Indian J Pure Appl Math (2026) 57:1819–1831
https://doi.org/10.1007/s13226-024-00744-8

ORIGINAL RESEARCH

Moduli of continuity of functions in Hölder's class and solution of Rayleigh, Van der Pol and Duffing equations by sixth-kind Chebyshev wavelets

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Received: 4 December 2023 / Accepted: 11 December 2024 / Published online: 27 March 2025
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Abstract In this paper, sixth-kind Chebyshev wavelet is considered. The moduli of continuity of a functions in Hölder's class by sixth-kind Chebyshev wavelets have been determined. The current study examines the solution of Rayleigh, Van der Pol and Duffing differential equations. The sixth-kind Chebyshev wavelets collocation approach is introduced to solve the Rayleigh differential equation, Van der Pol differential equation, and Duffing differential equation. The solution obtained by sixth-kind Chebyshev wavelets method are compared with solution obtained by Algebraic method (AGM) and ODE45 method. The proposed algorithms have provided accurate results, even using few terms of the wavelet expansion. The numerical results demonstrate that our method is simply applicable, efficient, powerful and very precise at the small number of basis functions.

Keywords Chebyshev wavelet of sixth-kind · Modulus of continuity · Wavelet approximation · Orthonormal basis · Differential equations · Collocation points

Mathematics Subject Classification 42C40 · 65T60 · 65L10 · 65L60

1 Introduction

Orthogonal polynomials are successfully employed for the numerical solutions of various differential equations. Chebyshev polynomials play significant roles in numerical analysis and approximation theory. There are well-known four kinds of Chebyshev polynomials, which are specific types of Jacobi polynomials. These kinds of polynomials have been extensively used in a variety of papers related to numerical analysis. The other two kinds of Chebyshev polynomials, namely, the fifth and sixth kinds of Chebyshev polynomial. These two classes are symmetric like the first and second kinds of Chebyshev polynomials. In fact, they are particular polynomials of the so-called "generalized ultraspherical polynomials. Regrading these polynomials, their theoretical, as well as practical aspects, have attracted a significant attention from several authors. Wavelets were introduced relatively recently, in the beginning of the 1980s. They attracted considerable interest from the mathematical community and from members of many diverse disciplines in which wavelets had promising applications. Wavelets can be used to characterize several function spaces: the Lebesgue spaces, Besov, Holder class, Lipschitz class, Hardy, Sobolev and Triebel-Lizorkin spaces etc. Wavelets [3,25] have a wide range of uses, including as a basis for several function spaces, method for solving differential and integral equations, and an analysis of time-frequency composition of signals. From a historical point of view, the first reference to the

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wavelet goes back to the early twentieth century when Alfred Haar wrote his dissertation titled "On the theory of the orthogonal function systems" in 1909 [11]. His research on orthogonal systems of functions led to the development of a set of rectangular basis functions. Later, an entire wavelet family, the Haar wavelet, was named based on this set of functions, and it is also the simplest wavelet family developed till this date.

Several researchers, including J. Boman, A. Babenko, M. Pavlovic, and others, have studied moduli of continuity [5]. The modulus of continuity is a mathematical concept used to describe the regularity or smoothness of a function. It provides a way to quantify how much a function can vary within a certain range of inputs. Wavelets [17,18] help in the most accurate representation of functions in different spaces like, Sobolev space, Lipschitz class, Hölder's class, Besov space, etc. Several academics, including Debnath [8], Meyer [19], Morlet [20], and Devore [9], have studied the wavelet-based approximation of functions that belong to a specific class. The defect in piecewise constant wavelets is that they are very poor at approximation of smooth functions. Haar wavelet contains a non-smooth character. This is a deficiency of the Haar wavelet to estimate the moduli of continuity and the approximation of the smooth function by it. This weak point is almost removed by considering other wavelets like, Chebyshev wavelet [1,10,18,22,24], Legendre wavelet [16,18], Jacobi wavelet [14], Euler wavelet [21], Bernstein wavelet [12], Lucas wavelet [13], Vieta–Fibonacci wavelet [15], etc. and more accurate moduli of continuity and approximations of functions are obtained.

Nonlinear oscillations provides an interesting potential example of the mathematical description of the nonlinear behavior of many phenomena in science, physics, and practical engineering. The homotopy perturbation method, and the Hamitonian approach, are two main analytical tools for solving nonlinear differential equations. The current study examines the solution of Rayleigh, Van der Pol and Duffing differential equations. In many engineering and physical applications, the Duffing oscillator is one of the most important models. Duffing equation is adopted in many problems such as optical stability, electrical circuit, oscillation of plasma, and the buckled beam. Nayfeh and Mickens utilized different approaches to obtain approximate solutions for nonlinear oscillators. Additionally, there are many other methods available in the literature for solving differential equations. The key feature of wavelet collocation method is that it simplifies the problem to a set of algebraic equations and consequently these equations are solved accurately as desired by researchers. The effectiveness and convenience of the wavelet solutions in comparison to the numerical results have been demonstrated. Therefore, this method is an effective implementation for resolving this kind of nonlinear problems.

This paper is organized as follows: Sect. 2 is introductory in which the moduli of continuity, sixth-kind Chebyshev wavelet and approximation of function in $L^2_\omega[0,1]$ class, are defined. In Sect. 3, theorem concerning the moduli of continuity of $f - S_{2^k,M}$ has been established with its detail proof . In Sect. 4, the process of solving the differential equations using the sixth-kind Chebyshev collocation method has been introduced and provides several numerical examples to demonstrate the accuracy of the proposed method. Finally, the main conclusions are summarized in Sect. 5.

2 Definitions and preliminaries

2.1 The sixth-kind Chebyshev wavelets and its properties

A function $\Psi \in L^2(\mathbb{R})$ is said to be mother wavelet if it satifies the 'admissibility condition' i.e.

$$C_\Psi := \int_{-\infty}^{\infty} \frac{|\hat{\Psi}(w)|^2}{|w|} dw < \infty.$$

$$C_\Psi := \int_{-\infty}^{\infty} \frac{|\hat{\Psi}(w)|^2}{|w|} dw \text{ is finite } \implies \hat{\Psi}(0) = \int_{-\infty}^{\infty} \Psi(t) dt = 0 \qquad \text{(Chui [6])}$$

Wavelets constitute a family of functions constructed from dialation and translation of mother wavelet $\Psi \in L^2(R)$. We define wavelet

$$\Psi_{b,a}(t) = |a|^{\frac{-1}{2}} \Psi\left(\frac{t-b}{a}\right), \quad a \neq 0. \qquad \text{(Daubechies [7])}$$

where a and b are called dialation and translation parameter respectively. Discretizing the parameters via $a = a_0^{-k}$, $b = nb_0a_0{}^{-k}$, $a_0 > 1, b_0 > 0$, the following family of discrete wavelets are constructed:

$$\Psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \Psi\left(a_0^k t - nb_0\right).$$

Now, taking $a_0 = 2$, $\Psi(t) = \sqrt{2}\,\tilde{Y}_m(t+1)\chi_{\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]}(t)$ & $b_0 = 2$, the sixth-kind Chebyshev wavelet over the interval [0, 1], denoted by $\Psi_{n,m}(t)$, is defined as

$$\Psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} \tilde{Y}_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t \le \frac{n}{2^{k-1}}; \\ 0, \quad \text{otherwise}, \end{cases}$$

where $n = 1, 2, \ldots, 2^{k-1}$, $m = 0, 1, \ldots, M-1$ is the degree of the sixth-kind Chebyshev polynomial and t is the normalized time [2,23]. Moreover,

$$\tilde{Y}_m(t) = \frac{1}{\sqrt{h_m}}\, Y_m(t), \quad h_m = \begin{cases} \frac{\pi}{2^{(2m+3)}}, & \text{if } m \text{ even}; \\ \frac{\pi(m+3)}{2^{(2m+3)}(m+1)}, & \text{if } m \text{ odd}. \end{cases}$$

Here, Y_m denotes the sixth-kind Chebyshev polynomials, which are orthogonal with respect to the weight function $\omega(t) = t^2\sqrt{1-t^2}$ over $[-1, 1]$, and satisfy the recurrence relation:

$$Y_{m+1}(t) = t\, Y_m(t) - \frac{(m+1-(-1)^m)\,(m+2-(-1)^m)}{4(m+1)(m+2)}\, Y_{m-1}(t), \quad m \ge 1,$$

with the initial values $Y_0(t) = 1$, $Y_1(t) = t$.

2.2 Sixth-kind Chebyshev wavelet expansion and approximation

The function $f \in L^2_\omega[0, 1]$ can be expressed via the sixth-kind Chebyshev wavelets series as

$$f(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m} \Psi_{n,m}(t), \quad \text{where } c_{n,m} = \langle f, \Psi_{n,m} \rangle_{L^2_\omega[0,1]} = \int_0^1 f(t) \Psi_{n,m}(t) \omega(t) dt \tag{2.1}$$

The $(2^{k-1}, M)^{th}$ partial sums of above series (2.1) is given by

$$S_{2^{k-1},M}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \Psi_{n,m}(t) = C^T\, \Psi(t) \tag{2.2}$$

in which C and $\Psi(t)$ are $2^{k-1}M$ column vectors of the form

$$\begin{aligned} C^T &= \left[c_{1,0}, c_{1,1}, \ldots c_{1,M-1}, c_{2,0}, c_{2,1} \ldots, c_{2,M-1}, \ldots, c_{2^{k-1},0}, \ldots, c_{2^{k-1},M-1}\right] \quad \text{and} \\ \Psi(t) &= \left[\Psi_{1,0}, \Psi_{1,1}, \ldots, \Psi_{1,M-1}, \Psi_{2,0}, \Psi_{2,1}, \ldots, \Psi_{2,M-1}, \ldots, \Psi_{2^{k-1},0}, \ldots, \Psi_{2^{k-1},M-1}\right]^T \end{aligned}$$

The sixth-kind Chebyshev wavelet approximation $E_{2^{k-1},M}(f)$ of a function $f \in L^2_\omega[0, 1]$ by $(2^{k-1}, M)^{th}$ partial sums $S_{2^{k-1},M}(f)$ of its sixth-kind Chebyshev wavelet series is given by

$$E_{2^{k-1},M}(f) = \min_{S_{2^k,M}(f)} \|f - S_{2^{k-1},M}(f)\|_2, \text{ where } \|f\|_2 = \left(\int_0^1 |f(t)|^2\, \omega(t)\, dt\right)^{\frac{1}{2}} \tag{2.3}$$

If $E_{2^{k-1},M}(f) \to 0$ as $k, M \to \infty$ then $E_{2^{k-1},M}(f)$ is called the best approximation of f of order $(2^{k-1}, M)$ (Zygmund [26]).



2.3 Modulus of continuity and function of Hölder's class

The modulus of continuity of a function $f \in L^2_\omega[0,1]$ is defined as

$$\Omega(f,\delta) = \sup_{0<h\le\delta} ||f(\cdot+h) - f(\cdot)||_2$$
$$= \sup_{0<h\le\delta} \left(\int_0^1 |f(t+h) - f(t)|^2 \, \omega(t) dt \right)^{\frac{1}{2}}. \tag{2.4}$$

It is remarkable to note that $\Omega(f,\delta)$ is a non-decresing function of δ and $\Omega(f,\delta) \to 0$ as $\delta \to 0^+$ (Chui [6]).

A function f is said to be in Hölder's class $H^\alpha[0,1]$ of order α, $0 < \alpha \le 1$, if f satisfies

$$f(x+t) - f(x) = O\left(|t|^\alpha\right), \quad \forall x, t, x+t \in [0,1]. \tag{2.5}$$

3 Modulus of continuity and error estimation

In this paper, following theorem has been proved.

Theorem 3.1 If a function $f \in L^2_\omega[0,1]$ such that $f \in H^\alpha[0,1]$ and its sixth-kind Chebyshev wavelets expansion be

$$f(t) = \sum_{n=0}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m} \Psi_{n,m}(t)$$

having $(2^k, M)^{th}$ partial sums

$$\left(S_{2^k,M}(f)\right)(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \Psi_{n,m}(t)$$

then the Moduli of continuity of $f - \left(S_{2^k,M}(f)\right)$ satisfies

$$\Omega\left(\left(f - S_{2^{k-1},M} f\right), \frac{1}{2^{k-1}}\right) = \sup_{0<h\le\frac{1}{2^{k-1}}} ||\left(f - S_{2^{k-1},M} f\right)(t+h) - \left(f - S_{2^{k-1},M} f\right)(t)||_2$$
$$= O\left(\frac{1}{2^{(\alpha+2)k}} \; \frac{1}{(M-1)^{\frac{3}{2}}}\right), \quad M > 1.$$

Proof of theorem 3.1 Consider

$$f(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m} \Psi_{m,n}(t), \quad \text{where} \quad c_{n,m} = \langle f, \Psi_{n,m} \rangle_{\omega_k}.$$
$$c_{n,m} = \int_0^1 f(t) \Psi_{n,m}(t) \omega_k(t) dt$$
$$= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t) \, 2^{\frac{k}{2}} \, \tilde{Y}_m(2^k t - 2n + 1) \, \omega(2^k t - 2n + 1) dt$$
$$= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t) \, 2^{\frac{k}{2}} \, \frac{1}{\sqrt{h_m}} \, Y_m(2^k t - 2n + 1) \, \omega(2^k t - 2n + 1) dt$$
$$= \frac{2^{\frac{k}{2}}}{\sqrt{h_m}} \int_\pi^0 f\left(\frac{\cos\theta + 2n - 1}{2^k}\right) Y_m(\cos\theta) \, \omega(\cos\theta) \frac{(-\sin\theta)}{2^k} d\theta$$
$$= \frac{1}{2^{\frac{k}{2}}} \frac{1}{\sqrt{h_m}} \int_0^\pi f\left(\frac{\cos\theta + 2n - 1}{2^k}\right) Y_m(\cos\theta) \, \cos^2\theta \, \sin^2\theta \, d\theta \tag{3.1}$$

Case 1 when m is even

$$
\begin{aligned}
c_{n,m} &= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2^{2m+3}}{\pi}}\int_0^\pi f\left(\frac{\cos\theta+2n-1}{2^k}\right)\frac{\sin(m+2)\theta}{2^m\,\sin 2\theta}\,\cos^2\theta\,\sin^2\theta\,d\theta\\
&= \frac{1}{2^{\frac{k}{2}}\sqrt{8\pi}}\int_0^\pi f\left(\frac{\cos\theta+2n-1}{2^k}\right)(\cos m\theta-\cos(m+4)\theta)\,d\theta\\
&= \frac{1}{2^{\frac{k}{2}}\sqrt{8\pi}}\left[-\int_0^\pi f'\left(\frac{\cos\theta+2n-1}{2^k}\right)\left(\frac{\sin m\theta}{m}-\frac{\sin(m+4)\theta}{m+4}\right)\frac{(-\sin\theta)}{2^k}d\theta\right]\\
&= \frac{1}{2^{\frac{3k}{2}}\sqrt{8\pi}}\left[\int_0^\pi f'\left(\frac{\cos\theta+2n-1}{2^k}\right)\left(\frac{\cos(m-1)\theta}{2\,m}-\frac{\cos(m+1)\theta}{2\,m}\right.\right.\\
&\quad\left.\left.-\frac{\cos(m+3)\theta}{2(m+4)}+\frac{\cos(m+5)\theta}{2(m+4)}\right)d\theta\right]\\
&= \frac{1}{2^{\frac{3k}{2}}\sqrt{8\pi}}\left[-\int_0^\pi f''\left(\frac{\cos\theta+2n-1}{2^k}\right)\left(\frac{\sin(m-1)\theta}{2\,m\,(m-1)}-\frac{\sin(m+1)\theta}{2\,m\,(m+1)}\right.\right.\\
&\quad\left.\left.-\frac{\sin(m+3)\theta}{2(m+4)(m+3)}+\frac{\sin(m+5)\theta}{2(m+4)(m+5)}\right)\frac{(-\sin\theta)}{2^k}\,d\theta\right]\\
&= \frac{1}{2^{\frac{5k}{2}}\sqrt{8\pi}}\left[\int_0^\pi f''\left(\frac{\cos\theta+2n-1}{2^k}\right)\left(\frac{\cos(m-2)\theta-\cos m\theta}{4\,m\,(m-1)}-\frac{\cos m\theta-\cos(m+2)\theta}{4\,m\,(m+1)}\right.\right.\\
&\quad\left.\left.-\frac{\cos(m+2)\theta-\cos(m+4)\theta}{4(m+4)(m+3)}+\frac{\cos(m+4)\theta-\cos(m+6)\theta}{4(m+4)(m+5)}\right)d\theta\right]\\
&= \frac{1}{2^{\frac{5k}{2}}\sqrt{8\pi}}\left[\int_0^\pi\left(f''\left(\frac{\cos\theta+2n-1}{2^k}\right)-f''\left(\frac{2n-1}{2^k}\right)\right)\left(\frac{\cos(m-2)\theta-\cos m\theta}{4\,m\,(m-1)}\right.\right.\\
&\quad\left.\left.-\frac{\cos m\theta-\cos(m+2)\theta}{4\,m\,(m+1)}-\frac{\cos(m+2)\theta-\cos(m+4)\theta}{4(m+4)(m+3)}+\frac{\cos(m+4)\theta-\cos(m+6)\theta}{4(m+4)(m+5)}\right)d\theta\right]\\
|c_{n,m}| &\le \frac{1}{2^{\frac{5k}{2}}\sqrt{8\pi}}\left[\int_0^\pi\left|f''\left(\frac{\cos\theta+2n-1}{2^k}\right)-f''\left(\frac{2n-1}{2^k}\right)\right|\left|\frac{1}{4\,m\,(m-1)}-\frac{1}{4\,m\,(m+1)}\right.\right.\\
&\quad\left.\left.-\frac{1}{4(m+4)(m+3)}+\frac{1}{4(m+4)(m+5)}\right|d\theta\right]\\
&\le \frac{1}{2^{\frac{5k}{2}}\sqrt{8\pi}}\int_0^\pi\frac{1}{2^{k\alpha}}\left(\frac{1}{4m(m-1)}+\frac{1}{4m(m+1)}+\frac{1}{4(m+4)(m+3)}+\frac{1}{4(m+4)(m+5)}\right)d\theta\\
&\le \sqrt{\frac{\pi}{8}}\frac{1}{2^{\frac{(2\alpha+5)k}{2}}}\;\frac{1}{m\,(m-1)} \qquad (3.2)
\end{aligned}
$$

Case 2 when m is odd

$$
\begin{aligned}
c_{n,m} &= \frac{1}{2^{\frac{k}{2}}}\frac{1}{\sqrt{h_m}}\int_0^\pi f\left(\frac{\cos\theta+2n-1}{2^k}\right)Y_m(\cos\theta)\,\cos^2\theta\,\sin^2\theta\,d\theta \qquad \text{by Eq. (3.1)}\\
&= \frac{2^{\frac{-k}{2}}}{\sqrt{h_m}}\int_0^\pi f\left(\frac{\cos\theta+2n-1}{2^k}\right)\frac{(m+3)\cos\theta\sin(m+2)\theta-\sin(m+3)\theta}{(m+1)\,2^m\sin 2\theta\cos\theta}\cos^2\theta\sin^2\theta d\theta\\
&= \frac{1}{2^{\frac{k+1}{2}}}\frac{1}{\sqrt{\pi\,(m+1)(m+3)}}\left[\int_0^\pi f\left(\frac{\cos\theta+2n-1}{2^k}\right)\left(\frac{m+3}{2}\cos m\theta-\cos(m+2)\theta\right.\right.\\
&\quad\left.\left.-\frac{m+1}{2}\cos(m+4)\theta\right)d\theta\right]\\
&= \frac{1}{2^{\frac{3k+1}{2}}}\frac{1}{\sqrt{\pi\,(m+1)(m+3)}}\left[\int_0^\pi f'\left(\frac{\cos\theta+2n-1}{2^k}\right)\left(\frac{m+3}{2m}\sin m\theta\,\sin\theta\right.\right.\\
&\quad\left.\left.-\frac{1}{m+2}\sin(m+2)\theta\,\sin\theta-\frac{m+1}{2(m+4)}\sin(m+4)\theta\,\sin\theta\right)d\theta\right]
\end{aligned}
$$

$$= \frac{2^{\frac{-(3k+1)}{2}}}{\sqrt{\pi\,(m+1)(m+3)}}\Bigg[\int_0^{\pi} f'\left(\frac{\cos\theta+2n-1}{2^k}\right)\left(\frac{m+3}{4m}\left(\cos(m-1)\theta\ -\cos(m+1)\theta\right)\right.$$
$$\left.-\frac{1}{2(m+2)}\left(\cos(m+1)\theta\ -\cos(m+3)\theta\right)-\frac{m+1}{4(m+4)}\left(\cos(m+3)\theta\ -\cos(m+5)\theta\right)\right)d\theta\Bigg]$$
$$= \frac{2^{\frac{-(5k+1)}{2}}\pi^{\frac{-1}{2}}}{\sqrt{(m+1)(m+3)}}\Bigg[\int_0^{\pi} f''\left(\frac{\cos\theta+2n-1}{2^k}\right)\sin\theta\left\{\frac{m+3}{4m}\left(\frac{\sin(m-1)\theta}{m-1}-\frac{\sin(m+1)\theta}{m+1}\right)\right.$$
$$\left.-\frac{1}{2(m+2)}\left(\frac{\sin(m+1)\theta}{m+1}-\frac{\sin(m+3)\theta}{m+3}\right)-\frac{m+1}{4(m+4)}\left(\frac{\sin(m+3)\theta}{m+3}-\frac{\sin(m+5)\theta}{m+5}\right)\right\}d\theta\Bigg]$$
$$= \frac{2^{\frac{-(5k+1)}{2}}\pi^{\frac{-1}{2}}}{\sqrt{(m+1)(m+3)}}\Bigg[\int_0^{\pi}\left\{f''\left(\frac{\cos\theta+2n-1}{2^k}\right)-f''\left(\frac{2n-1}{2^k}\right)\right\}\sin\theta$$
$$\times\left\{\frac{m+3}{4m}\left(\frac{\sin(m-1)\theta}{m-1}-\frac{\sin(m+1)\theta}{m+1}\right)-\frac{1}{2(m+2)}\left(\frac{\sin(m+1)\theta}{m+1}-\frac{\sin(m+3)\theta}{m+3}\right)\right.$$
$$\left.-\frac{m+1}{4(m+4)}\left(\frac{\sin(m+3)\theta}{m+3}-\frac{\sin(m+5)\theta}{m+5}\right)\right\}d\theta\Bigg]$$

$$|c_{n,m}| \le \frac{2^{\frac{-(2k\alpha+5k+1)}{2}}\pi^{\frac{1}{2}}}{2\sqrt{2}m(m-1)} = \sqrt{\frac{\pi}{8}}\ \frac{1}{2^{\frac{(2\alpha+5)k+1}{2}}}\frac{1}{m(m-1)} \tag{3.3}$$

From (3.2) and (3.3)

$$|c_{n,m}| \le \sqrt{\frac{\pi}{8}}\ \frac{1}{2^{\frac{(2\alpha+5)k+1}{2}}}\frac{1}{(m-1)^2}, \qquad \forall\, m>1.$$
$$f(x)-S_{2^{k-1},M}(f)(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty}c_{n,m}\Psi_{n,m}(t)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\Psi_{n,m}(t)$$
$$= \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}c_{n,m}\Psi_{n,m}(t),$$
$$||f-S_{2^{k-1},M}(f)||_2^2 = \int_0^1 |f(t)-S_{2^{k-1},M}(f)(t)|^2dt$$
$$= \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}|c_{n,m}|^2, \quad \text{by orthonormality of } \{\Psi_{n,m}\} \tag{3.4}$$

By equation (3.4), the upper bound for $|c_{n,m}|$ is independent of n; hence

$$||f-S_{2^{k-1},M}(f)||_2^2 \le \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\frac{\pi}{8}\ \frac{1}{2^{(2\alpha+5)k+1}}\frac{1}{(m-1)^4}$$
$$= \frac{1}{2^{(2\,\alpha+4)\,k+2}}\frac{\pi}{8}\left(\sum_{m=M}^{\infty}\frac{1}{(m-1)^4}\right)$$
$$||f-S_{2^{k-1},M}(f)||_2^2 \le \frac{\pi}{2^{(2\,\alpha+4)\,k+5}}\left(\frac{1}{(M-1)^4}+\int_M^{\infty}\frac{1}{(m-1)^4}dm\right), \quad \text{by the Cauchy integral test}$$
$$\le \frac{\pi}{2^{(2\,\alpha+4)\,k+5}}\left(\frac{1}{(M-1)^4}+\frac{1}{3\,(M-1)^3}\right)$$
$$\le \frac{\pi}{2^{(2\,\alpha+4)\,k+4}}\frac{1}{3\,(M-1)^3}$$
$$||f-S_{2^{k-1},M}(f)||_2 = O\left(\frac{1}{2^{(\alpha+2)\,k}}\ \frac{1}{(M-1)^{\frac{3}{2}}}\right)$$

$$\Omega\left(\left(f - S_{2^{k-1},M} f\right), \frac{1}{2^{k-1}}\right) = \sup_{0<h\leq\frac{1}{2^{k-1}}} ||\left(f - S_{2^{k-1},M} f\right)(t+h) - \left(f - S_{2^{k-1},M} f\right)(t)||_2$$
$$\leq ||\left(f - S_{2^{k-1},M} f\right)||_2 + ||\left(f - S_{2^{k-1},M} f\right)||_2$$
$$= 2||\left(f - S_{2^{k-1},M} f\right)||_2$$
$$= O\left(\frac{1}{2^{(\alpha+2)k}} \frac{1}{(M-1)^{\frac{3}{2}}}\right)$$

Thus, this theorem is completely established.

□

4 Algorithm of the sixth-kind Chebyshev wavelets collocation method for solving differential equations

The Rayleigh differential equation is defined by the following non-linear equation [4]:

$$y''(t) + \left(\beta - \gamma\,(y'(t))^2\right) y'(t) + \lambda^2\, y(t) = 0, \quad y(0) = a, \quad y'(0) = 0. \tag{4.1}$$

Here, a, β, γ & λ are constant numbers.

Let $y(t)$ be the solution of Rayleigh differential equation then the sixth-kind Chebyshev wavelet expansion of $y(t)$ is

$$y(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \Psi_{n,m}(t) \tag{4.2}$$

This solution converges to the exact solution of equation (4.1). The sixth-kind Chebyshev wavelet collocation method are used to determine the unknown coefficients $c_{n,m}$. A total of $2^{k-1}M$ conditions must exists for the determination of coefficients:

$$c_{1,0},\ c_{1,1}, \ldots, c_{1,M-1},\ c_{2,0},\ c_{2,1} \ldots, c_{2,M-1}, \ldots, c_{2^{k-1},0},\ c_{2^{k-1},1}, \ldots, c_{2^{k-1},M-1}.$$

Equation (4.1) has two initial conditions by which two following conditions are obtained:

$$y(0) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \Psi_{n,m}(0) = a, \quad y'(0) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \Psi'_{n,m}(0) = 0 \tag{4.3}$$

It is important to note that an additional $2^{k-1}M - 2$ conditions are needed to determine all the unknown coefficients. These extra conditions can be obtained using Eqs. (4.1) and (4.2). Put the value of $y(t)$ from Eq. (4.2) into equation (4.1), we obtain:

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \Psi''_{n,m}(t) + \left[\beta - \gamma \left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \Psi'_{n,m}(t)\right)^2\right] \left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \Psi'_{n,m}(t)\right)$$
$$+ \lambda^2 \left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \Psi_{n,m}(t)\right) = 0 \tag{4.4}$$

By evaluating the above equation at discreate points, also known as collocation points:

$$t_j = \frac{j}{2^{k-1}M}, \quad j = 3, 4, \ldots, 2^{k-1}M$$

$2^{k-1}M - 2$ equations are obtained. Hence the values of unknown coefficients $c_{n,m}$ are obtained by these $2^{k-1}M$ equations. By substituting the values of $c_{n,m}$ into Eq. (4.2), the explicit form of the approximate solution of Eq. (4.1) are obtained. This algorithm is also applicable to higher-order differential equations

4.1 Numerical examples

Several numerical examples are given to demonstrate the effectiveness and applicability of the sixth-kind Chebyshev wavelet collocation method for solving the differential equations.

Example 1 Rayleigh equation [4]

$$y''(t) + \left(\beta - \gamma\,(y'(t))^2\right) y'(t) + \lambda^2\, y(t) = 0, \quad y(0) = a, \quad y'(0) = 0. \tag{4.5}$$

The solution obtained by Algebraic method(AGM) of equation (4.5) for $a = 0.1$, $\beta = 0.2$, $\gamma = 0.1$, & $\lambda = 1.2$ is expressed as

$$y(t) = 0.10035\, e^{-0.1\,t}\, \cos(1.1958\, t - 0.08343) \tag{4.6}$$

For $k = 1$, $M = 5$ the wavelet solution of equation (4.5) is given by

$$\begin{aligned} y(t) &= c_{1,0}\Psi_{1,0}(t) + c_{1,1}\Psi_{1,1}(t) + c_{1,2}\Psi_{1,2}(t) + c_{1,3}\Psi_{1,3}(t) + c_{1,4}\Psi_{1,4}(t) \\ &= 0.55c_{10} + \big(6.38t - 3.19\big)c_{11} + \big(9.02(2t-1)^2 - 4.51\big)c_{12} + \big(14.74(2t-1)^3 \\ &\quad - 18.42t + 9.21\big)c_{13} + \big(36.10(2t-1)^4 - 36.10(2t-1)^2 + 6.77\big)c_{14} \end{aligned} \tag{4.7}$$

Put the value of $y(t)$ from Eqs. (4.7) into (4.5), we obtain:

$$\begin{aligned} &0.79c_{10} + 72.2c_{12} - \big(0.1\big(6.38c_{11} + c_{13}(88.4(2t-1)^2 - 18.4) + c_{14}(288.8(2t-1)^3 - 288.8t \\ &\quad + 144.43) + c_{12}(72.21t - 36.10)\big)^2 - 0.2\big)\big(6.38c_{11} + c_{13}(88.44(2t-1)^2 - 18.42) \\ &\quad + c_{14}(288.8(2t-1)^3 - 288.8t + 144.4) + c_{12}(72.2t - 36.1)\big) + 1.4c_{12}(9(2t-1)^2 - 4.5) \\ &\quad + c_{14}(1733.19(2t-1)^2 - 288.86) + 1.44c_{13}(14.74(2t-1)^3 - 18.42t + 9.21) \\ &\quad + 1.4c_{14}(36.1(2t-1)^4 - 36.1(2t-1)^2 + 6.7) + 1.4c_{11}(6.3t - 3.19) + c_{13}(707t - 353) = 0 \end{aligned} \tag{4.8}$$

From Eq. (4.7) and using initial condition $y(0) = 0.1$, $y'(0) = 0$ we obtain following two equations

$$0.55536c_{10} - 3.191538c_{11} + 4.513517c_{12} - 5.527906c_{13} + 6.770275c_{14} = 0.1 \tag{4.9}$$

$$6.383076c_{11} - 36.108133c_{12} + 70.020148c_{13} - 144.432533c_{14} = 0 \tag{4.10}$$

Now, substitute $t = 0.3,\ 0.6\ \&\ 0.9$ in equation (4.8) we obtain following three equations respectively

$$\begin{aligned} &0.799719c_{10} - 1.838326c_{11} + 67.796631c_{12} - 137.566152c_{13} - 8.79363c_{14} \\ &\quad - \left(\frac{1}{10}\big(6.383076c_{11} - 14.443253c_{12} - 4.274914c_{13} + 39.285649c_{14}\big)^2 - 0.2\right) \\ &\quad \times \big(6.383076c_{11} - 14.443253c_{12} - 4.274914c_{13} + 39.285649c_{14}\big) = 0 \end{aligned} \tag{4.11}$$

$$\begin{aligned} &0.799719c_{10} + 0.919163c_{11} + 66.23676c_{12} + 68.273624c_{13} - 211.78489c_{14} \\ &\quad - \left(\frac{1}{10}\big(6.383076c_{11} + 7.221627c_{12} - 14.888495c_{13} - 26.575586c_{14}\big)^2 - 0.2\right) \\ &\quad \times \big(6.383076c_{11} + 7.221627c_{12} - 14.888495c_{13} - 26.575586c_{14}\big) = 0 \end{aligned} \tag{4.12}$$

$$\begin{aligned} &0.799719c_{10} + 3.676652c_{11} + 74.036117c_{12} + 283.283533c_{13} + 818.146174c_{14} \\ &\quad - \left(\frac{1}{10}\big(6.383076c_{11} + 28.886507c_{12} + 38.179407c_{13} + 32.352887c_{14}\big)^2 - 0.2\right) \\ &\quad \times \big(6.383076c_{11} + 28.886507c_{12} + 38.179407c_{13} + 32.352887c_{14}\big) = 0 \end{aligned} \tag{4.13}$$

Solving equations from (4.9) to (4.13), we obtain the values of wavelet coefficients c_{nm}

$$\begin{aligned} c_{10} &= 0.13747724916088395059778442, \quad c_{11} = -0.00970345904173570757458944 3 \\ c_{12} &= -0.00143838772153311242707 34, \quad c_{13} = 0.00015915634173585026045547 17 \end{aligned}$$

Table 1 Comparison between AGM solution, ODE45 solution and SKCWS for different values of M

t	AGM	ODE45 solution	SKCWS M = 5	SKCWS M = 10	SKCWS M = 20
0	0.100000956188466	0.100000000000000	0.100000000000000	0.100000000000000	0.100000000000000
0.1	0.099286590497708	0.099285632499393	0.099266351642267	0.099285632638449	0.099285632661720
0.2	0.097172604676769	0.097171593329309	0.097109373632995	0.097171593453176	0.097171593505711
0.3	0.093716814611015	0.093715681532980	0.093603251387730	0.093715681606020	0.093715681686203
0.4	0.088994997626277	0.088993632329606	0.088833150081550	0.088993632358050	0.088993632464238
0.5	0.083099457817024	0.083097676981338	0.082895214649072	0.083097677056291	0.083097677186460
0.6	0.076137384454333	0.076134902501202	0.075896569784445	0.076134902783482	0.076134902935197
0.7	0.068229030945966	0.068225438292694	0.067955319941355	0.068225438790760	0.068225438960665
0.8	0.059505743832085	0.059500497402557	0.059200549333024	0.059500498009399	0.059500498194664
0.9	0.050107872851699	0.050100302973674	0.049772321932209	0.050100302872411	0.050100303069778

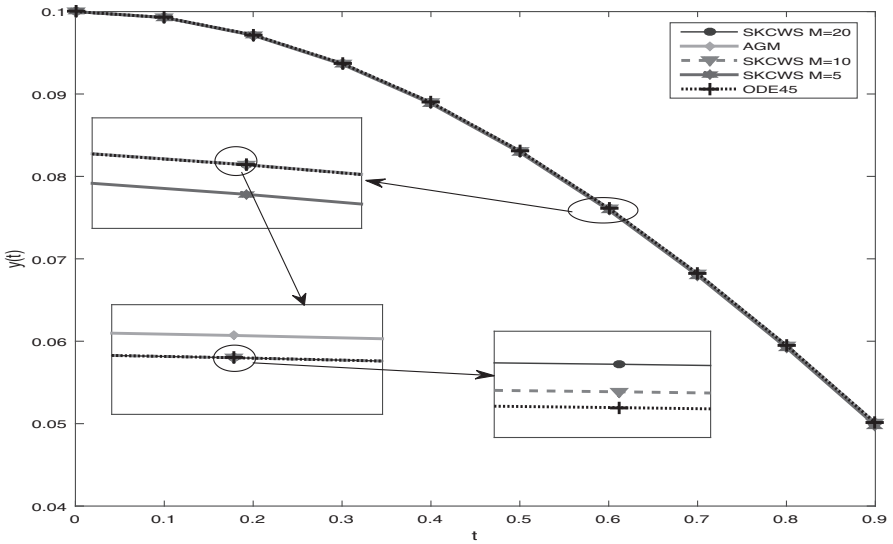


Fig. 1 The graphs of AGM, ODE45 and SKCW solutions for different values of M

$$c_{14} = 0.000007918748807997981369726858478299 1$$

Substitute the value of $c_{10}, c_{11}, c_{12}, c_{13}$ and c_{14} in Eq. (4.7) we obtain the explicit form of the approximate solution of Eq. (4.5), as given by:

$$y(t) = 0.004575t^4 + 0.009619t^3 - 0.074373t^2 + 2.721362758816350e^{-18}t + 0.1 \tag{4.14}$$

Example 2 Van der Pol equation [4]

$$y''(t) + \epsilon\left(1 - y^2(t)\right)y'(t) + \lambda^2\, y(t) = 0, \;\; y(0) = a, \;\; y'(0) = 0. \tag{4.15}$$

The AGM solution to equation (4.15) with $a = 0.2,\ \epsilon = 0.15,\ \&\ \lambda = 1.2$ is expressed as

$$y(t) = 0.20036\, e^{-0.072\, t}\, \cos(1.19784\, t - 0.060036) \tag{4.16}$$

Appling the sixth-kind Chebyshev wavelet collocation method for $k = 1$ and $M = 5$, the values of wavelet expansion coefficients $c_{n,m}$ are given by

$$c_{10} = 0.27375523605802070812860196, \quad c_{11} = -0.019707225435698519340145 6$$
$$c_{12} = -0.00295630802194479076396 22, \quad c_{13} = 0.00030860904832525651063211$$
$$c_{14} = 0.00001774453471460292554859700814417$$

Table 2 Comparison between AGM, ODE45 and SKCWS for different values of M

t	AGM	ODE45 solution	SKCWS M = 5	SKCWS M = 10
0	0.199999027533838	0.200000000000000	0.200000000000000	0.200000000000000
0.1	0.198567635235302	0.198568605599175	0.198531684849528	0.198568606009983
0.2	0.194321218208140	0.194322217806090	0.194202604259809	0.194322218722375
0.3	0.187360659964115	0.187361891714251	0.187145008317728	0.187361893295431
0.4	0.177824158862874	0.177826240779417	0.177515750835954	0.177826243177704
0.5	0.165884721195364	0.165888991658761	0.165496289352942	0.165888994816412
0.6	0.151747193060854	0.151756022293163	0.151292685132933	0.151756025979187
0.7	0.135644880015303	0.135661930944525	0.135135603165952	0.135661934840801
0.8	0.117835808655094	0.117866196234086	0.117280312167809	0.117866200066327
0.9	0.098598688638811	0.098648999797871	0.098006684580100	0.098649002987500

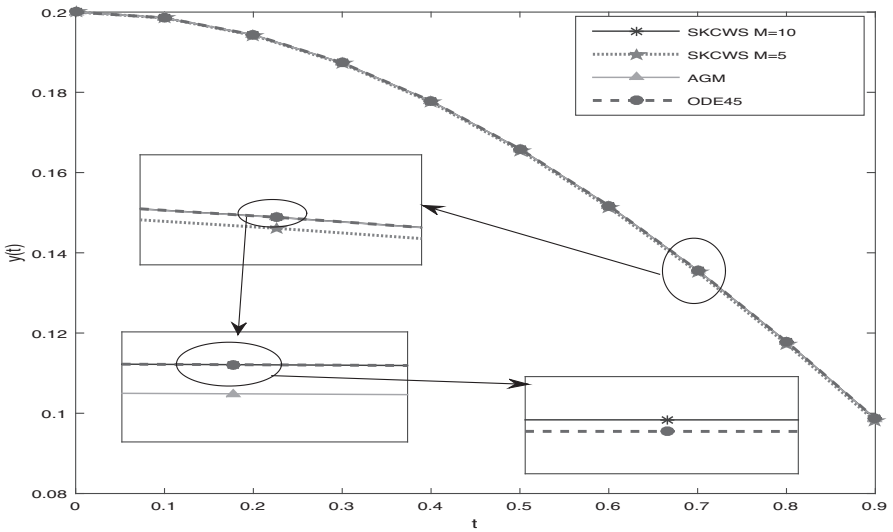


Fig. 2 The graphs of AGM, ODE45 and SKCW solutions for different values of M

Substitute the value of $c_{10}, c_{11}, c_{12}, c_{13}$ and c_{14} in Eq. (4.7) we obtain the explicit form of the approximate solution of Eq. (4.15), as given by:

$$y(t) = 0.010252t^4 + 0.015891t^3 - 0.148523t^2 + 2.303074748677947e^{-18}t + 0.2 \tag{4.17}$$

Example 3 Duffing equation [4]

$$y''(t) + \beta^2 \left(y(t) + \lambda^2 y^3(t) \right) = \mu \sin(\omega t), \quad y(0) = a, \quad y'(0) = 0. \tag{4.18}$$

The AGM solution to Eq. (4.18) with $a = 0.15,\ \beta = 0.3,\ \lambda = 0.2,\ \mu = 0.2$ & $\omega = 2$ is expressed as

$$y(t) = 0.372137\, e^{-0.000101788\, t} \cos(0.30036\, t - 1.15592) - 0.051154 \sin(2\, t) \tag{4.19}$$

Appling the sixth-kind Chebyshev wavelet collocation method for $k = 1$ and $M = 5$, the values of wavelet expansion coefficients $c_{n,m}$ are given by

$$c_{10} = 0.29561779570236462847840172, \quad c_{11} = 0.00656363089555501638482858166$$
$$c_{12} = 0.00193600436551014815282515, \quad c_{13} = 0.00029014338216021411767821278$$
$$c_{14} = -0.00005326683070134908652174693688256 6$$

Substitute the value of $c_{10}, c_{11}, c_{12}, c_{13}$ and c_{14} in Eq. (4.7) we obtain the explicit form of the approximate solution of Eq. (4.18), as given by:

$$y(t) = -0.03077t^4 + 0.09576t^3 - 0.01988t^2 - 1.340208992993567e^{-19}t + 0.15 \tag{4.20}$$

Table 3 Comparison between AGM solution, ODE45 solution and SKCWS for different values of M

t	AGM	ODE45 solution	SKCWS M = 5	SKCWS M = 10
0	0.149999773234504	0.150000000000000	0.150000000000000	0.150000000000000
0.1	0.149995493435698	0.149998974439890	0.149893825070754	0.149998973448585
0.2	0.150251686614910	0.150258824622949	0.149921427377184	0.150258822213409
0.3	0.151148326174406	0.151159515839737	0.150546604624755	0.151159511633753
0.4	0.153033863720252	0.153049490542307	0.152159297271216	0.153049484252600
0.5	0.156210990939470	0.156231430399258	0.155075588526599	0.156231422045832
0.6	0.160924231503128	0.160949848214318	0.159537704353220	0.160949838049746
0.7	0.167349857974298	0.167381004470568	0.165714013465680	0.167380992789800
0.8	0.175588536003741	0.175625550606869	0.173699027330864	0.175625537442350
0.9	0.185660989362109	0.185704192472385	0.183513400167940	0.185704177749735

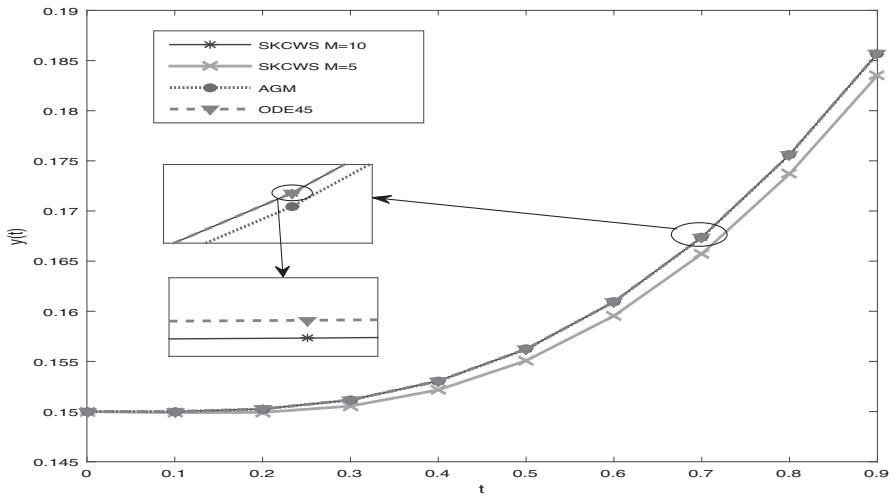


Fig. 3 The graphs of AGM, ODE45 and SKCW solutions for different values of M

5 Conclusion

In this paper, we have proposed a fast and precise algorithm based on the sixth-kind Chebyshev wavelet collocation method. This algorithm has been applied to solve the Rayleigh, Van der Pol, and Duffing differential equations. A key advantage of the method is its ability to transform the problem into a system of algebraic equations, significantly simplifying the computational process. Several numerical examples have been presented to demonstrate the efficiency of the proposed algorithm. Figures 1, 2, and 3 illustrate that the method yields excellent results, which is a significant contribution of this research. The calculations performed in Sect. 4 confirm that the sixth-kind Chebyshev wavelet collocation method provides high accuracy and is particularly well-suited for solving boundary value problems and higher-order differential equations. Despite its effectiveness, a potential limitation is that solving the system of equations arising from Eq. (4.4) can occasionally be challenging. While this issue is rare, it presents an opportunity for future research to further refine and improve the method.

5.1 Scope of future work

The scope of future works related to this research is outlined as follows:

(i) To estimate the moduli of continuity of $f - S_{2^{k-1},M}(f)$ in the sobolev space and the solution of the following fractional Riccati equation:

$$D^{\gamma} y(x) + (y(x))^2 - 1 = 0, \quad y(0) = 0, \quad 0 < \gamma \leq 1,$$

using Fibonacci wavelets.

(ii) To develop an estimator of $f - S_{2^{k-1},M}(f)$ in the sobolev space and solution of fractional Riccati equation:

$$D^{\gamma} y(x) + (y(x))^2 - 2y(x) - 1 = 0, \quad y(0) = 0, \quad 0 < \gamma \leq 1,$$

using Euler wavelets.

(iii) An algorithm has been developed to solve Rayleigh, Van der Pol and Duffing differential equation using the sixth-kind Chebyshev wavelet collocation method. The methodology introduced in this chapter is extendable to a numerical algorithm for solving fractional-order nonlinear ordinary differential equations.

Let us consider a class of fractional initial value problems expressed as follows:

$$D^{\gamma} y(x) + f(x, y, y') = 0, \quad x \in (0, 1], \quad 1 < \gamma \leq 2, \tag{5.1}$$

subject to initial condition

$$y(0) = a, \ y'(0) = b, \quad a, \ b \in \mathbb{R}.$$

Here, D^{γ} denotes the left-sided Caputo fractional derivative. The fractional order differential equation (5.1) is recommended to be solved in the future using the sixth-kind Chebyshev wavelet collocation method.

(iv) Continuing the work mentioned in point (iii), it is recommended to solve a class of fractional initial value problems of the form:

$$F(x, D^{\delta} y(x), D^{\delta_1} y(x), \ldots, D^{\delta_n} y(x)) = 0, \quad y^{(i)}(0) = y_i, \quad 0 < i \leq n,$$

where $n = \lceil \gamma \rceil$ is the smallest integer greater than or equal to δ, and $0 < \delta_1 < \delta_2 < \cdots < \delta_n \leq \delta$, using the method proposed in this research paper.

Acknowledgements Shyam Lal, one of the authors, is thankful to DST-CIMS for encouragement to this work. Deepak Kumar Singh, one of the authors, is grateful to U.G.C. (India) for providing financial assistance in the form of UGC Non-NET Fellowship for his research work. Authors are grateful to the referee for his valuable comments and suggestions to improve the quality of this research paper.

Data availability No data associated in the manuscript.

Declarations

Competing interests The authors declared that they have no conflict of interest.

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