Indian J Pure Appl Math (2026) 57:1832–1843
https://doi.org/10.1007/s13226-024-00692-3

ORIGINAL RESEARCH

Spline approximation methods for second order singularly perturbed convection-diffusion equation with integral boundary condition

A. Puvaneswari · T. Valanarasu

Received: 7 December 2023 / Accepted: 27 August 2024 / Published online: 11 September 2024
© The Indian National Science Academy 2024

Abstract In this work, we considered a Singular Perturbation Problem (SPP) with an integral boundary condition. Two numerical methods, namely Variable Mesh Spline Approximation Method (VMSAM) and Cubic B-Spline Collocation Method (CBSCM) are discussed to obtain second order convergence in the supremum norm. Numerical examples are presented to validate the obtained results.

Keywords Singular perturbation · Integral boundary condition · Variable mesh · Spline difference scheme · Collocation points · B-Spline functions

Mathematics Subject Classification 65L10 · 65L20 · 65L50 · 65L70

1 Introduction

Differential equations are said to be singularly perturbed due to the existence of a small positive parameter ε in the highest derivative term of the equation. This type of problem occurs in various fields of applied mathematics and engineering applications. For instance, fluid mechanics, chemical reactor theory, the Naviour-Stokes equations of fluid flow at high Reynold's number, control theory, electrical networks and other physical models. Generally, these problems have boundary layers in their solutions. Problems with singular perturbations have solutions that are multiscale in nature, with one component varying smoothly and the other component rapidly. This makes numerical analysis of this kind of problem more difficult than it is for ordinary problems. Classical numerical methods would fail to capture the layer behaviour. To overcome this difficulty, we should have a greater number of mesh points in the layer region. For this, we have to adopt layer adopted meshes and we have to develop suitable numerical methods for the singular perturbation problem which are ε-uniform convergence.

According to [1,2], Boundary Value Problems (BVPs) with integral boundary conditions are an important type of problem that can be found in a number of applications, including electrochemistry, thermoelasticity, heat conduction, underground water flow, and population dynamics. In fact, integral boundary condition boundary value problems have attracted a lot of attention. For example, in a study of temperature distribution of the rod where the temperature at one end of the rod say $x = 1$ depends not only on $u(1)$, but on the integrated effect of

Communicated by G.D. Veerappa Gowda.

A. Puvaneswari
Department of Mathematics, University College of Engineering, Anna University, BIT Campus, Tiruchirappalli, Tamilnadu 620 024, India
E-mail: puvaneswari@aubit.edu.in

T. Valanarasu (✉)
Department of Mathematics, CDOE, Bharathidasan University, Tiruchirappalli, Tamilnadu 620 024, India
E-mail: valanarasu@bdu.ac.in

temperature distribution $u(x)$ across the entire length of the rod, weighted by $g(x)$, the mathematical model is the boundary value problem with integral boundary condition. Many works have examined problems involving ordinary and partial differential equations with integral boundary conditions [3,4]. Still, the problems addressed in all of these papers are of the regular kind (i.e., they are not singularly perturbed).

Studies of singularly perturbed differential equations with non-integral initial/boundary conditions have a large literature (see previous studies [5–7] and the references therein). A singular perturbation problem with an integral boundary condition usually occurs when different timelines or length scales dominate a system's behaviour and this kind of problems exist in almost all the fields of science and engineering like control systems and optimization, biological and chemical transport, electrostatics and electrical circuits, thermal conduction and heat transfer. Integral boundary conditions are usually present in all these cases because it is necessary for taking into consideration global system features or non-local impacts. Hence in recent years, attention has been given for studying BVPs with integral boundary conditions which are singularly perturbed [8–10].

Motivated by the works of [8] and [9], we considered the following class of singularly perturbed boundary value problem with integral boundary condition:

$$\mathscr{L}u(x) \equiv -\varepsilon u''(x) + p(x)u'(x) = f(x),\ x \in \Omega = (0,1), \tag{1.1}$$

$$u(0) = A_1, \tag{1.2}$$

$$\mathscr{L}_1 u(1) \equiv u(1) - \varepsilon \int_0^1 g(x)u(x)dx = A_2, \tag{1.3}$$

where ε $(0 < \varepsilon \ll 1)$ is a singular perturbation parameter, $p(x) \geq \alpha > 0,\ g(x)$ is non-negative with $\int_0^1 g(x)dx < 1$ and A_1 and A_2 are given constants and $\bar{\Omega} = [0,1]$. We assume that $p(x)$ and $f(x)$ are sufficiently smooth functions on $\bar{\Omega}$ such that the solution $u(x)$ exhibits boundary layer at the end point $x = 1$. We employ the maximum norm $||v||_\infty = \max|v(x)|$ for any continuous function $v(x)$ defined on the appropriate interval. C (Sometimes subscripted) stands for a common positive constant that is independent of ε and the mesh dimension N throughout the study.

In this paper, two different numerical methods, namely VMSAM and CBSCM are suggested to solve the BVP (1.1)–(1.3). VMSAM [11–13] is based on a spline difference scheme with the integral boundary condition (1.3) discretized by the Trapezoidal rule on the non-uniform meshes (variable meshes). Consequently, we can get a greater number of mesh points in the layer region which will help us to study the layer behavior and leads to a good approximation of the solution. CBSCM is based on B-spline functions with the integral boundary condition (1.3) discretized by the Trapezoidal rule on a piecewise uniform mesh (Shishkin mesh). The B-spline collocation technique applied to differential equations yields banded matrices with fewer bands than the full matrices that may be obtained by using, for instance, polynomials, trigonometric functions, and other well-known non-piecewise approximates.

The paper describes the following: In the following section, some preliminary results and the derivative bounds for the solution of the problem are defined. Section 3 describes the mesh construction strategy for both methods and in Sect. 4, difference schemes are derived for both methods. Section 5 discusses the error estimate of both methods, following test problems in Sect. 6.

2 Results on continuous problem

In this section, we state maximum principle, stability result and the derivative bounds for the solution of the BVP (1.1)–(1.3) as in [9].

Lemma 2.1 [9] (Maximum Principle) *Let $\mathscr{L}$ and $\mathscr{L}_1$ be the differential operators in* (1.1)–(1.3) *and $\varphi(x) \in C^2(\bar{\Omega})$. If $\varphi(0) \geq 0$, $\mathscr{L}_1\varphi(1) \geq 0$ and $\mathscr{L}\varphi(x) \geq 0, \forall x \in \Omega$, then $\varphi(x) \geq 0,\ \forall x \in \bar{\Omega}$.*

Lemma 2.2 [9] (Stability Result) *The solution $\varphi(x)$ of problem* (1.1)–(1.3) *satisfies the bound*

$$|\varphi(x)| \leq C \max\{|\varphi(0)|, |\mathscr{L}_1|, ||f||\}, \forall x \in \bar{\Omega}. \tag{2.1}$$

Lemma 2.3 [9] (Derivative Estimate) *Let $\varphi(x)$ be the solution of* (1.1)–(1.3). *Then*

$$|\varphi^{(i)}(x)| \leq C\varepsilon^{-i} \ \text{ for } \ i = 1(1)3, x \in \Omega.$$

Lemma 2.4 [9] *If $v(x)$ and $w(x)$ are the solution of smooth and layer components of $\phi(x)$, we get the following bounds:*

$$||v^{(i)}||_{\bar{\Omega}} \leq C(1+\varepsilon^{(2-i)})$$
$$|w^{(i)}(x)| \leq C\varepsilon^{-i}e^{-\alpha(1-x)/\varepsilon}, \text{ for } 0 \leq i \leq 4, \ \forall x \in \Omega.$$

3 Mesh construction

In this section, we discussed two different ways of mesh construction for our numerical methods. For VMSAM, variable mesh is constructed and for CBSCM, Shiskhin mesh is used as described below. For the considered problem (1.1)–(1.3), the boundary layer exists at the right end of the interval [0, 1].

3.1 VMSAM

We divide the interval into three subintervals [13] as

$[0, \varrho]$, $[\varrho, 1-\varrho]$ and $[1-\varrho, 1]$ where $\varrho = \varepsilon \log\left(\frac{1}{\varepsilon}\right)$ which denotes the width of the boundary layer. Each subintervals consists of n_1, n_2 and n_3 number of points in the above subdomains respectively such that $n_1 + n_2 + n_3 = N$. Let $\tilde{h}_1$ and κ be the assumed positive constants. The non uniform grids in the subdomain $[0, \varrho]$ is defined as

$$h_i = \varsigma\tilde{h}_i, \quad i = 1, \ldots, n_1$$
$$\varsigma = \frac{\varrho}{\tilde{\varsigma}},$$
$$\tilde{\varsigma} = \tilde{h}_1 + \tilde{h}_2 + \tilde{h}_3 + \cdots + \tilde{h}_{n_1},$$

where

$$\tilde{h}_i = \tilde{h}_{i-1} + \kappa \left[\frac{\tilde{h}_{i-1}}{\varepsilon}\right] \min\left(\tilde{h}_{i-1}^2, \varepsilon\right), \quad i = 1, 2, \ldots, n_1.$$

Since the layer exists in the right side of the interval [0, 1], we take the mirror image of the above for the interval $[1-\varrho, 1]$. In the subdomain $[0, 1-\varrho]$, the grid is uniform and is defined as follows:

$$h_i = \frac{1-\varrho}{n_1+n_2}, \quad i = 1, \ldots, 3N/4,$$

and hence we define $x_0 = 0$, $x_i = x_{i-1} + h_i$, $i = 1, \ldots, N$, and $\bar{\Omega}^N = \{x_i\}_{i=0}^N$.

3.2 CBSCM

For this method, we divide the interval [0, 1] into two subintervals $[0, 1-\varrho]$ and $[1-\varrho, 1]$ both subinterval consists of $N/2$ mesh points with $\varrho = \min\left(\frac{1}{2}, \frac{\varrho_0\varepsilon}{\alpha}\ln N\right)$, $\varrho_0 \geq 2$.

Define

$$x_i = \begin{cases} i\bar{h}, & i = 0, 1, 2, \ldots, N/2 \\ x_{N/2} + (i - N/2)\bar{h}, & i = N/2+1, \ldots, N \end{cases}$$

where

$$\bar{h} = \begin{cases} h_1 = \dfrac{2(1-\varrho)}{N}, & i = 1, 2, \ldots, N/2 \\ h_2 = \dfrac{2\varrho}{N}, & i = N/2+1, \ldots, N \end{cases}$$

is the piecewise uniform mesh spacing and $\bar{\Omega}^N = \{x_i\}_{i=0}^N$.

4 Difference schemes

4.1 Derivation of the difference scheme for VMSAM

In this section, using cubic spline difference scheme, the approximate solution of the differential equation and the integral boundary condition of the given problem (1.1)–(1.3) are derived.

Let $x_0 = 0,\ x_N = 1, \quad x_i = x_0 + \sum_{m=1}^{i} h_m, \quad i = 1, \ldots, N, h_i = x_i - x_{i-1}$ be the mesh. For the given values $u(x_0), u(x_1), \ldots, u(x_N)$ of a function $u(x)$ at the nodal points $x_0, x_1, \ldots, x_N$ there exists an interpolating cubic spline function $\mathscr{P}_i(x)$ with the following properties:

1. $\mathscr{P}_i(x)$ coincides with a polynomial of degree three on each subintervals $[x_{i-1}, x_i],\ i = 1, \ldots, N$,
2. $\mathscr{P}_i(x) \in C^2(\bar{\Omega})$,
3. $\mathscr{P}_i(x_i) = u(x_i), \quad i = 0, 1, \ldots, N$.

Then the cubic spline function can be written as follows:

$$\begin{aligned}\mathscr{P}_i(x) = {} & \frac{(x_i - x)^3}{6h_i}\mathscr{M}_{i-1} + \frac{(x - x_{i-1})^3}{6h_i}\mathscr{M}_i + \left(u(x_{i-1}) - \frac{h_i^2}{6}\mathscr{M}_{i-1}\right)\left(\frac{x_i - x}{h_i}\right) \\ & + \left(u(x_i) - \frac{h_i^2}{6}\mathscr{M}_i\right)\left(\frac{x - x_{i-1}}{h_i}\right),\end{aligned} \tag{4.1}$$

where $x \in [x_{i-1}, x_i]$ and $\mathscr{M}_i = \mathscr{P}_i''(x_i),\ i = 0, \ldots, N$.

The derivation of the difference scheme is as follows:
Differentiating (4.1) and denoting the nodal interpolants of $u(x)$ by u_i's, we get

$$\mathscr{P}_i'(x) = -\frac{(x_i - x)^2}{2h_i}\mathscr{M}_{i-1} + \frac{(x - x_{i-1})^2}{2h_i}\mathscr{M}_i + \left(\frac{u_i - u_{i-1}}{h_i}\right) - \left(\frac{\mathscr{M}_i - \mathscr{M}_{i-1}}{6}\right)h_i.$$

Since $\mathscr{P}_i(x) \in C^2(\bar{\Omega})$, we have $\mathscr{P}_i'(x_i) = \mathscr{P}_{i+1}'(x_i)$. This gives

$$\frac{h_i}{6}\mathscr{M}_{i-1} + \frac{h_i + h_{i+1}}{3}\mathscr{M}_i + \frac{h_{i+1}}{6}\mathscr{M}_{i+1} = \frac{u_{i+1} - u_i}{h_{i+1}} - \frac{u_i - u_{i-1}}{h_i}.$$

where

$$\mathscr{M}_i = \frac{1}{\varepsilon}(-f_i + p_i u_i').$$

$f_i = f(x_i)$ and similarly $p_i = p(x_i)$. As discussed in [11,12], we can obtain the following linear system of equations by applying Taylor series expansion

$$\mathscr{L}^N u_i = H_i, \quad i = 1, \ldots, N-1, \tag{4.2}$$

where

$$\begin{aligned}\mathscr{L}^N u_i &= E_i u_{i-1} + F_i u_i + G_i u_{i+1}, \\ H_i &= \frac{h_i}{6}f_{i-1} + \frac{h_i + h_{i+1}}{3}f_i + \frac{h_{i+1}}{6}f_{i+1}, \\ E_i &= -\frac{h_{i+1} + 2h_i}{6(h_i + h_{i+1})}p_{i-1} - \frac{h_{i+1}}{3h_i}p_i + \frac{h_{i+1}^2}{6h_i(h_i + h_{i+1})}p_{i+1} - \frac{\varepsilon}{h_i}, \\ F_i &= \frac{h_i + h_{i+1}}{6h_{i+1}}p_{i-1} + \frac{h_{i+1}^2 - h_i^2}{3h_i h_{i+1}}p_i - \frac{h_i + h_{i+1}}{6h_i}p_{i+1} + \frac{\varepsilon}{h_i} + \frac{\varepsilon}{h_{i+1}}, \\ G_i &= -\frac{h_i^2}{6h_{i+1}(h_i + h_{i+1})}p_{i-1} + \frac{h_i}{3h_{i+1}}p_i + \frac{2h_{i+1} + h_i}{6(h_i + h_{i+1})}p_{i+1} - \frac{\varepsilon}{h_{i+1}}.\end{aligned} \tag{4.3}$$

Using the boundary condition (1.2), the scheme (4.2) can take the form

$$F_1u_1 + G_1u_2 = H_1 - E_1A_1. \tag{4.4}$$

The integral boundary condition (1.3) can be approximated using Trapezoidal rule in the following way:

$$\begin{aligned}
&u_N - \varepsilon \sum_{i=1}^{N} \frac{g_{i-1}u_{i-1} + g_iu_i}{2} h_i = A_2,\\
&u_N - \frac{\varepsilon}{2}[(g_0u_0 + g_1u_1)h_1 + (g_1u_1 + g_2u_2)h_2 + (g_2u_2 + g_3u_3)h_3 + (g_3u_3 + g_4u_4)h_4 + \cdots\\
&+ (g_{N-2}u_{N-2} + g_{N-1}u_{N-1})h_{N-1} + (g_{N-1}u_{N-1} + g_Nu_N)h_N] = A_2.
\end{aligned}$$

Replacing u_0 by A_1 and after collecting the terms containing $u_1, u_2, \ldots, u_{N-1}$, we get

$$\begin{aligned}
u_N = \gamma[A_2 + \frac{\varepsilon}{2}[&g_0A_1h_1 + g_1u_1(h_1 + h_2) + g_2u_2(h_2 + h_3) + g_3u_3(h_3 + h_4) + \cdots\\
&+ g_{N-2}u_{N-2}(h_{N-2} + h_{N-1}) + g_{N-1}u_{N-1}(h_{N-1} + h_N)]]
\end{aligned} \tag{4.5}$$

where $\gamma = \dfrac{2}{2 - \varepsilon g_N h_N}$. When $i = N - 1$ in equation (4.2) takes the form

$$E_{N-1}u_{N-2} + F_{N-1}u_{N-1} + G_{N-1}u_N = H_{N-1}.$$

Substituting u_N and after simplification, we get

$$\begin{aligned}
&G_{N-1}\gamma\varepsilon[u_1g_1(h_1 + h_2) + u_2g_2(h_2 + h_3) + u_3g_3(h_3 + h_4) + \cdots + u_{N-3}g_{N-3}(h_{N-3} + h_{N-2})]\\
&\quad + u_{N-2}(E_{N-1} + G_{N-1}\gamma\varepsilon g_{N-2}(h_{N-2} + h_{N-1})/2) + u_{N-1}(F_{N-1} + G_{N-1}\gamma\varepsilon g_{N-2}(h_{N-1} + h_N))/2\\
&= H_{N-1} - G_{N-1}\gamma A_2 - G_{N-1}\gamma g_0A_1h_1/2.
\end{aligned} \tag{4.6}$$

Equations (4.2) along with (4.4) and (4.6) forms a linear system of $N - 1$ equations with $N - 1$ unknowns $u_1, u_2, \ldots, u_{N-1}$. After solving this $N - 1$ equations, we can obtain the values of $u_1, u_2, \ldots, u_{N-1}$. Using this $N - 1$ values, we can find the value of u_N from equation (4.5). Thus obtained the approximate value of (1.1)-(1.3).

4.1.1 Derivation of the difference scheme for CBSCM

Let $L_2(\bar{\Omega})$ be a vector space of all the square integrable function on $\bar{\Omega}$ and let X be a linear subspace of $L_2(\bar{\Omega})$. For $k = \{-1, 0, 1, \ldots, N + 1\}$, cubic B-splines are defined as

$$\mathscr{P}_k(x) = \frac{1}{\bar{h}^3}\begin{cases}
(x - x_{k-2})^3, & x_{k-2} \le x \le x_{k-1},\\
\bar{h}^3 + 3\bar{h}^2(x - x_{k-1}) + 3\bar{h}(x - x_{k-1})^2 - 3(x - x_{k-1})^3, & x_{k-1} \le x \le x_k,\\
\bar{h}^3 + 3\bar{h}^2(x_{k+1} - x) + 3\bar{h}(x_{k+1} - x)^2 - 3(x_{k+1} - x)^3, & x_k \le x \le x_{k+1},\\
(x_{k+2} - x)^3, & x_{k+1} \le x \le x_{k+2},\\
0, & \text{otherwise}
\end{cases}$$

Let $x_0, x_1, x_2, \ldots, x_N$ be the collection of mesh points on $\bar{\Omega}$ and let it be denoted by π. By introducing four additional points x_{-2}, x_{-1}, x_{N+1} and x_{N+2} where $x_{-2} < x_{-1} < x_0$ and $x_N < x_{N+1} < x_{N+2}$. Then $\pi = \{x_{-2}, x_{-1}, x_0, \ldots, x_N, x_{N+1}, x_{N+2}\}$. Further, we can say that each of the above function $\mathscr{P}_i(x)$ is twice continuously differentiable on the real line. Each $\mathscr{P}_i(x)$ is a piecewise cubic polynomial at π and $\mathscr{P}_i(x) \in X$. Let $\Psi = \{\mathscr{P}_{-1}, \mathscr{P}_0, \ldots, \mathscr{P}_{N+1}\}$ and let span $(\Psi) = \phi_3(\pi)$. The function Ψ are linearly independent on [0, 1] and hence $\phi_3(\pi)$ is $(N + 3)$ dimensional. Let T be a linear operator whose domain is X and range is also in X. Define

$$\mathscr{S}(x) = \sum_{i=-1}^{N+1} d_i\mathscr{P}_i(x) \tag{4.7}$$

where d_i's are real coefficients to be determined. Two more cubic B-splines $\mathscr{P}_{-1}$, $\mathscr{P}_{N+1}$ are introduced to satisfy the boundary conditions. Hence, the given problem (1.1)–(1.3) has taken the form

$$\mathscr{L}\mathscr{S}(x_i) = f(x_i), \quad 0 \le i \le N \tag{4.8}$$

$$\mathscr{S}(x_0) = A_1, \tag{4.9}$$

$$\mathscr{S}(x_N) - \varepsilon \sum_{i=1}^{N} \frac{g_{i-1}\mathscr{S}_{i-1} + g_i \mathscr{S}_i}{2} h_i = A_2. \tag{4.10}$$

Solving the equations (4.8) and using the values of B-spline functions $\mathscr{P}_k$'s and their derivatives at the mesh points, we obtain a system of $N+1$ linear equations in $N+3$ unknowns.

$$\left(\frac{-6\varepsilon}{h_i^2} - \frac{3p_i}{h_i}\right) d_{i-1} + \frac{12\varepsilon}{h_i^2} d_i + \left(\frac{-6\varepsilon}{h_i^2} + \frac{3p_i}{h_i}\right) d_{i+1} = f_i, \quad i = 0(1)N$$

which can be written, in general

$$E_i d_{i-1} + F_i d_i + G_i d_{i+1} = f_i, \qquad i = 0(1)N \tag{4.11}$$

$$\text{where } E_i = \frac{-6\varepsilon}{h_i^2} - \frac{3p_i}{h_i}; \quad F_i = \frac{12\varepsilon}{h_i^2} \quad \text{and} \quad G_i = \frac{-6\varepsilon}{h_i^2} + \frac{3p_i}{h_i}$$

Using (4.9), Equation (4.11), when $i = 0$ becomes

$$d_0(-4E_0 + F_0) + d_1(-E_0 + G_0) = f_0 - A_1 E_0. \tag{4.12}$$

Similary using (4.10), Equation (4.11), when $i = N$ becomes

$$\begin{aligned}
&d_0 G_N \frac{\gamma\varepsilon}{2}(g_1h_1 + g_1h_2) + d_1 G_N \frac{\gamma\varepsilon}{2}(4g_1h_1 + 4g_1h_2 + g_2h_2 + g_2h_3)\\
&+ d_2 G_N \frac{\gamma\varepsilon}{2}(g_1h_1 + g_1h_2 + 4g_2h_2 + 4g_2h_3 + g_3h_3 + g_3h_4)\\
&+\\
&\vdots\\
&+ d_{N-2} G_N \frac{\gamma\varepsilon}{2}(g_{N-3}h_{N-3} + g_{N-3}h_{N-2} + 4g_{N-2}h_{N-2} + 4g_{N-2}h_{N-1} + g_{N-1}h_{N-1} + g_{N-1}h_N)\\
&+ d_{N-1}(G_N \frac{\gamma\varepsilon}{2}(g_{N-2}h_{N-2} + g_{N-2}h_{N-1} + 4g_{N-1}h_{N-1} + 4g_{N-1}h_N + g_N h_N) - \gamma G_N + E_N)\\
&+ d_N(G_N \frac{\gamma\varepsilon}{2}(g_{N-1}h_{N-1} + g_{N-1}h_N + 4g_N h_N) - 4\gamma G_N + F_N) = f_N - G_N A_2 \gamma - G_N \frac{\gamma\varepsilon}{2} g_0 A_1 h_1.
\end{aligned} \tag{4.13}$$

where $\gamma = \dfrac{2}{2 - \varepsilon g_N h_N}$.

Equations (4.11) along with (4.12) and (4.13) together form a system of $N+1$ linear equations with $N+1$ unknowns which can be written in the matrix form as $Ad = \delta$ where

$$A = \begin{pmatrix}
-4E_0 + F_0 & -E_0 + G_0 & 0 & 0 & \dots & \dots & \dots & 0\\
E_1 & F_1 & G_1 & 0 & 0 & \dots & \dots & 0\\
0 & E_2 & F_2 & G_2 & 0 & \dots & \dots & 0\\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & E_{N-1} & F_{N-1} & G_{N-1}\\
A_N(x_1) & A_N(x_2) & \cdots & A_N(x_i) & \cdots & \cdots & A_N(x_{N-1}) & A_N(x_N)
\end{pmatrix}$$

$$A_N(x_1) = G_N \frac{\gamma\varepsilon}{2} g_1(h_1 + h_2)$$

$$A_N(x_2) = G_N\frac{\gamma\varepsilon}{2}(4g_1(h_1+h_2)+g_2(h_2+h_3)$$
$$A_N(x_3) = G_N\frac{\gamma\varepsilon}{2}(g_1(h_1+h_2)+4g_2(h_2+h_3)+g_3(h_3+h_4)$$
$$\vdots$$
$$A_N(x_{N-2}) = G_N\frac{\gamma\varepsilon}{2}(g_{N-3}(h_{N-3}+h_{N-2})+4g_{N-2}(h_{N-2}+h_{N-1})+g_{N-1}(h_{N-1}+h_N))$$
$$A_N(x_{N-1}) = G_N\frac{\gamma\varepsilon}{2}(g_{N-2}(h_{N-2}+h_{N-1})+4g_{N-1}(h_{N-1}+h_N)+g_Nh_N)-\gamma G_N+E_N$$
$$A_N(x_N) = G_N\frac{\gamma\varepsilon}{2}(g_{N-1}(h_{N-1}+h_N)+4g_Nh_N)-4\gamma G_N+F_N$$

The column vectors d and δ are given as $d = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_N \end{pmatrix}$, $\delta = \begin{pmatrix} f_0 - A_1E_0 \\ f_1 \\ f_2 \\ \vdots \\ f_N - G_NA_2\gamma - G_N\frac{\gamma\varepsilon}{2}g_0A_1h_1 \end{pmatrix}$.

Here the coefficient matrix A is strictly diagonally dominant with the assumption that $0 < \bar{h} < 2\varepsilon/p(x_i)$. After solving the system of equations, we get $d_0, d_1, d_2, \ldots, d_N$ from which we can evaluate the values of d_{-1} and d_{N+1}. Hence the method of collocation using a basis of cubic B-splines applied to problems (1.1)–(1.3) has a unique solution $\mathscr{S}(x)$ given by (4.7).

5 Error estimates

5.1 VMSAM

We will make use of comparison functions and discrete maximum principle in order to derive the error estimate of the solution. Now we state the following Lemmas as in [14] which are essential to prove the final result.

Lemma 5.1 (Discrete Maximum Principle) *Let $\{\varphi_i\}$ be a set of approximate values of the mesh points x_i, with $\varphi_i \geq 0, \mathscr{L}_1^N\varphi_N \geq 0$ and $\mathscr{L}^N\varphi_i \geq 0,\ 1 = 1, \ldots, N-1$, then $\varphi_i \geq 0,\ 1 = 0, 1, \ldots, N$*

Lemma 5.2 *If $\mathscr{R}_1(h,\varepsilon) \geq 0$ and $\mathscr{R}_2(h,\varepsilon) \geq 0$ are such that*

$$\mathscr{L}^N(\mathscr{R}_1(h,\varepsilon)\phi_k + \mathscr{R}_2(h,\varepsilon)\psi_k) \geq \mathscr{L}^N(\pm e_k) = \pm\tau_k(u), \quad k = 1, \ldots, N-1$$
$$\mathscr{L}_1^N(\mathscr{R}_1(h,\varepsilon)\phi_N + \mathscr{R}_2(h,\varepsilon)\psi_N) \geq \mathscr{L}_1^N(\pm e_N) = \pm\tau_N(u),$$

then the discrete maximum principle implies that

$$|e_k| \leq \mathscr{R}_1(h,\varepsilon)|\psi_1(k)| + R_2(h,\varepsilon)|\psi_2(k)|,$$

where $e_k = |\varphi(x_k) - \varphi_k|$ for each k and ψ_1 and ψ_2 are two comparison functions.

The following lemma is the extension of Lemma 2.4 in [14] for the properties of exact solution of (1.1)–(1.3).

Lemma 5.3 *If $u(x)$ satisfies* (1.1)*, then*

$$u(x) = w(x) + z(x)$$
$$\text{where, } w(x) = -\varepsilon\frac{u'(1)}{p(1)}\exp\left(\frac{-p(1)(1-x)}{\varepsilon}\right)$$
$$|z^i(x)| \leq C\left(1 + \varepsilon^{-i+1}\exp\left(\frac{-C_1(1-x)}{\varepsilon}\right)\right)$$

$i = 0(1)4$, C is some constant and C_1 is a positive constants independent of h and ε.

We use the following two comparison functions (as in [14]).

$$\psi_1(x_i) = -1 + x_i \quad \text{and} \quad \psi_2(x_i) = C_2 \exp\left[-\frac{1-x_i}{\varepsilon}\right].$$

Now, to estimate the truncation error of the scheme (4.2). Consider the case in which $Ch_c^2 \le \varepsilon$. We have

$$\tau_i(u) = \mathcal{T}_{0,i}u_i + \mathcal{T}_{1,i}u_i' + \mathcal{T}_{2,i}u_i'' + \mathcal{T}_{3,i}u_i''' + \text{remainder terms}, \quad i = 1, \ldots, N-1,$$

where

$$\begin{aligned}
\mathcal{T}_{0,i} &= E_i + F_i + G_i,\\
\mathcal{T}_{1,i} &= h_{i+1}G_i - h_iE_i + \left[\frac{h_i}{6}p_{i-1} + \frac{h_i + h_{i+1}}{3}p_i + \frac{h_i+1}{6}p_{i+1}\right],\\
\mathcal{T}_{2,i} &= \left(\frac{h_i^2}{2}E_i + \frac{h_{i+1}^2}{2}G_i\right) + \varepsilon\left[\frac{h_i}{6} + \frac{h_i+h_{i+1}}{3} + \frac{h_{i+1}}{6}\right] - \left[\frac{h_i^2}{6}p_{i-1} - \frac{h_{i+1}^2}{6}p_{i+1}\right]
\end{aligned}$$

and

$$\mathcal{T}_{3,i} = \left(\frac{h_{i+1}^3}{6}G_i - \frac{h_i^3}{6}E_i\right) + \varepsilon\left[\frac{h_i^2}{6} - \frac{h_{i+1}^2}{6}\right] + \frac{h_i^3}{12}p_{i-1} + \frac{h_{i+1}^3}{12}p_{i+1}.$$

Using (4.3), we find that $\mathcal{T}_{0,i} = 0$, $\mathcal{T}_{1,i} = 0$, $\mathcal{T}_{2,i} = 0$ and $|\mathcal{T}_{3,i}| \le Ch_c^3$. From Lemma 5.3, we have

$$w'''(x_i) = \left(\frac{p(1)}{\varepsilon}\right)^2 u'(1)\exp\left(\frac{-p(1)(1-x_i)}{\varepsilon}\right)$$
$$\text{hence, } |\tau_i(w)| \le Ch_c^3\varepsilon^{-2}\exp\left(-\frac{p(1)(1-x_i)}{\varepsilon}\right),\ Ch_c^2 \le \varepsilon.$$

Again

$$|z_i'''| \le C\left[1 + \varepsilon^{-2}\exp\left(-\frac{C_1(1-x_i)}{\varepsilon}\right)\right]$$
$$\Longrightarrow \quad |\tau_i(z)| \le Ch_c^3\left[1 + \varepsilon^{-2}\exp\left(-\frac{C_1(1-x_i)}{\varepsilon}\right)\right],\ Ch_c^2 \le \varepsilon.$$

Then

$$\tau_i(u) = \tau_i(w) + \tau_i(z) \text{ gives}$$
$$|\tau_i(u)| \le Ch_c^3\varepsilon^{-2}\left[1 + \exp\left(-\frac{C_1(1-x_i)}{\varepsilon}\right)\right],\ Ch_c^2 \le \varepsilon.$$

Similarly we can prove the result for the case $Ch_c^2 \ge \varepsilon$ and we reached the following theorem.

Theorem 5.4 *If $u(x_i)$ is the solution of the BVP* (1.1)–(1.3) *and u_i, $i = 0, 1, \ldots, N$ is the numerical solution obtained by the VMSAM, then we have*

$$\max_i |u(x_i) - u_i| \le Ch_c^2\left[1 + \exp\left(\frac{-C_3(1-x_i)}{\varepsilon}\right)\right].$$

5.2 CBSCM

Lemma 5.5 *The B-splines $\mathscr{P}_{-1}, \mathscr{P}_0, \mathscr{P}_1, \ldots, \mathscr{P}_{N+1}$ satisfy the inequality*

$$\sum_{i=-1}^{N+1} |\mathscr{P}_i(x)| \leq 10, \quad x \in \bar{\Omega}.$$

Theorem 5.6 *Let $\mathscr{S}(x)$ be the collocation approximation from the space of cubic splines $\phi_3(\pi)$ to the solution $u(x)$ of the considered problem* (1.1)–(1.3)*. If $f \in C^2[0,1]$, then the parameter uniform error estimate is given by*

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq k\leq N} |u(x_k) - \mathscr{S}(x_k)| \leq CN^{-2}(\ln N)^3$$

where C is a positive constant independent of ε and N.

Proof Let $U(x)$ be the unique spline interpolate from $\phi_3(\pi)$ to the solution $u(x)$ of BVP (1.1)–(1.3) defined by

$$U(x) = \sum_{k=-1}^{N+1} \bar{d}_k \mathscr{P}_k(x). \tag{5.1}$$

If $f(x) \in C^2[0,1]$, then $u(x) \in C^4[0,1]$ and from de Boor-Hall error estimates [15] that

$$||D^j(u-U)||_\infty \leq C_j ||u^{(4)}|| \bar{h}^{4-j}, \quad j = 0, 1, 2, 3. \tag{5.2}$$

where $\tilde{h} = \max\{h_1, h_2\}$ and C_j are constants independent of ε and N. Then from the estimate (5.2) follows as

$$\begin{aligned}
|\mathscr{L}u(x_k) - \mathscr{L}U(x_k)| &= |-\varepsilon u''(x_k) - p(x_k)u'(x_k) - (-\varepsilon U''(x_k) - p(x_k)U'(x_k))| \\
&\leq |-\varepsilon(u''(x_k) - U''(x_k)) - p(x_k)(u'(x_k) - U'(x_k))| \\
&\leq |\varepsilon(u''(x_k) - U''(x_k))| + |p(x_k)||u'(x_k) - U'(x_k)| \\
&\leq C_0 \varepsilon \tilde{h}^2 ||u^4|| + C_1 |p(x_k)| \tilde{h}^3 ||u^4|| \\
|\mathscr{L}u(x_k) - \mathscr{L}U(x_k)| &\leq C(\varepsilon \tilde{h}^2 + ||p||_\infty \tilde{h}^3) ||u^4|| \\
&\leq C(\varepsilon \tilde{h}^2 + ||p||_\infty \tilde{h}^3)(1 + \varepsilon^{-4} \exp(-\alpha(1-x_k)/\varepsilon)).
\end{aligned} \tag{5.3}$$

Now we have two cases to discuss:

Case (i): $\varrho = 1/2$. Then $1/2 \leq (2\varepsilon/\alpha^*) \ln N$ which gives $\varepsilon^{-1} \leq C \ln N$. Therefore, we obtain

$$|\mathscr{L}u(x_k) - \mathscr{L}U(x_k)| \leq CN^{-2}(\ln N)^3.$$

Case (ii): $\varrho = (2\varepsilon/\alpha) \ln N$. If k satisfies $1 \leq k \leq N/2$, we have $\tilde{h} = 2\varrho/N = N^{-1}(4\varepsilon/\alpha) \ln N$, and hence $\dfrac{\tilde{h}}{\varepsilon} = CN^{-1} \ln N$. Then (5.3) becomes

$$|\mathscr{L}u(x_k) - \mathscr{L}U(x_k)| \leq CN^{-2}(\ln N)^2.$$

Further, if k satisfies $N/2 \leq k \leq N$, then by using the lemma discussed in [16], we can obtain from (5.3)

$$|\mathscr{L}u(x_k) - \mathscr{L}U(x_k)| \leq CN^{-2}.$$

From the above three equations, we get

$$|\mathscr{L}u(x_k) - \mathscr{L}U(x_k)| \leq CN^{-2}(\ln N)^3$$

and hence we have

$$|\mathscr{L}\mathscr{S}(x_k) - \mathscr{L}U(x_k)| = |f(x_k) - \mathscr{L}U(x_k)| = |\mathscr{L}u(x_k) - \mathscr{L}U(x_k)| \leq CN^{-2}(\ln N)^3. \tag{5.4}$$

Suppose that $\mathscr{L}U(x_k) = \tilde{h}^2 \bar{f}(x_k), \quad \forall\ 0 \le k \le N$ with the boundary conditions $U(x_0) = 0$, $U(x_N) = A_2 + \varepsilon \int_0^1 g(x)u(x)dx$ leads to the linear system $A\bar{d} = \bar{\delta}$, which in turn results into

$$A(d - \bar{d}) = \delta - \bar{\delta} \tag{5.5}$$

where $d - \bar{d} = (d_0 - \bar{d_0}, d_1 - \bar{d_1}, \ldots, d_N - \bar{d_N})^T$,

$\delta - \bar{\delta} = (\tilde{h}^2(f(x_0) - \bar{f}(x_0)), \tilde{h}^2(f(x_1) - \bar{f}(x_1)), \ldots, \tilde{h}^2(f(x_N) - \bar{f}(x_N)))^T$.

Using the equation (5.4), we get

$$||\delta - \bar{\delta}||_\infty \le CN^{-4}(\ln N)^3. \tag{5.6}$$

Again, by estimation given in [17], we have

$$|d_k - \bar{d_k}| \le CN^{-2}(\ln N)^3,\ 0 \le k \le N.$$

Also from the boundary condition and from the above bounds, $|d_{-1} - \bar{d}_{-1}|$ and $|d_{N+1} - \bar{d}_{N+1}|$ give the same estimates as above. Hence, we have

$$\max_{-1\le k\le N+1} |d_k - \bar{d_k}| \le CN^{-2}(\ln N)^3.$$

From the above result and Lemma 5.5, we have

$$|\mathscr{S}(x) - U(x)| \le \max_{-1\le i\le N+1} |d_k - \bar{d_k}| \sum_{i=-1}^{N+1} |\mathscr{B}_i(x)| \le CN^{-2}(\ln N)^3,$$

which gives

$$\max_{0\le i\le N} |\mathscr{S}(x_i) - U(x_i)| \le CN^{-2}(\ln N)^3.$$

Then by triangle inequality, we get

$$\sup_{0<\varepsilon\le 1} \max_{0\le k\le N} |u(x_k) - \mathscr{S}(x_k)| \le CN^{-2}(\ln N)^3.$$

□

6 Numerical experiments

In this section, we present two numerical examples to illustrate the method discussed in this paper.

Example 6.1 [9]

$$-\varepsilon u''(x) + u'(x) = 1 \quad x \in \Omega,$$
$$u(0) = 0,\ u(1) - \varepsilon \int_0^1 \frac{x}{2} u(x)dx = 0.$$

Example 6.2

$$-\varepsilon u''(x) + (1 + x)u'(x) = e^{-4x}, \quad x \in \Omega,$$
$$u(0) = 0,\ u(1) - \varepsilon \int_0^1 \frac{x}{2} u(x)dx = 1.$$

The exact solutions of the above examples are not available. Hence using the double mesh principle, for both examples, a finite set of values $\varepsilon \in R_\varepsilon = \{2^{-1}, 2^{-2}, \ldots, 2^{-24}\}$, the maximum pointwise errors E^N which is defined as $E^N_\varepsilon = \max_{x_i\in\bar{\Omega}^N} |U^N(x_i) - U^{2N}(x_i)|$ and $E^N = \max_\varepsilon E^N_\varepsilon$ are estimated where $U^N(x_i)$ and $U^{2N}(x_i)$ are the numerical solution evaluated using N and $2N$ mesh points. Uniform rate of convergence is derived using the formula $P^N = \log_2\left(\frac{E^N}{E^{2N}}\right)$. The approximate solutions are depicted in Fig. 1 for Example 6.1 and Example 6.2 using the derived methods VMSAM and CBSCM. In the following tables, computed errors and computed order of convergence for the discussed methods are provided and they are compared with the results available in [9] for both the example problems (Tables 1, 2).

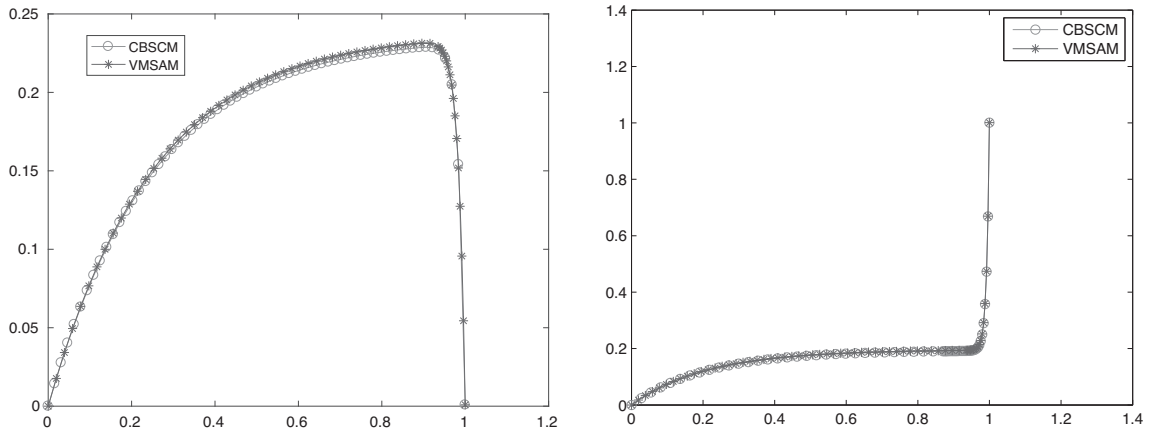


Fig. 1 Solution graph of Example 6.1 and Example 6.2 using VMSAM and CBSCM for $\varepsilon = 2^{-6}$ with $N = 64$.

Table 1 Comparitive values of E^N and P^N for Example 6.1 using VMSAM and CBSCM and Finite Difference Scheme [9]

	Number of mesh points N					
	64	128	256	512	1024	2048
	VMSAM					
E^N	1.6596e−01	4.5369e−02	7.9842e−03	2.0061e−03	4.5215e−04	1.0497e−04
P^N	1.8711	2.5065	1.9927	2.1495	2.1069	–
	CBSCM					
E^N	1.5541e−01	6.0517e−02	1.9414e−02	6.4524e−03	2.0807e−03	6.3617e−04
P^N	1.3606	1.6402	1.5892	1.6328	1.7096	–
	Finite difference scheme [9]					
E^N	4.362e−02	2.848e−02	1.760e−02	1.039e−02	5.956e−03	3.343e−03
P^N	0.6150	0.6942	0.7598	0.8034	0.8332	–

Table 2 Comparitive values of E^N and P^N for Example 6.2 using VMSAM and CBSCM and Finite difference scheme

	Number of mesh points N					
	64	128	256	512	1024	2048
	VMSAM					
E^N	1.0095e−01	2.4951e−02	5.3525e−03	1.2480e−03	2.9391e−04	6.6801e−05
P^N	2.0164	2.2208	2.1006	2.0861	2.1374	–
	CBSCM					
E^N	4.6837e−01	2.9386e−01	1.4704e−01	5.7284e−02	1.8193e−02	5.7423e−03
P^N	0.6725	0.9990	1.3600	1.6547	1.6637	—
	Finite difference scheme					
E^N	9.0451e−02	6.8923e−02	4.7314e−02	2.9966e−02	1.7927e−02	1.0321e−02
P^N	0.3922	0.5427	0.6590	0.7412	0.7966	—

7 Conclusion

Two different numerical methods consists of VMSAM on non-uniform mesh and CBSCM on Shishkin mesh are considered to solve singularly perturbed convection diffusion problem with integral boundary condition, whose solutions exhibit boundary layer at the end point $x = 1$. For VMSAM, in the construction of mesh, we have chosen $\tilde{h}_1 = 0.00001$ and $\kappa = 1$ for both examples. We note that the increase in the value of κ will lead to more concentration of points near the boundary regions and for a fixed κ, the increase in the value of $\tilde{h}_1$ leads to the same result. For CBSCM, Shishkin mesh is used. Both methods are of second order accurate and are validated by solving two example problems and the error of the scheme is measured using the discrete maximum norm.

From the results obtained, we can conclude that both methods are efficient one and the order of convergence in VMSAM is superior than CBSCM.

Declarations

Competing interests The author(s) declare that they do not have any competing interests.

References

1. J.R. Cannon, The solution of the heat equation subject to the specification of energy, *Quarterly of Applied Mathematics*, 21(2) (1963), 155–160.
2. L. Formaggia, F. Nobile, A. Quarteroni and A. VENEZIANI, Multi scale modeling of the circulatory system: a preliminary analysis, *Computing and Visualization in Science*, 2 (1999), 75–83. https://doi.org/10.1007/ s007910050030 <error l="303" c="Invalid
command: paragraph not started." />
3. A. Bouziani and N.E. Benouar, Mixed problem with integral conditions for a third order parabolic equation. *Kobe J. Math.* 15 (1998), 47-58.
4. T. Jankowski, Extensions of quasilinearization method for differential equations with integral boundary conditions. *Mathematical and Computer Modelling*, 37 (2003), 155–165.
5. H.G. Roos, M. Stynes, L. Tobiska, Numerical methods for singularly perturbed differential equations, *Springer Series in Computational Mathematics*, vol. 24. Berlin: Springer-Verlag; (1996).
6. R.E. O'Malley, Singular perturbation methods for ordinary differential equations, *Applied Mathematical Sciences*, vol. 89 (1991), New York: Springer.
7. J.J.H. Miller, E. O'Riordan and G.I. Shishkin, Fitted numerical methods for singular perturbation problems. *Singapore: World Scientific Co., 1996*.
8. M. Cakir, G.M. Amiraliyev, A finite difference method for the singularly perturbed problem with nonlocal boundary condition, *Applied Mathematics and Computation*, 160 (2005), 539-549. https://doi.org/10.1016/j.amc.2003.11.035
9. V. Raja and A. Tamilselvan, A difference scheme on shiskin mesh for convection diffusion problems with integral boundary condition, *AIP Conference Proceedings* 2277, 190001 (2020); https://doi.org/10.1063/5.0025595.
10. G.M. Amiraliyev, I.G. Amiraliyeva and M.Kudu, A numerical treatment for singularly perturbed differential equations with integral boundary condition. *Applied Mathematics and Computation* 185, (2007), 574–582.
11. A. Puvaneswari, A. Ramesh Babu and T.Valanarasu, Cubic spline scheme on variable mesh for singularly perturbed periodical boundary value problem, *NOVI SAD Journal of Mathematics*, 50(1), (2020), 157-172.
12. A. Puvaneswari and T. Valanarasu, Cubic spline approximation method for periodic singular perturbation problem, *South East Asian J. of Mathematics & Mathematical Sciences*, 21, Proceedings (2022), 65-78.
13. M.K. Kadalbajoo and K.C. Patidar, Variable mesh spline approximation method for solving singularly perturbed turning point problems having boundary layer(s), *An International Journal of Computers and Mathematics with Applications*, 42, (2001), 1439-1453.
14. R. Bruce Kellogg and Alice Tsan, Analysis of Some Difference Approximations for a Singular Perturbation Problem Without Turning Points, *Mathematics of Computation,* 32, (1978), 1025-1039.
15. P.M. Prenter, Spline and variational methods, *John Wiley and Sons, New York*, (1975).
16. M.K. Kadalbajoo and K.C. Patidar, ε-Uniform fitted mesh finite difference methods for general singular perturbation problems, *Applied mathematics and computation*,179 (2006), 248-266.
17. J.M. Varah, A lower bound for the smallest singular value of a matrix, *Linear Algebra Appl.*, 11, (1975), 3-5.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.