a generic cubic threefold X. Let $C \subset X$ be the set of opposite vertices of triangles in X with L as an edge. Then C is a curve of degree 10.*

As a corollary, we can give an explicit answer to Problem 1.3 in the case of cubic *threefold*.

Corollary 1.7 *Let $X \subset \mathbb{P}^4$ be a generic cubic threefold and L a generic line on X. Let S be a generic surface of degree d on X. Then the number of triangles on X with L an edge such that the opposite vertex lies on S is*

$$\frac{10d}{3}.$$

Remark 1.8 (i) Since C is of degree 10 and S is of degree d, it is tentative to conclude that C intersects with S at $10d$ points (counted with multiplicities). However, a moment of thinking will convince the reader that it is not true. Since $\dim C + \dim S < \dim \mathbb{P}^4$, we cannot work on the intersection of C and S as in $\mathbb{P}^4$. Instead, one needs to regard that the intersection of C and S happens in X, which is a 3-dimensional variety. This poses the subtlety of the problem. See Section 4.3 for further discussions. (ii) It may be a priori not so clear why the degree d of the surface S must be divided by 3. This is in fact a consequence of a theorem of Lefschetz on hyperplane sections [7]. See Theorem 4.12 for the precise statement.

2 Triangles with a given edge

The main purpose of this section is to prove Theorem 1.4. The $n = 2$ case is classical, but the proofs are inspiring, so we review them in Section 2.1. The higher dimensional cases are more intriguing and are presented in Section 2.2.



2.1 The $n = 2$ case: cubic surfaces

In this subsection, we review the proof of Theorem 1.4 in the cubic surface case, namely, the number of triangles with a given edge in any *smooth* cubic surface is 5 (Proposition 2.3). The proof that we follow here is presented in [9, §7]. The reasons why we devote this subsection to such a well-known result are two-fold. First, This result gives the special case of our project in the 2-dimensional case and the proof as is presented in [9] is elementary and has inspired us in higher dimensional cases. Second, the results of this subsection, especially Theorem 2.1, are actually used in further sections (c.f. Lemma 3.3). In the end, we count the number of triangles in a nonsingular cubic surface(Corollary2.4).

Let $S \subset \mathbb{P}^3$ be a cubic surface. Contrary to the presentation in [9, §7], at this stage, we do *not* assume S is smooth. This complicates a little the proof of Theorem 2.1. However, this more general statement will be used in later parts of this paper so we need to prove the difference part.

Theorem 2.1 *There exists at least one line L on S.*

Proof Here we follow closely the proof as is presented in [9, Proposition 7.2]. But there are some slight differences since we are not assuming that the cubic surface is smooth here. Looking closely at the proof in [9, Proposition 7.2], we find that we only need the existence of a nonsingular point $P \in S$ (instead of the assumption that S is everywhere smooth). Here, we prove that a nonsingular point always exists unless when the cubic surface is a triple plane.

Lemma 2.2 *Let S be a cubic surface defined by $f(x_0, \ldots, x_3) = 0$. Then there does not exist any nonsingular point in S if and only if there is a linear function $L(x_0, \ldots, x_3)$ such that $f = L^3$.*

Proof The direction ($\Leftarrow$) is clear. Let us show the direction ($\Rightarrow$). If there does not exist any nonsingular point at S, then

$$S \subset \left\{\frac{\partial f}{\partial x_0} = 0\right\} \cap \left\{\frac{\partial f}{\partial x_1} = 0\right\} \cap \left\{\frac{\partial f}{\partial x_2} = 0\right\} \cap \left\{\frac{\partial f}{\partial x_3} = 0\right\}.$$

Let $Q_i = \frac{\partial f}{\partial x_i}$. Because S is a cubic hypersurface, the polynomials Q_0, Q_1, Q_2 and Q_3 will not be constantly equal to 0 at the same time. We may assume that Q_0 is not constantly equal to 0. Since Q_0 is a quadratic polynomial and S is a cubic hypersurface, we can know that the fact $S \subset \{Q_0 = 0\}$. Then, we divide it into 2 cases: case(i): if $S = \{Q_0 = 0\}$, then it implies that $S = Q \cup L$, where Q is a quadratic surface that is equal to $\{Q_0 = 0\}$ and L is a plane contained in Q. The last claim shows that Q is a union L with another plane L'. Then S must contain a double plane L with another (simple) plane L'. However, if $L' \neq L$ then all the points on $L' - L$ are nonsingular, which contradicts our assumption that S contains no nonsingular points. Hence, in this case, S must be a triple plane. Case(ii): if $S \subsetneq \{Q_0 = 0\}$. Since Q has a subsqace, Q can be written in the form of $Q = L \cup L'$. Hence, we have $S = L$ as a set. Then $S = L^3$. As a result, S must be a triple plane, as desired. □

Therefore, if S does not have any nonsingular point, then S is a triple plane and it has lots of lines, as desired. Then the proof of [9, Proposition 7.2] works exactly the same if S contains a nonsingular point. □

The following proposition is stated and proved in [9, Proposition 7.3].

Proposition 2.3 *Let S be a* smooth *cubic surface. Given a line $L \subset S$, there exist exactly* 5 *triangles with L as an edge.*

Corollary 2.4 *There are exactly* 45 *triangles on S.*

Proof From Proposition 2.3, we know that there are 5 triangles on each given edge. From [9, (7.6)], we know that there are a total of 27 lines on S. Every triangle consists of 3 different lines. So the number of the triangles equals $\frac{5\times 27}{3} = 45$. □

2.2 Higher dimensional cases

The study of triangles with a given edge in a cubic hypersurface of higher dimension is quite similar to that in dimension 2. In particular, the idea of the proof of Proposition 2.3 still works in higher dimensions. However, the classification of the residual conics is more complicated (see Lemma 2.5). Since a triangle determines a plane

in $\mathbb{P}^{n+1}$, let us first study the planes in $\mathbb{P}^{n+1}$ containing a given line. We know that the planes in $\mathbb{P}^n$ that pass through a given line can be parametrized by $\mathbb{P}^{n-1}$.

Let L be a line in X. Let $H \subset \mathbb{P}^{n+1}$ be a plane containing L. The intersection of H with X is a cubic curve C. Since the line L is on C, C is the union of a line L and a conic Q. The triangles with L as a given edge correspond to the planes such that the conic Q is the union of two distinct lines, different from L. In what follows, we will calculate explicitly the locus of the planes satisfying the above condition.

Let us assume that the line L is defined by $\{x_0 = x_1 = \ldots = x_{n-1} = 0\}$. Any points on L can be written as the form of $[0 : 0 : \ldots : x_n : x_{n+1}]$. Since $L \subset X$, the defining polynomial of the cubic hypersurface X can be written as

$$f = x_0 Q_0 + x_1 Q_1 + \ldots + x_{n-1} Q_{n-1},$$

where $Q_i = Q_i(x_0, \ldots, x_{n+1})$ are quadratic polynomials. Given an $\mathbf{a} = [a_0 : a_1 : \ldots : a_{n-1}] \in \mathbb{P}^{n-1}$. We identify $\mathbf{a}$ with the point $[a_0 : \ldots : a_{n-1} : 0 : 0] \in \mathbb{P}^{n+1}$ and we use the same notation for the point. Because $\mathbf{a}$, L generate the plane $P_{\mathbf{a}}$, we have $P_{\mathbf{a}} = \{[a_0 R : a_1 R : \ldots : a_{n-1} R : S : T] | [R : S : T] \in \mathbb{P}^2\}$. Combined with the expression of f, we have

$$\begin{aligned} & f|_{P_{\mathbf{a}}}(R, S, T) \\ &= a_0 R Q_0(a_0 R, \ldots, a_{n-1} R, S, T) + \ldots + a_{n-1} R Q_{n-1}(a_0 R, \ldots, a_{n-1} R, S, T) \\ &= R \cdot Q_{\mathbf{a}}(R, S, T), \end{aligned} \tag{1}$$

where $Q_{\mathbf{a}}(R, S, T)$ is a quadratic form in R, S, T whose coefficients are polynomials in $\mathbf{a}$. Let us write

$$Q_{\mathbf{a}}(R, S, T) = (R, S, T) \begin{pmatrix} F(\mathbf{a}) & G_1(\mathbf{a}) & G_2(\mathbf{a}) \\ G_1(\mathbf{a}) & L_1(\mathbf{a}) & L_3(\mathbf{a}) \\ G_2(\mathbf{a}) & L_3(\mathbf{a}) & L_2(\mathbf{a}) \end{pmatrix} \begin{pmatrix} R \\ S \\ T \end{pmatrix}. \tag{2}$$

Then F is a cubic polynomial and G_1, G_2 are quadratic forms, and L_1, L_2, L_3 are linear, with $\mathbf{a}$ as variables. The cubic curve defined by Equation (2) must contain the line $R = 0$, which is exactly the line L. The residual curve is a conic curve defined by the polynomial $Q_{\mathbf{a}}(R, S, T)$. We classify the cubic curves defined by Equation (2) into four types according to the expression of $Q_{\mathbf{a}}(R, S, T)$. We have the following result.

Lemma 2.5 *Exactly one of the four cases happens:*

(i) $P_{\mathbf{a}} \subset X$.
(ii) $P_{\mathbf{a}} \cap X = L \cup L_1 \cup L_2$, *and* $L_1 = L_2$ *or* $L_1 = L$ *or* $L_2 = L$.
(iii) $P_{\mathbf{a}} \cap X = L \cup L_1 \cup L_2$, *and* L_1, L_2, L_3 *are three different lines which are coplanar.*
(iv) $P_{\mathbf{a}} \cap X = L \cup Q$ *where* Q *is a* nonsingular *conic curve.*

The triangles with L as an edge corresponds to the case (iii). To start with, let us first discuss cases (i),(ii) and (iii).

Proposition 2.6 *The planes in cases (i)(ii)(iii) are parametrized by a quintic hypersurface in* $\mathbb{P}^{n-1}$.

Proof These three cases are characterized by the condition $\det Q_{\mathbf{a}} = 0$. So we can get the following equation.

$$\begin{aligned} & F(\mathbf{a})(L_1(\mathbf{a}) \cdot L_2(\mathbf{a}) - L_3(\mathbf{a})^2) - G_1(\mathbf{a})(G_1(\mathbf{a}) \cdot L_2(\mathbf{a}) - G_2(\mathbf{a}) \cdot L_3(\mathbf{a})) \\ & + G_2(\mathbf{a})(G_1(\mathbf{a}) \cdot L_3(\mathbf{a}) - L_1(\mathbf{a}) \cdot G_2(\mathbf{a})) = 0. \end{aligned}$$

The equation is a degree 5 polynomial in $a_0 \ldots a_{n-1}$. So the equation defines a quintic hypersurface in $\mathbb{P}^{n-1}$. So the planes containing L that satisfy case (i),(ii) and (iii) can be seen as a quintic hypersurface in $\mathbb{P}^{n-1}$.

Then we need to eliminate the cases (i) and (ii), so we can get the planes that satisfy case (iii).

Lemma 2.7 *Suppose the cubic hypersurface X is chosen generically. In that case, the planes in the case (i) form a codimension at least* 2 *subvariety of* $\mathbb{P}^{n-1}$, *while the planes in the case (ii) form a codimension* 3 *subvariety of* $\mathbb{P}^{n-1}$.

Proof Since the rank function is an upper semi-continuous function with respect to the coefficients of the defining equation of the cubic hypersurface, it suffices to check the statement for a specific cubic hypersurface. Let $f = x_0x_2^2 + x_1x_3^2 + \ldots + x_{n-2}x_n^2 + x_{n-1}x_{n+1}^2$ and $\mathbf{a} = [\mathbf{a}_0 : \mathbf{a}_1 : \ldots \mathbf{a}_{n-1} : 0 : 0]$. Hence, $P_{\mathbf{a}} = [\mathbf{a}_0R : \mathbf{a}_1R : \ldots : S : T]$. Then the interaction of $f(x)$ and $P_{\mathbf{a}}$ is $f|_{P_{\mathbf{a}}}(R, S, T) = R[(\mathbf{a}_0\mathbf{a}_2^2 + \mathbf{a}_1\mathbf{a}_3^2 + \ldots + \mathbf{a}_{n-3}\mathbf{a}_{n-1}^2)R^2 + \mathbf{a}_{n-2}S^2 + \mathbf{a}_{n-1}T^2]$. Hence, $Q_{\mathbf{a}}$ for this equation is

$$\begin{pmatrix} \Sigma & 0 & 0 \\ 0 & \mathbf{a}_{n-2} & 0 \\ 0 & 0 & \mathbf{a}_{n-1} \end{pmatrix}$$

When the rank of the above matrix equals 1, then $a_{n-2} = a_{n-1} = 0$, indicating that the codimension equals 2.

The condition $R|Q_{\mathbf{a}}(R, S, T)$ is equivalent to saying that the coefficients of S^2, ST, T^2 vanish (i.e., $L_1(\mathbf{a}) = L_2(\mathbf{a}) = L_3(\mathbf{a}) = 0$). For a generic choice of X, they are 3 independent linear conditions. So codimension of the locus of $\mathbf{a} \in \mathbb{P}^{n-1}$ that satisfies $R|Q_{\mathbf{a}}$ is 3. □

Corollary 2.8 *For a generic cubic hypersurface X of dimension $n \geq 2$, the triangles with L as a given edge form a nonempty open subset of a quintic hypersurface in $\mathbb{P}^{n-1}$.*

Remark 2.9 It is useful to notice that for a generic cubic *threefold*, Lemma 2.7 says that there are no planes in the cases (i) and (ii). Therefore, Proposition 2.6 and Lemma 2.7 show that the triangles with a generic given edge in X form a quintic plane curve. This observation is used in Section 4.

3 Varieties of triangles

The main purpose of this section is to prove Theorem 1.2. We use the Grassmainnians for a useful construction in this section, so let us first briefly recall the basic properties of Grassmannians. A more detailed presentation can be found in, for instance, [7, 1.5]

Let W be the variety of all the triangles in X, and F be the variety of all the lines in X.

Proposition 3.1 *For the generic cubic hypersurface X, the dimension of F, the set of points of a variety, equals $2n - 4$.*

Proof Let V be the set of all the homogeneous polynomials of degree 3 in $x_0, x_1, \ldots, x_{n+1}$. Let $\mathbb{P}(V)$ be the projective space associated with V. Then we can view $\mathbb{P}(V)$ as a space parametrizing cubic hypersurfaces in $\mathbb{P}^{n+1}$. Let us consider the following construction:

$$\begin{array}{ccccc} & & I = \{(X, L) \in \mathbb{P}(V) \times Gr(2, n+2) : L \subset X\} & & \\ & \swarrow^{p} & & \searrow^{q} & \\ \mathbb{P}(V) & & & & Gr(2, n+2) \end{array}$$

Let N be the dimension of $\mathbb{P}(V)$.

Lemma 3.2 *For any $L_0 \in Gr(2, n+2)$, the dimension of $q^{-1}(L_0)$ equals to $N - 4$.*

Then in order to calculate the dimension of F, we should first calculate the dimension of I. From [8, Ex. II.3.22], the dimension of I equals the dimension of $Gr(2, n+2)$ plus the dimension of the fiber q. We know that the dimension of $Gr(2, n+2) = 2n$ while from Lemma 3.2, the dimension of the fiber of q equals $N - 4$. As a result, the dimension of I equals $2n + N - 4$. Then we know that $p^{-1}(X_0)$ is equivalent to the set of all the lines in a given hypersurface X which is the same as F. Since $F(X) = p^{-1}(X)$, we need to calculate the dimension of the fibers of p. To this end, we need the following:

Lemma 3.3 *For any cubic hypersurface of dimension $n \geq 2$, there is a line on it. (Corollary of Theorem 2.1)*

Proof Let X be a cubic hypersurface. Let V be a generic 3-dimensional linear subspace. Then $X \cap V$ is a cubic surface. From Theorem 2.1, we can know that for any cubic surface, there must be a line. So the line is also in X. □

By Lemma 3.3, every cubic hypersurface can be projected via the map p. Therefore, the map $p : I \to \mathbb{P}(V)$ is a surjective. Hence, by [8, Ex. II.3.22], the dimension of F is equal to $2n - 4$. □

Theorem 3.4 *For a generic cubic hypersurface X, the dimension of the variety W is $3n - 6$.*

Proof Let $\tilde{W} = \{(L_1, L_2, L_3) \in F \times F \times F : L_1, L_2, L_3$ are distinct and coplanar $\}$. Let $\pi : \tilde{W} \to W$ be the map that sends the ordered list (L_1, L_2, L_3) onto the triangle formed by L_1, L_2 and L_3 as edges. Then π is a $6 : 1$ map. Hence, by [8, Ex. II.3.22], the dimension of $\tilde{W}$ is equal to the dimension of W. Hence, it suffices to calculate the dimension of $\tilde{W}$. Consider the projection $pr_1 : \tilde{W} \to F$ which sends (L_1, L_2, L_3) to L_1. The fiber over $L_1 \in F$ is $\{(L_1, L_2, L_3) : L_2, L_3$ are distinct and different with L_1 and coplanar to $L_1\}$. Then pr_1 is $2 : 1$ to the set of the triangles with a given edge. Hence, by [8, Ex. II.3.22] the dimension of the fiber of pr_1 equals the dimension of the set of the triangles with a given edge which is equal to $n - 2$ by Corollary 2.8. As a result, the dimension of $\tilde{W} = (2n - 4) + (n - 2) = 3n - 6$. So, the dimension of W is also $3n - 6$. □

4 Locus of the opposite vertices

The main goal of this section is to give a response to Problem 1.3 in the case of a generic cubic *threefold*, namely, Corollary 1.7. The strategy is as follows. First, we prove the finitely of the result of this question. Then, we study the locus of all the opposite vertices of triangles with a given edge. By a blow-up construction in Section 4.1, we calculate that this locus is a degree 10 curve in $\mathbb{P}^4$ (Theorem 1.6). Then by considering the intersection of this locus and the given surface, we get Corollary 1.7.

Proposition 4.1 *There are finitely many triangles satisfying the restrictions in Problem 1.3.*

Proof Let V be the locus of opposite vertices in $\mathbb{P}^{n+1}$. Then we just need to show that the intersection of V and S has dimension 0. Any plane passing through L in $\mathbb{P}^{n+1}$ can be projected to a point on $\mathbb{P}^{n-1}$. Let V' be the image of $V - (V \cap L)$ through the map. Any point on V represents a vertex of a triangle with L as an edge. Every point on $V - (V \cap L)$, together with L, can determine a plane that intersects X at a triangle. So every point on V' represents a plane in $\mathbb{P}^{n+1}$ that intersects X at a triangle. Hence, V' can be seen as the set of the triangles with L s a given edge. From the Corollary 2.8, we know that the set of the triangles with L as a given edge is an open quintic hypersurface in $\mathbb{P}^{n-1}$. We define a map ϕ from $V - V \cap L$ to V'. Similar to Lemma 4.9, the map ϕ from $V - V \cap L$ to V is a one-to-one map. Hence, the dimension of V equals the dimension of V' which also equals $n - 2$. Since V, S both lie in X and $n - 2 + 2 = n$ and since S is generic in X, the dimension of the intersection of V and L is 0. □

4.1 A blow-up construction

Let L be a line in $\mathbb{P}^4$, and a projective plane $\mathbb{P}^2$ which does not intersect with L. Any plane that includes L will intersect with the $\mathbb{P}^2$ at only one point. Together with L, any point on $\mathbb{P}^4$ can determine a plane. So we can construct a map from the points on $\mathbb{P}^4$ except those on L to the points in $\mathbb{P}^2$. However, the map is defined by L. In order to solve this, we can blow up $\mathbb{P}^4$ along L, through which we pull all the planes that pass through L apart. We define the image of $\mathbb{P}^4$ after the blow-up as $\mathcal{P}$. Let p be the map from $\mathcal{P}$ to $\mathbb{P}^4$ and q be the map from $\mathcal{P}$ to $\mathbb{P}^2$.

There is a standard construction in algebraic geometry to make the above discussion rigorous. To this end, we define

$$\mathcal{P} = \{([aR : bR : cR : S : T], [a : b : c]) \in \mathbb{P}^4 \times \mathbb{P}^2\}.$$

And $p : \mathcal{P} \to \mathbb{P}^4$ and $q : \mathcal{P} \to \mathbb{P}^2$ are the two projections. We study the geometric behaviors of the maps p and q.

Lemma 4.2 *The map $p : \mathcal{P} \to \mathbb{P}^4$ is the blow-up of $\mathbb{P}^4$ along L.*

Proof We need to show that the fibers over points on L are isomorphic to $\mathbb{P}^2$ whereas the fibers over other points are only one point. For a given point x on $\mathbb{P}^4$. Let the coordinate of x be $[x_0 : x_1 : x_2 : x_3 : x_4]$. If the point x is on L, then $(x_0, x_1, x_2) = 0$. Hence, $p^{-1}(x) = \{([a : b : c], [0 : 0 : 0 : x_3, x_4])\} \cong \mathbb{P}^2$. On the other hand, if the point x is not on L, then $(x_0, x_1, x_2) \neq 0$. Hence, $p^{-1}(x) = \{([x_0 : x_1 : x_2], [x_0 : x_1 : x_2 : x_3 : x_4])\}$ is a point. □

Definition 4.3 A $\mathbb{P}^2$-bundle over X is a map $\pi : Y \to X$ such that there is an open covering $X = \bigcup_{i\in I} U_i$, such that over each U_i, the restriction $\pi_{U_i} : Y_{U_i} \to U_i$ is isomorphic to the trivial bundle, i.e., $Y_{U_i} \cong U_i \times \mathbb{P}^2$.

Lemma 4.4 *The map $q : \mathcal{P} \to \mathbb{P}^2$ is a $\mathbb{P}^2$-bundle.*

Proof The projective plane $\mathbb{P}^2$ can be covered with 3 pieces of $\mathbb{A}^2$. Let us denote them U_0, U_1 and U_2, where $U_i := \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 : x_i \neq 0\}$. To prove that $q : \mathcal{P} \to \mathbb{P}^2$ is a $\mathbb{P}^2$ bundle, we need to prove that q is a trivial bundle over U_0, U_1, U_2, in other words, $q|_{U_i} : \mathcal{P}_{U_i} \to U_i$ is isomorphic to $U_i \times \mathbb{P}^2 \to U_i$. Let us check this on U_0, and similar arguments for U_1 and U_2. Let $(x, y) \in U_0 \cong \mathbb{A}^2$. Then in the homogeneous coordinate form, this point is $[1 : x : y] \in \mathbb{P}^2$. From the defintion of the map q we know that $q^{-1}(U_0) = \{([1 : x : y], [R : xR : yR : S : T])\}$. Let ϕ be a map

$$q^{-1}(U_0) \to U_0 \times \mathbb{P}^2.$$

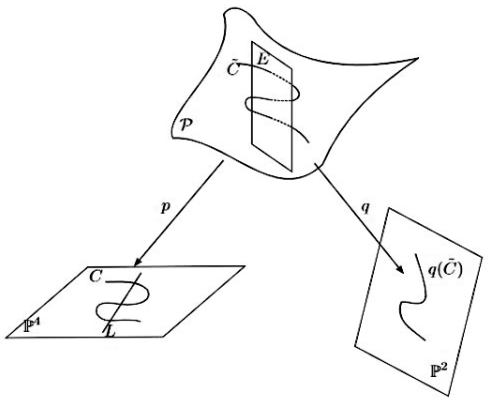
$$([1 : x : y], [R : xR : yR : S : T]) \mapsto ((x, y), [R : S : T]).$$

The inverse of ϕ is defined as the following:

$$U_0 \times \mathbb{P}^2 \to q^{-1}(U_0)$$

$$((x, y), [R : S : T]) \mapsto ([1 : x : y], [R : xR : yR : S : T]).$$

So we can know that ϕ is bijective . Hence, $q^{-1}(U_0) \cong U_0 \times \mathbb{P}^2$. Since $\mathbb{P}^2 = U_0 \cup U_1 \cup U_2$, the map $q : \mathcal{P} \to \mathbb{P}^2$ is a $\mathbb{P}^2$-bundle. □



The blow-up structure of $p : \mathcal{P} \to \mathbb{P}^4$ and the $\mathbb{P}^2$-bundle structure of $q : \mathcal{P} \to \mathbb{P}^2$ give us the following useful theorem that relates the curves in $\mathbb{P}^4$ and its image in $\mathbb{P}^2$. For a rigorous proof of this result, see [6, Theorem 6.7, Corollary 6.7.1] or [4, (5.13)].

Theorem 4.5 *([6]) With the notations as in the above blow-up construction, let C be a curve in $\mathbb{P}^4$ such that its strict transformation $\tilde{C}$ intersects the exceptional divisor at m points, counted with multiplicities. Assume that $q : \tilde{C} \to q(\tilde{C})$ is generically one-to-one. Then we have*

$$\deg C = \deg q(\tilde{C}) + m.$$

4.2 Degree of the locus of the opposite vertices

Our result in this section has been shown from the paper of [12]. Hence, we just give a short proof here. We denote $C \subset X \subset \mathbb{P}^4$ as the locus of opposite vertices of triangles with L as an edge. In other words,

$$C = \{x \in \mathbb{P}^4 : x = L_1 \cap L_2, \{L_1, L_2, L\} \text{ is a triangle on } X\}.$$

In this subsection, we are going to calculate the degree of C. In the spirit of Theorem 4.5, we need to calculate the intersection number of the strict transformation $\tilde{C}$ and the exception divisor E.

Proposition 4.6 *For a generic cubic threefold X, the intersection number of $\tilde{C}$ and E is* 5.

Proof It is not hard to see that the defining equation of $\tilde{C}$ is

$$\begin{cases} g_1R + g_2S + g_3T = 0 \\ g_2R + g_4S + g_6T = 0 \\ g_3R + g_6S + g_5T = 0 \end{cases}.$$

It is not hard to see that the defining equation of E is $R = 0$. Then $\tilde{C} \cap E$ is given by the following system of equations on $\mathcal{P}$:

$$\begin{cases} g_1R + g_2S + g_3T = 0 \\ g_2R + g_4S + g_6T = 0 \\ g_3R + g_6S + g_5T = 0 \\ R = 0 \end{cases}.$$

This system of equations is equivalent to the following :

$$\begin{cases} g_2S + g_3T = 0 \\ g_4S + g_6T = 0 \\ g_6S + g_5T = 0 \end{cases}. \tag{3}$$

Consider the coefficients of S, T, we can get the following matrix:

$$\begin{pmatrix} g_2 & g_3 \\ g_4 & g_6 \\ g_6 & g_5 \end{pmatrix}. \tag{4}$$

Given $\mathbf{a}$ in $\mathbb{P}^2$, the system of equations (3) has a nonzero solution in (S, T) if and only if the matrix (4) has rank ≤ 1. Then for any 2×2 submatrix, its determinant must equal 0. Hence,

$$\begin{cases} g_2g_6 - g_3g_4 = 0 \\ g_2g_5 - g_3g_6 = 0 \\ g_4g_5 - g_6^2 = 0 \end{cases}. \tag{5}$$

Lemma 4.7 *For a generic choice of g_i, $i \in [2, 6]$, the system of equations (5) has exactly* 5 *solutions in $\mathbb{P}^2$.*

Proof The first and second equations have 9 solutions in the projective plane by Bézout's theorem. Let $a_1, a_2 \ldots a_9$ be the 9 solutions. Then we need to find out how many of the 9 solutions satisfy the third function.

Case(i): Let $\mathbf{a}$ be one of $a_1, a_2 \ldots a_9$ that satisfies $g_2 = 0$. Then the first two equations are equivalent to

$$\begin{cases} g_2 = 0 \\ g_3g_4 = 0 \\ g_3g_6 = 0 \end{cases}.$$

The $\{g_3g_4 = 0\}$ can be seen as a conic with a line. Similarly, the $\{g_3g_6 = 0\}$ can be seen as the same conic with another line. Hence, the intersection of $\{g_3g_4 = 0\}$ and $\{g_3g_6 = 0\}$ is a conic and a point in the middle. Then we need to consider the equation $g_2 = 0$. It represents another conic curve. For a generic choice of g_2, the point in the middle is not in g_2. Then the intersection is 4 points because it can be seen as the intersection of 2 different conic curves. May assume a_1, a_2, a_3, a_4 are the 4 points satisfying $g_2 = 0$. For a generic choice of g_6, none of the four points above satisfy $g_4g_5 - g_6^2 = 0$. Hence, a_1, a_2, a_3, a_4 do not satisfy the system of equations (5).

Case(ii): From case(i), we know that there are 4 points that satisfy $g_2 = 0$, and then there are 5 points that do not satisfy $g_2 = 0$. Let us show that these five points satisfy automatically the third equation. In fact, $g_6 = \frac{g_3g_4}{g_2}$ and $g_5 = \frac{g_3g_6}{g_2}$. It satisfies $g_4g_5 - g_6^2 = 0$ automatically. Hence, these 5 solutions satisfy the system of equations (5).

In conclusion, the system of equations (5) has exactly 5 solutions in $\mathbb{P}^2$. □

Since every $\mathbf{a}$ corresponds to the only solution of (S, T) up to a scalar (thus the only solution in $\mathbb{P}^1$), the system of equations (3) has 5 different solutions. Hence, $\tilde{C}$ intersects E at 5 different points. □

Theorem 4.8 *The degree of C is* 10.

Proof Any point in $\mathbb{P}^2$ represents a plane passing through L in $\mathbb{P}^4$. So $q(\tilde{C})$ represents the plane through which also intersects with the curve C. Since C is the locus of opposite vertices of triangles with L as a given edge, the planes through L which also intersects with C can be seen as the set of triangles with L as a given edge. From the discussion in Section 2, the set of triangles with L as a given edge has degree 5. Hence, the degree of $q(\tilde{C})$ is equal to 5.

Lemma 4.9 *The map $q : \tilde{C} \to q(\tilde{C})$ is generically one-to-one.*

Proof Let $\tilde{C}^\circ := \tilde{C} - E$ be the complement of $\tilde{C} \cap E$ in $\tilde{C}$. Since $\tilde{C} \cap E$ has only finitely many points (in fact there are 5 by Proposition 4.6). To show that $q : \tilde{C} \to q(\tilde{C})$ is generically one-to-one, it suffices to show that $q : \tilde{C}^\circ \to q(\tilde{C}^\circ)$ is one-to-one. From Lemma 4.2, we know that $\mathcal{P}$ is a blow-up along L, so $\mathcal{P} - E \cong \mathbb{P}^4 - L$. Hence, we just need to prove that the map from $C - L$ to $q(\tilde{C}^\circ)$ is one-to-one. □

Lemma 4.10 *There does not exist any plane in $\mathbb{P}^4$ that intersects with the generic cubic hypersurface X at* 2 *triangles.*

Proof From Lemma 2.7, we know that there does not exist any plane that is included in X. If there exist 2 triangles in a plane, then the intersection of the plane and X have at least 4 different lines. However, the intersection of a cubic hypersurface and a plane is a cubic curve. As a result, there does not exist any plane in $\mathbb{P}^4$ that intersects with the cubic hypersurface X at 2 triangles.

Let us continue the proof of Lemma 4.9. If there exist 2 points on $C - L$ that are projected to the same point on $q(\tilde{C})$, the 2 points together with L must form the same plane. Hence, there are 2 triangles in the same plane since both the 2 points are on C. It contradicts to Lemma 4.10. As a result, the map $q : \tilde{C} \to q(\tilde{C})$ is generically one-to-one. □

From the Theorem 4.5 and Proposition 4.6, we can calculate the degree of C as

$$\deg C = \deg q(\tilde{C}) + m = 5 + 5 = 10.$$

This completes the proof of Theorem 4.8. □

4.3 Counting triangles

Theorem 4.11 *Let S be a generic surface of degree d on X. The number of triangles on X with L an edge such that the opposite vertex lies on S is $\frac{10d}{3}$.*

Proof To calculate the number of triangles with L as an edge and have the opposite vertex lying on S, we just need to find out the number of intersections of S and C. To this end, we will need the following theorem of Lefschetz which is special to the 3-dimensional case.

Theorem 4.12 *(Lefschetz [7]) Let $X \subset \mathbb{P}^4$ be a hypersurface and $S \subset X$ be a surface in X. Then there is a hypersurface $V \subset \mathbb{P}^4$ such that $S = V \cap X$.*

Now let V be a hypersurface in $\mathbb{P}^4$ such that $S = V \cap X$. Since the degree of X is 3 and the degree of S is d, the degree of V equals $\frac{d}{3}$. That means that the cohomology class of S in X is $\frac{d}{3}$ times the generator of $H^2(X, \mathbb{Z})$. On the other hand, since the degree of C is 10, the homology class of C in X is 10 times the generator of $H_2(X, \mathbb{Z})$. Hence, the number of intersections of $C \cap S$ equals $\frac{10d}{3}$. As a result, the number of triangles on X with L an edge such that the opposite vertex lies on S is $\frac{10d}{3}$. □

Acknowledgements I would like to express my heartfelt gratitude to my mentor, Prof. Deng Weibing, for his invaluable guidance and unwavering support throughout my journey in mathematics. I am also deeply indebted to my family and friends for their unwavering encouragement and belief in my abilities. Their constant support has been a source of motivation and strength during this research endeavor. I thank the anonymous referee for his precious suggestions that helped improve the presentation of the article. This research represents a significant step forward in my understanding of mathematical phenomena that is invaluable to me in my mathematics journey.

Competing interests There are no competing interests.

References

1. C. Bai, *On the geometry of higher dimensional Voisin map*, preprint 2024.
2. A. Beauville, *Variétés kählériennes dont la première classe de Chern est nulle*, J. Differential Geometry 18(1983), 755-782.
3. A. Beauville, R. Donagi, *The variety of lines of a cubic fourfold*, C. R. Acad. Sci. Paris 301 Séries I, (1985), 703-706.
4. C. H. Clemens, P. A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Annals of Mathematics, 95(2), 281-356, 1972.
5. O. Debarre, L. Manivel, *Sur la variété des espaces linéaires contenus dans une intersection complète*, Math. Ann. 312 (1998), 549-574.
6. W. Fulton, *Intersection theory*, A Series of Modern Surveys in Mathematics, Springer, 1984.
7. P. A. Griffiths, J. Harris, *Principals of algebraic geometry*, Wiley Classics Library Edition, 1994.
8. R. Hartshorne. *Algebraic Geometry*, Graduate texts in Mathematics 52, Springer (1997).
9. M. Reid, *Undergraduate Algebraic Geometry*, London Mathematical Society Student Texts, Series Number 12.
10. C. Voisin, *Triangle varieties and surface decomposition of hyper-Kähler manifolds*, Recent Developments in Algebraic Geometry: To Miles Reid for his 70th Birthday, edited by Hamid Abban; Gavin Brown; Alexander Kasprzyk; Shigefumi Mori, London Mathematical Society Lecture Note Series Cambridge University Press.
11. I. Safarevic, *Basic Algebraic Geometry 1*, Beijing World Publishing Corporation, 1998.
12. Clemens, H., Griffiths, P. A. (1972). *The intermediate Jacobian of the cubic is threefold.* Annals of Mathematics, 95(2), 281-356.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.