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ORIGINAL RESEARCH

On the self-complementary power graph of finite groups

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Abstract The power graph of a group G (denoted by $P(G)$) is the graph whose vertex set is G and two distinct vertices are adjacent if one is the power of the other. By removing the identity from a group's power graph, $P(G)$, one can obtain the group's proper power graph. A graph is self-complementary if it is isomorphic to its complement. In this paper, we observe that if G is a p-group, then $P^*(G)$ is never self-complementary. For the case of an EPPO group G, $P^*(G)$ is not self-complementary. Moreover, we show that if G is a group having two or more distinct prime divisors, then there does not exist any finite group G except possibly $G \ncong C_n \rtimes C_{p_k^{\alpha_k}}$ with $p_k \nmid n$ and $C_n \rtimes Q_{2^r}$ with $2 \nmid n$ and Q_{2^r} is the generalized quarternion group such that its proper power graph is self-complementary. The self-complementary index of a graph Γ, denoted by $s(\Gamma)$, is a graph parameter that measures the closeness of a graph to being self-complementary. In this paper, we provide suitable sharp bounds (upper and lower) for the self-complementary index of $P^*(G)$. Besides that, we look into the finite groups whose power graphs are self-complementary in the sense of different types of forbidden subgraphs and graph theory properties such as regularity, unicyclicity, 2-connectedness, strongly regularity, etc.

Keywords Power graph · EPPO group · Nilpotent group · Unicyclic · Self-complementary graph · Self-complementary index

Mathematics Subject Classification 20D60 · 05C25

1 Introduction

Graphs defined on groups have been very well studied through various graph constructions over the last few decades (see Reference [2] for more details). A power graph of groups is one such graph, which was first defined in 2002. This type of graph was initially discovered as directed graphs of finite semigroups by Kelarev and Quinn in [16]. For a given semigroup S, the directed power graph of S (denoted by $\overrightarrow{\mathcal{P}}(S)$) is a graph (V, E) where $V = S$ and $E = \{(a, b) : a \neq b, b = a^m \ for\ some\ m \in \mathbb{N}\}$. The corresponding underlying graph was developed in 2009 by Chakrabarty *et al.* as the notion of an undirected power graph of semigroups in [5]. For a given semigroup S, the undirected power graph of S (denoted by $\mathcal{P}(S)$) is a graph (V, E) where $V = S$ and

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$E = \{\{a, b\} : a \neq b, and\ either\ a = b^m\ or\ b = a^n,\ \ for\ some\ m, n \in \mathbb{N}\}$. Later on, this definition was adopted to describe the power graph of a finite group. The proper power graph of a group G (defined by Doostabadi et al. [11]), denoted by $P^*(G)$, is the graph obtained by removing the identity element from the power graph of G. More than 100 academic publications have been written that describe power graphs in various ways.

A graph is called self-complementary if it is isomorphic to its complement, or, in other words, there exists an isomorphism, say $f : G \to \overline{G}$, which can be represented as a permutation that is called a complementing permutation. The order of any self-complementary graph is either 0 or 1 (mod 4). In particular, for the regular graph, its order will be 1 (mod 4). Also, a self-complementary graph of order n has $\frac{n(n-1)}{4}$ vertices.

In 1962, Sachs started the study of self-complementary graphs. The diameter of a self-complementary graph is either 2 or 3 demonstrate by Sachs [23] in 1962. Later in 1974, Clapham and in 1975, Camion studied the Hamiltonicity of self-complementary graphs and concluded that self-complementary graphs are connected as well as tracable in [4,8]. In [22,23] the authors proved the result regarding the cycle structure of complementing permutation which says: If G is a self-complementary graph of order n and $f(G) = \overline{G}$, then if $n \equiv 0(mod\ 4)$ each cycle of f is of length divisible by 4 and if $n \equiv 1(mod\ 4)$ then f has exactly one cycle of length 1 and all other cycles have length divisible by 4. In 1974, [14] Gibbs concluded about the induced paths of self-complementary graphs, which is called the "Decomposition Theorem of Self-Complementary Graphs": If G is self-complementary and has $4k$ vertices, then G possesses a collection of vertex disjoint induced paths P_4. In 1976 [9,10] Claphamand and Kleitman studied the degree sequence of self-complementary graphs. Now we recall a theorem that was concluded by Rao in 1977.

Theorem 1.1 *A self-complementary graph of order $n > 5$ always contains a graph cycle of length l for every integer $3 \leq l \leq (n-2)$.*

In 1979, Chao and Whitehead discussed the chromaticity of a self-complementary graph in [6]. Gangopadhyay studied the characterizations of bipartite self-complementary bipartitioned sequences in 1982 in [13]. In 1985, Kotzig's solved three open questions for self-complementary graphs in [17]. In [7], Chia and Lim solved the enumeration problem for the vertex transitive self-complementary digraph of order prime. In 1988, [20] Mathon constructed a self-complementary strongly regular graph of order less than 51. Moreover, several works have been done for self-complementary graphs in [1,24,26].

In this paper, we study the existence of a group whose proper power graph is self-complementary. In this context, we observe that if G is a p-group, then $P^*(G)$ is never self-complementary. For the case of an EPPO group (a group whose every non-identity element is of prime power order) G, $P^*(G)$ is never self-complementary. Moreover, we show that if G is a group having two or more distinct prime divisors, then there does not exist any finite group G except possibly $G \ncong C_n \rtimes C_{p_k^{\alpha_k}}$ with $p_k \nmid n$ and $C_n \rtimes Q_{2^r}$ with $2 \nmid n$ and Q_{2^r} such that its proper power graph is self-complementary. The self-complementary index of a graph is the largest order of an induced subgraph of that graph whose complement is also induced in that graph. Clearly, the self-complementary index of a complete graph is 1. Here we provide a sharp bound for this index for the proper power graph of any finite group other than the cyclic p-group. In the last section, we discuss the existence of the self-complementary proper power graph in the sense of different graph theoretic properties as well as in the sense of different forbidden subgraphs.

2 Self-complementary power graph and self-complementary index

A graph is called self-complementary if it is isomorphic to its complement. The order of any self-complementary graph is either 0 or 1 (mod 4). In particular, if the graph is regular, then its order is 1 (mod 4).

In this section, we investigate the existence of finite groups whose proper power graph is self-complementary. It is clear that if G is cyclic, then $P^*(G)$ is connected, but $\overline{P^*(G)}$ is disconnected. Hence, the proper power graph of a cyclic group is never a self-complementary graph. So, we look for groups that are non-cyclic and have a self-complementary proper power graph.

When a graph is not self-complementary, the natural question arises: how close is it to being a self-complementary graph? This measurement is called the self-complementary index of the graph. The self-complementary index, which is denoted by $s(G)$ of a graph G, is the maximum order of an induced subgraph of G whose complement is also an induced subgraph of G. In this section, we will discuss the self-complementary index of power graphs.

Proposition 2.1 *The proper power graph of a finite p-group is never self-complementary.*

Proof Let G be a finite p-group. If possible, let $P^*(G)$ be self-complementary. Then, its diameter is either 2 or 3. This implies $P^*(G)$ is connected, and hence G is either a cyclic group or generalized quaternion group (by Theorem 2.6 in [11]). But for both of these groups, $\overline{P^*(G)}$ is disconnected. This contradicts that $P^*(G)$ is self-complementary. Thus, there does not exist any finite p-group whose proper power graph is self-complementary. □

Proposition 2.2 *Let G be an EPPO group such that $|\pi(G)| \geq 2$, then $P^*(G)$ is not self-complementary.*

Proof Let G be an EPPO group. First, suppose that $\overline{P^*(G)}$ contains $K_{1,2}$. As $|\pi(G)| \geq 2$, $o(G)$ has at least one odd prime divisor, say p. Now let $p' \neq p$ be another prime divisor of $o(G)$. Now, let v be an element of order p' and x, y be two elements of order p such that $\langle x \rangle = \langle y \rangle$. In this case, $v \sim x$, y and $x \nsim y$ in $\overline{P^*(G)}$ which form the star $K_{1,2}$. But $P^*(G)$ does not contain such a complete bipartite subgraph since every element of G is of prime power order. Thus, any two adjacent vertices must lie in the same cyclic subgroup of prime power order. Hence, any three nonidentity vertices form K_3 or $\overline{K_3}$. This concludes the theorem. □

Theorem 2.1 *Let G be a finite group such that $o(G)$ has more than one distinct prime divisor, then $P^*(G)$ is never self-complementary except possibly $G \ncong C_n \rtimes C_{p_k^{\alpha_k}}$ with $p_k \nmid n$ and $C_n \rtimes Q_{2^r}$ with $2 \nmid n$ and Q_{2^r} is the generalized quaternion group.*

Proof If possible, let $P^*(G)$ be self-complementary. Then, by [23], the diameter of $P^*(G)$ must be either 2 or 3. Here, we consider two cases:
Case I: $diam(P^*(G)) = 2$.
As in [11], G is a nilpotent group such that each Sylow subgroup of G is either cyclic or generalized quaternion group. Thus, $G \cong C_n \times Q_{2^k}$ where $2 \nmid n$. If $G \cong C_n \times Q_{2^k}$ where $2 \nmid n$, then $\overline{P^*(G)}$ is connected. Hence, by Corollary 1.9 in [18], $\overline{P^*(G)}$ has diameter 3; thus, a contradiction arises. Therefore, in this case, there does not exist any group such that $P^*(G)$ is self-complementary.
Case II: $diam(P^*(G)) = 3$:
In this case, G is a non-nilpotent group, and for each $p_i \in \pi(G)$, G has a unique subgroup of order p_i. Again, $diam(\overline{P^*(G)}) = 3$ as $P^*(G)$ is self-complementary. Thus, by [18], only possibilities for G are $C_n \rtimes C_{p_k^{\alpha_k}}$ with $p_k \nmid n$ or $C_n \rtimes Q_{2^r}$ with $2 \nmid n$. Moreover, in this case, there are no other groups whose proper power graph is self-complementary. This completes the proof. □

Remark 2.1 Based on the above Theorem 2.1 and Proposition 2.2, it is clear that if G is a non-EPPO and $\pi(G) \geq 2$, then $P^*(G)$ is never self-complementary except possibly for the groups $C_n \rtimes C_{p_k^{\alpha_k}}$ with $p_k \nmid n$ and $C_n \rtimes Q_{2^r}$ with $2 \nmid n$, where Q_{2^r} is the generalized quaternion group. So if the groups of these forms are self-complementary, then they must have orders either $4k$ or $4k+1$. Now let us check whether the possible cases for G are by taking one example: Let $G \cong C_7 \rtimes C_3$. Then $P^*(G)$ is the disjoint union of $K_6 \cup 7K_2$. So it does not contain a claw, whereas $\overline{P^*(G)}$ contains a claw. Thus, for this group again, $P^*(G)$ is not self-complementary. So, naturally, a question arises:

Problem 2.1 *Does there exist any group G whose proper power graph $P^*(G)$ is self-complementary?*

2.1 Lower bounds for the self-complementary index of proper power graphs

In this section, we derive some lower bounds for the self-complementary index of proper power graphs. Recall that a group G is called an EPPO group if the order of each element of G is a prime power, and G is called an EPO group if the order of each element of G is a prime. Before going to the main results of this section, we recall a result from group theory.

Theorem 2.2 (Kulakoff's theorem) *Let G be a p-group of order p^α. Then*

(a) *the number of subgroups of order p^β for some β satisfying $1 < \beta \leq \alpha$ is congruent to $1 (mod\ p)$,*
(b) *if G has unique subgroup of order p^β for some β satisfying $1 < \beta \leq \alpha$, then G is cyclic or $\beta = 1$ and $p = 2$, G is the generalized quaternion group Q_{2^α}.*

Theorem 2.3 *Let G be a non-cyclic p-group, then $s(P^*(G)) \geq (p-1)$. Moreover, the equality holds if G is a p-group of exponent p.*

Proof First, we observe that $P^*(G)$ is disconnected [11], and the number of components is the number of distinct subgroups of order p. By Kulakoff's theorem, the number of distinct subgroups of order p of G is $\equiv 1(mod\ p)$, and since G is non-cyclic, it has at least $p+1$ distinct subgroups of order p. Thus, $P^*(G)$ contains K_{p-1} and $\overline{K_{p-1}}$. Hence, $s(P^*(G)) \geq p-1$.

Next, suppose that G is a p-group of exponent p. If $s(P^*(G))$ is at least p then $P^*(G)$ has a subgraph $G_1 \cup G_2$ of order at least p such that both G_1, G_2 are subgraphs of K_p and $G_1 \cup G_2$ is the disjoint. Now the complement of $G_1 \cup G_2$ is also contained in $P^*(G)$ but $\overline{G_1 \cup G_2}$ is a connected graph, and hence it lies in a single component of $P^*(G)$. As each component of $P^*(G)$ is K_{p-1} and the order of $\overline{G_1 \cup G_2}$ is at least p, it cannot be contained in $P^*(G)$. Thus $s(P^*(G)) = p-1$. □

Lemma 2.1 *Let G be a cyclic group of order pq, then $s(P^*(G)) = 3$.*

Proof Let $G \cong C_{pq}$. If possible, let $s(P^*(G)) \geq 4$. Then, $P^*(G)$ must contain some self-complementary subgraph of order 4. So $P^*(G)$ contains either $\{K_4, \overline{K_4}\}$ or $\{P_4\}$ or $\{claw, co-claw\}$ or $\{C_4, 2K_2\}$ or $\{paw, co-paw\}$ or $\{diamond, co-diamond\}$. By [19], $P(G)$ is a cograph as well as a chordal graph, so $P^*(G)$ does not contain P_4, C_4. Since every element of G is of order p, q, or pq, $P^*(G)$ does not contain $\overline{K_4}$. Next, we claim that $P^*(G)$ cannot contain co-diamond, i.e., the disjoint union $K_2 \cup \{u\} \cup \{v\}$. As $u \nsim v$, let $o(u) = p$, $o(v) = q$. But C_{pq} contains only the elements of order p, q, pq, which implies at least one endpoint of K_2 must be adjacent to any one of u, v, and therefore $P^*(G)$ cannot contain co-diamond. For a similar reason, $P^*(G)$ does not contain co-paw. Now, if possible, let it contain claws. Now, in a claw in $P^*(G)$, the vertex of degree 3 must be of order pq, and the order of a leaf is p, q, or pq. But we cannot make a claw with elements of these orders. Thus, $P^*(G)$ is claw-free. Therefore, $P^*(G)$ does not contain any of the 4-vertex graph and its complement. This implies $s(P^*(G) \leq 3$.

On the other hand, it is obvious that $P^*(G)$ contains P_3. Now as p, q both are distinct primes, let $p < q$, and choose $x \in C_{pq}$ such that $o(x) = p$, and another two vertices $a \sim b$ of order q. Thus, $P^*(G)$ contains an induced subgraph, which is the disjoint union of $K_2 \cup \{x\}$, i.e., the complement of P_3. Hence, $s(P^*(G)) \geq 3$. Hence, if $G \cong C_{pq}$, then $s(P^*(G)) = 3$. □

Theorem 2.4 *If G is a nilpotent group other than p-group, then $s(P^*(G)) \geq 3$. The equality holds if and only if $G \cong C_{pq}$.*

Proof Let p, q be two distinct primes such that $pq|o(G)$ and since G is nilpotent, it contains an element of order pq. Hence $P^*(G)$ contains P_3. Since $p \neq q$, one of them must be an odd prime, say p. Let $a, b \in G$ with $o(a) = p$ and $o(b) = q$. Then $\{a, a^{-1}\} \cup \{b\}$ forms $\overline{P_3}$ in $P^*(G)$. Hence, $s(P^*(G)) \geq 3$.

Next, let $s(P^*(G)) = 3$. Clearly, as G is nilpotent, then $o(G)$ cannot have more than two distinct prime divisors; else, $P^*(G)$ contains P_4, which implies $s(P^*(G)) \geq 4$. Thus $o(G) = p^\alpha q^\beta$. Again, by [19], if G is a group other than C_{pq}, then $P^*(G)$ contains P_4. Hence $s(P^*(G)) \geq 4$. On the other hand by Lemma 2.1 if $G \cong C_{pq}$ then $s(P^*(C_{pq})) = 3$. This concludes the theorem. □

Theorem 2.5 *Let G be a finite group that is not a p-group, then $s(P^*(G)) \geq 2$. Moreover, the equality holds if and only if G is an EPO group of order $2^k 3^s$.*

Proof Let p, q be two distinct prime divisors of $o(G)$. Let $a, b \in G$ such that $o(a) = p, o(b) = q$. Since $a \nsim b$ in $P^*(G)$, so they form $\overline{K_2}$. On the other hand, $P^*(G)$ contains K_2. Hence $s(P^*(G)) \geq 2$.

Next, let G be an EPO group of order $2^k 3^s$. Then the order of each nonidentity element of G is either 2 or 3. If possible, let $s(P^*(G)) \geq 3$, then $P^*(G)$ must contain $\{K_3, \overline{K_3}\}$ or $\{P_3, \overline{P_3}\}$. But clearly, $P^*(G)$ does not contain K_3. Next, we check for P_3 and its complement. If possible, let $P^*(G)$ contain P_3, then any 3 vertices must belong to the same cyclic p-subgroup and hence form a triangle, although this is not possible as every element has order 2 or 3. Hence $s(P^*(G)) \leq 2$ and hence $s(P^*(G)) = 2$ in this case.

On the other side, let $s(P^*(G)) = 2$. Now, if possible, let $|\pi(G)| \geq 3$. Let p, q, and r be three distinct prime divisors of $o(G)$. Then at least one of p, q, r (say r) must be ≥ 5, and let a be an element of order r. Then $P^*(G)$ contains K_3 by taking 3 element from $\langle a \rangle$. Also, let b, c be the elements in G such that $o(b) = p$, $o(c) = q$. Now $\{a, b, c\}$ forms $\overline{K_3}$. Which implies $s(P^*(G)) \geq 3$. Therefore, $o(G)$ must have exactly two distinct prime divisors. Again, using Theorem 2.4, we conclude that G is a non-nilpotent group of order $p^k q^s$. Now, if G contains an element, say x, of order pq, then again Lemma 2.1 implies $P^*(G) \geq 3$. Thus, G must be an EPPO group of order $p^k q^s$. But G cannot contain any element of order ≥ 4. In that case, $P^*(G)$ must contain K_3 and $\overline{K_3}$. This gives us that G must be an EPO group of order. This completes the proof of the theorem. □

2.2 Upper bounds for the self-complementary index of proper power graphs

To compute the upper bound, we recall the definition of the induced number of a graph. The induced number of a graph Γ, which is denoted by $m(\Gamma)$, is the minimum order of the graph that contains both Γ and its complement as induced subgraphs. In [1], Akiyama *et al.* proved that, if Γ is a graph with n vertices, then the self-complementary index and the induced number of Γ satisfy $s(\Gamma)+m(\Gamma) = 2n$. So, to provide an upper bound for the self-complementary index of a proper power graph, we obtain a suitable lower bound for the induced number of that graph. Further, we use $\omega(\Gamma)$ and $\alpha(\Gamma)$ to denote the clique number and independence number, respectively, of a graph Γ.

Lemma 2.2 *Let G be a group with n elements. Then*

$$m(P^*(G)) \geq n - 1 + |\omega(P^*(G)) - \alpha(P^*(G))|.$$

Proof Here we need to find the minimum order graph, say Γ, in which both $P^*(G)$ and its complement are embedded. Let T and T' denote the largest independent subsets of $P^*(G)$ and $\overline{P^*(G)}$, respectively. We consider two cases, as follows: **Case I:** Let $|T'| > |T|$. Now we define a one-one correspondence from each member of T to a unique member of T'. So we need to add at least $|T'| - |T|$ more vertices to make Γ. Thus, Γ must contain at least $P^*(G) + |T'| - |T|$ vertices. But the cardinality of the set T is $\alpha(P^*(G))$, and as T' is the largest independent set in $\overline{P^*(G)}$, it forms the largest clique in $P^*(G)$. Hence $|T'| = \omega(P^*(G))$. Therefore, we obtain $m(P^*(G)) \geq |P^*(G)| + \omega(P^*(G)) - \alpha(P^*(G))$.
Case II: Let $|T'| < |T|$. Then, following the same approach as in Case I, we can show that $m(P^*(G)) \geq |P^*(G)| + \alpha(P^*(G)) - \omega(P^*(G))$. Thus, by combining both cases, we obtain the required result. □

Theorem 2.6 *Let G be a group with n elements. Then*

$$s(P^*(G)) \leq n - 1 - |\alpha(P^*(G)) - \omega(P^*(G))|.$$

Moreover, equality holds if $G \cong C_{p^r}$.

Proof We already know that $s(P^*(G)) = 2|(P^*(G))| - m(P^*(G))$. Thus, using Lemma 2.2, we get the required inequality.

Next suppose that $G \cong C_{p^r}$. So $P^*(G)$ is complete, and hence $s(P^*(G)) = 1$. Thus, $\alpha(P^*(G)) = 1$ and $\omega(P^*(G)) = p^r - 1$. So equality holds in this case. □

3 Regularity and unicyclicity

Theorem 3.1 *Let G be a finite group of order n, then $\overline{P^*(G)}$ is regular if and only if G is either C_{p^k} or a p-group of exponent p.*

Proof Let $\overline{P^*(G)}$ be regular. If possible, let $o(G)$ have two distinct prime divisors, say p, q. Let $a, b \in G$ such that $o(a) = p, o(b) = q$. Now $deg(a) = n - \frac{p-2}{2}$ and $deg(b) = n - \frac{q-2}{2}$. Since $\overline{P^*(G)}$ is regular, $deg(a) = deg(b)$, and hence $p = q$. Therefore, G is a p-group. Again, if G is non-cyclic, then G must be a p-group of exponent p; otherwise, if G has an element, say a, of order p^k and another element b of order p with $a \nsim b$. Then $deg(a) = n - \frac{p^k + p^{k-1} - 3}{2}$ and $deg(b) = n - \frac{p-2}{2}$. As $\overline{P^*(G)}$ is regular, we must have $k = 1$. This implies that G is a p-group of exponent p. Conversely, if $G \cong C_{p^k}$, then $\overline{P^*(G)}$ is an edgeless graph, so it is regular. Again if G is a non-cyclic p-group of exponent p, then the degree of each vertex is $n - \frac{p-2}{2}$. Thus, $\overline{P^*(G)}$ is regular. □

Corollary 3.1 *For any finite group G, $P^*(G)$ is self-complementary in the sense of regularity if and only if G is either C_{p^k} or a p-group of exponent p.*

Theorem 3.2 *If G is a finite group with $\overline{P^*(G)}$ being strongly regular, then G is a non-cyclic p-group of exponent p, and vice versa.*

Proof If $\overline{P^*(G)}$ is strongly regular, then it is regular, and so G is either C_{p^k} or a p-group of exponent p. But if G is cyclic group C_{p^k} then $\overline{P^*(G)}$ is trivial group so it is not strongly regular. But if G is a non-cyclic p-group with exponent p and a regular graph for each vertices, the number of neighbours is the same and the number of non-neighbours is equal to $\frac{p-2}{2}$. Hence, $\overline{P^*(G)}$ is strongly regular. □

A connected graph is called unicyclic if it contains exactly one cycle. Therefore, a connected unicyclic graph is a pseudotree, which is not a tree. "Unicyclic graphs except the cycles C_{4n+1} and C_{4n+2} are graceful"; this was a conjecture by Truszczynski [25], which was proved very recently by Gallian in [12]. In this section, we study the groups for which both $P^*(G)$ and $\overline{P^*(G)}$ are unicyclic.

Lemma 3.1 *Let G be a finite group. Then the proper power graph of G is unicyclic if and only if G is isomorphic to one of the following:*

(a) *the cyclic group* C_4.
(b) *the dihedral group* D_4.
(c) *an EPPO-group of order* 2^r3^s *such that every non-identity element of G is of order* 2, 3, *or* 4, *with Sylow 2-subgroup is normal and isomorphic to* C_4 *or* D_4.

Proof Let $P^*(G)$ be unicyclic. First, we observe that G cannot contain any element of order 5 and above, because in this case, $P^*(G)$ contains at least two K_3. Thus, $|G|$ is either 2^a, 3^b, or 2^m3^n.
Case I. If $|G| = 3^b$, then it is clear that G must be a 3-group of exponent 3. But for this group, $P^*(G)$ is acyclic. So we can discard this case.
Case II. Let $|G| = 2^a$. In this case, the order of an element in G cannot exceed 4, and G cannot have two distinct cyclic subgroups of order 4. Moreover, $|G| \leq 8$, because if $|G| \geq 16$, then G must contain an element of order 8 or two distinct subgroups of order 4. So either $|G| = 4$ or $|G| = 8$.

If $|G| = 4$, then $G \cong C_4$. Again, if $|G| = 8$, then $G \cong D_4$ (all other groups of order 8 have either an element of order 8 or two distinct subgroups of order 4).
Case III. Let us take $|G| = 2^m3^n$. Firstly, G does not have an element of order 6 or more. So, every element of G has order either 2, 4, or 3, and hence G is an EPPO group. Now the elements of order 3 are not a part of any cycle. So the unique cycle must come from a Sylow 2-subgroup. Following the arguments as in Case II, we conclude that the Sylow 2-subgroup is isomorphic to either C_4 or D_4. Beside the Sylow 2-subgroup must be unique; otherwise, any two Sylow 2-subgroups are isomorphic, so in that case we get at least 2 cycles in $P^*(G)$. This gives the group the required structure.

Conversely, if $G \cong C_4$ or D_4, then the result holds trivially. We now assume that G is an EPPO group of order 2^m3^n satisfying the stated conditions. As Sylow 3-subgroup is a 3-group of exponent 3, none of its elements is a part of a cycle. On the other hand, since the unique Sylow 2-subgroup of G is either C_4 or D_4. Thus, there exists only one cycle in C_4 or D_4. Hence, in this case, $P^*(G)$ is a unicyclic graph. □

Theorem 3.3 *There does not exist any finite group G such that both* $P^*(G)$ *and* $\overline{P^*(G)}$ *are unicyclic.*

Proof Suppose both $P^*(G)$ and $\overline{P^*(G)}$ are unicyclic. Then, by Lemma 3.1, G is either C_4 or D_4 or an EPPO group of order 2^r3^s such that every element of G is of order 2, 4, 3 with Sylow 2-subgroup normal and isomorphic to C_4 or D_4.

It is clear that if $G \cong C_4$, then $\overline{P^*(G)}$ is acyclic, and if $G \cong C_4$, then $\overline{P^*(G)}$ contains K_4. So next, we consider the final case. However, in that case, the elements of C_4 and C_3 are isolated in $\overline{P^*(G)}$, and in $\overline{P^*(G)}$, the elements of order 3 and 4 are adjacent, and together they form cycle C_4. Still, if we choose a, b such that $o(a) = 4$, $o(b) = 3$, then $\overline{P^*(G)}$ contains at least 3 C_4 as (a, b, a^2, b^2, a), (a, b, a^3, b^2, a), and (a^2, b, a^3, b^2, a). Hence, this completes the proof of the theorem. □

4 2-connectedness

A connected graph G is called 2-connected if it is still connected after deleting a single vertex of the graph G. If a graph is 2-connected, then any two distinct vertices of the graph lie in a cycle. In the current section, we will explore those nilpotent groups for which $P^*(G)$ is self-complementary in the sense of 2-connectedness. Clearly, as $\overline{P^*(G)}$ is disconnected for a cyclic group G, we are interested in the non-cyclic groups.

Proposition 4.1 *Let G be a finite p-group such that p is an odd prime. Then $\overline{P^*(G)}$ is 2-connected if and only if G is non-cyclic.*

Proof Let $\overline{P^*(G)}$ be 2-connected. Then G is a non-cyclic group.

Now let G be non-cyclic and let a, b be two distinct elements of G. If a, b are not elements of the same cyclic subgroup, then in $\overline{P^*(G)}$, the vertices a and b are contained in the 4-cycle $a \sim b \sim a^2 \sim b^2 \sim a$. But if $a \nsim b$, then either $b \in \langle a \rangle$ or $a \in \langle b \rangle$. Without loss of generality, we assume that $b \in \langle a \rangle$. Then there exists at least one $c \in G$ such that a, c generate distinct cyclic subgroups, and in this case, $a \sim c \sim b \sim c^2 \sim a$ is the cycle in $\overline{P^*(G)}$. Thus, $P^*(G)$ is 2-connected. □

Proposition 4.2 *Let G be a group of order power of 2. Then $\overline{P^*(G)}$ is 2-connected if and only if G is neither a cyclic group nor a generalized quaternion group.*

Proof First, we observe that $\overline{P^*(G)}$ cannot have any isolated vertices. So, the minimal subgroups of order 2 should not be unique, i.e., G cannot have a unique element of order 2; otherwise, this unique element will be contained in all other subgroups, which makes this vertex isolated in $\overline{P^*(G)}$. Let $x, y \in G$. Here, we consider two cases:

Case I: Let $\{x, y\}$ be an edge in $\overline{P^*(G)}$. We consider following subcases:

Subcase a. If both has order greater than 2, then there exists a cycle $x \sim y \sim x^2 \sim y^2 \sim x$.

Subcase b. If $o(x) = 2$ and $o(y)$ is at least 4, then by Theorem 2.2 there exists another element $c \in G$ of order 2 and hence we get a cycle with vertices x, y, c, x.

Subcase c. If both $o(x) = o(y) = 2$, then by Theorem 2.2 there exists $c \in G$ such that $o(c) = 2$ and, then the vertices x, y, c, x forms a cycle.

Case II: Let $x \nsim y$ in $\overline{P^*(G)}$. Then either $\langle x \rangle$ contained in $\langle y \rangle$ or the reverse. Now if both x and y have order 4 and above, then since the minimal subgroup is not unique, by Theorem 2.2 there exist at least two elements of order 2, say a, b. Then x, y lie on the cycle $x \sim a \sim y \sim b \sim x$. Next, suppose that $o(x) = 2$, then $\langle x \rangle$ lies in $\langle y \rangle$. Using the similar argument as above, we get that $x \sim a \sim y \sim b \sim x$.

The converse part is clear, as in both cases $\overline{P^*(G)}$ has an isolated vertex, so $\overline{P^*(G)}$ is disconnected in both cases. □

Corollary 4.1 *There does not exist any p-group G such that both $P^*(G)$ and $\overline{P^*(G)}$ are 2-connected.*

Proof We observe that if $\overline{P^*(G)}$ is 2-connected, which implies that G is a non-cyclic group other than a generalized quaternion group. But $P^*(G)$ is disconnected for these groups. Hence, we get the stated conclusion. □

Theorem 4.1 *Let G be a nilpotent group other than a p-group. Then $\overline{P^*(G)}$ is 2-connected if and only if G is non-cyclic.*

Proof Let G be non-cyclic, and let $p_1, p_2, \cdots, p_r$ be distinct prime divisors of $|G|$, and $P_1, P_2, \cdots, P_r$ be corresponding Sylow subgroups of G. Then $G \cong P_1 \times P_2 \times \cdots \times P_r$ with at least one Sylow subgroup is non-cyclic. Let P_i be non-cyclic. Now let $x \in G$ such that $o(x) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, $\alpha_i \geq 0$, and $y \in G$. We now consider two cases::

Case I. Suppose there exists some prime p_j that does not divide $o(y)$; but p_j divides $o(x)$.

If $x \sim y$ in $\overline{P^*(G)}$, then $x \sim y \sim x$ is a cycle containing both x, y.

On the other hand, if $x \nsim y$, then $\langle y \rangle \subseteq \langle x \rangle$ or reverse. Let us choose $\langle y \rangle \subseteq \langle x \rangle$. Now there exists some $a \in P_i$ with order p_i such that $\langle a \rangle$ does not belong to $\langle x \rangle$. Then x, y lie on the cycle $x \sim a \sim y \sim a^k \sim x$ for $p_i \geq 3$.

If $p_i = 2$, as P_i is non-cyclic, there exists some $t \in P_i$ such that $t \sim x$ in $\overline{P^*(G)}$. Now, by Kulakoff's theorem, we get another element $t_1 \in P_i$ such that x, y lie on $x \sim t \sim z \sim y \sim t_1 \sim x$, where $z \in P_j$.

Case II. Let both $o(x)$ and $o(y)$ have the same prime divisors. Let $o(y) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$. Similarly, if $x \sim y$ in $\overline{P^*(G)}$, we get x, y lie on the cycle. Now consider $x \nsim y$. In this case, just as in the previous case, we get that x, y lie either on $x \sim a \sim y \sim a^k \sim x$ where $a \in P_i$ and $p_i \geq 3$, or using the Kulakoff'theorem,em we can say that x, y lie on $x \sim a \sim y \sim b \sim x$ such that $a, b \in P_i$. □

Theorem 4.2 *Let G be a nilpotent group other than a p-group. Then $P^*(G)$ is always 2-connected.*

Proof Let P_i be the Sylow p_i-subgroups of G for $1 \leq i \leq k$ and $G \cong P_1 \times P_2 \times \cdots \times P_k$. Let $x, y \in G$ such that $o(x) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, $\alpha_i \geq 0$. We consider two cases here:

Case I. Let there exist some prime p_i such that p_i does not divide $o(y)$, but p_i divides $o(x)$.

If $x \sim y$, then $\langle y \rangle \subseteq \langle x \rangle$ and x, y lie on x, y, x. Let $x \nsim y$, then set $o(x) = p_i^{\alpha_i} l$. As G is nilpotent and $gcd(x^l, y) = 1$, so $[x^l, y] = 1$.

Now if $o(x), o(y)$ has a common factor, say p_j, with $o(z) = p_j$ and $\langle z \rangle$ contained in both $\langle x \rangle$ and $\langle y \rangle$, then x and y are contained in the cycle $x \sim x^l \sim x^l y \sim y \sim z \sim x$.

On the other hand, if $o(x)$ and $o(y)$ have no common factor, then $[x, y] = 1$ as G is nilpotent. Then x, y belong to the cycle $x \sim x^t \sim x^t y^l \sim y^l \sim y \sim x^{t-1} y \sim x^{t-1} \sim x$.

Case II. Let $o(x), o(y)$ have the same prime divisor. Then $o(y) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$. Clearly, $x \sim y$ implies both lie on a cycle.

Next, let $x \nsim y$. We consider four elements as $p = x^{p_2^{\alpha_2} \cdots p_k^{\alpha_k}}$, $q = y^{p_2^{\beta_2} \cdots p_k^{\beta_k}}$, $m = x^{p_1^{\alpha_1} p_3^{\alpha_3} \cdots p_k^{\alpha_k}}$ and $n = y^{p_1^{\beta_1} p_3^{\beta_3} \cdots p_k^{\beta_k}}$. Then x and y are contained in the cycle $x \sim p \sim pl \sim l \sim ql \sim q \sim y \sim n \sim ns \sim s \sim ms \sim m \sim x$ such that $o(l) = p_2$ and $o(s) = p_1$. □

Corollary 4.2 *Let G be a nilpotent group other than p-group. Then both $P^*(G)$ and $\overline{P^*(G)}$ are 2-connected if and only if G is non-cyclic.*

5 Forbidden induced subgraphs

In this section, we discuss some forbidden induced subgraphs of proper power graphs of finite groups and their complements. We know that perfect graphs, cographs, and threshold graphs are complementary; further perfect graphs and cographs are closed under taking induced subgraphs. Therefore, $P^*(G)$ is always complementary in the sense of perfectness. On the other hand, both $P^*(G)$ and $\overline{P^*(G)}$ are cographs if and only if $P(G)$ is a cograph. Previously, we gave some nice characterizations of groups whose power graph is a cograph; see [3,19]. Here we are interested in identifying the groups G for which both $P^*(G)$ and its complements belong to certain classes of forbidden subgraphs, such as bipartite graphs, claw-free graphs, line graphs, paw-free graphs, balanced graphs, etc.

Theorem 5.1 *Let G be a finite group, then $\overline{P^*(G)}$ is K_3-free if and only if $G \cong C_{p^k}$ or $C_{p^a q}$ where $k, r \geq 1$.*

Proof Let $\overline{P^*(G)}$ be K_3-free. If $|\pi(G)| \geq 3$, then clearly $\overline{P^*(G)}$ contains K_3. Next, let $|\pi(G)| = 1$, i.e., G be a p-group. Suppose G is neither cyclic nor generalized quaternion group. Then G contains at least 3 distinct elements of order p that are non-adjacent in $P^*(G)$; so K_3 must be an induced subgraph of $\overline{P^*(G)}$. Again, if G is generalized quaternion, then we can also find 3 distinct elements of order 4 such that they form K_3 in $\overline{P^*(G)}$. Thus, in any aspect we obtained, G must be cyclic. Now if G is C_{p^k}, then $\overline{P^*(G)}$ is null graph so is K_3-free.

Suppose $|\pi(G)| = 2$ and $|G| = p^a q^b$, where $p, q \in \pi(G)$. Firstly, let both $p$and q are odd prime. We claim that both the Sylow subgroups of G are cyclic and normal; otherwise, there exist two elements a, b say of order p (or q) such that $a \nsim b$ in $P^*(G)$ and c of order q (or p) such that together they form K_3. Therefore, $G \cong C_{p^a q^b}$. Next, we claim that either $a = 1$ or $b = 1$. Suppose both $a, b > 1$, then we get two elements a, b of order p^2 and q^2, respectively, and $o(a^p b^q) = pq$. But they form K_3 in $\overline{P^*(G)}$. Thus, $G \cong C_{p^r q}$ with $r \geq 1$.
Next let $|G| = 2^a q^b$. Clearly, the Sylow q-subgroup of G must be cyclic and normal. Suppose, the Sylow 2-subgroup of G is either non-cyclic or not normal. If the Sylow 2-subgroup of G is generalized quaternion group then G has at least two elements, say x, y, of order 4 such that $x \nsim y$ in $P^*(G)$. Since G contains an element say c of order q so the vertices $\{a, b, c\}$ form K_3 in $\overline{P^*(G)}$. Again when the Sylow 2-subgroup of G is not generalized quaternion, then G has at least two elements of order 2 that are non-adjacent in $P^*(G)$. Thus by following the same argument we can conclude that $\overline{P^*(G)}$ contains K_3. Hence $G \cong C_{2^a q^b}$. From the argument in p, q odd we obtain $G \cong C_{2^a q}$ or C_{2q^b}. Therefore, if $|G| = p^a q^b$ then $G \cong C_{p^r q}$ with $r \geq 1$. Now we prove that $\overline{P^*(G)}$ is triangle-free. If possible, let x, y, z be three vertices of a triangle in $\overline{P^*(G)}$. Then the order of none of x, y, z can divide the orders of the other two, but no such combinations are possible from the divisors of $p^r q$. Thus, $\overline{P^*(G)}$ is K_3-free. This completes the proof of the theorem. □

Corollary 5.1 *Let G be a finite group. Then $\overline{P^*(G)}$ is bipartite if and only if $G \cong C_{p^k}$ or $C_{p^a q}$.*

Proof Let $\overline{P^*(G)}$ be bipartite. Then $\overline{P^*(G)}$ does not contain any odd cycle. Since for any group G, $P^*(G)$ is perfect (see [19]), and the complement of a perfect graph is again perfect (by the Weak Perfect Graph Theorem). So $\overline{P^*(G)}$ is perfect. Thus, it does not contain any odd cycle of length 5 and above. Now $\overline{P^*(G)}$ is K_3-free also. This gives $G \cong C_{p^k}$ or C_{p^aq}.

Conversely, let G takes one of the given forms. We now show that $\overline{P^*(G)}$ is bipartite. If G is cyclic p-group C_{p^k}, then $\overline{P^*(G)}$ is the null graph $\overline{K}_{p^k-1}$, so it is bipartite.

If G is C_{p^aq}, then by Theorem 5.1, the graph $\overline{P^*(G)}$ is K_3-free, and by the perfectness of $\overline{P^*(G)}$ we conclude that it does not contain any odd cycle of length more than 3. Thus, in this case, $\overline{P^*(G)}$ is odd cycle-free, and hence $\overline{P^*(G)}$ is bipartite. □

Corollary 5.2 *Let G be a finite group, then $\overline{P^*(G)}$ is never a tree.*

In our next result, we identify proper power graphs that are complementary in the sense of bipartiteness. Before that, we recall a result by Simić:

Theorem 5.2 [21] *Both Γ and $\overline{\Gamma}$ are bipartite if and only if either of them is an induced subgraph of P_4 or $2K_2$.*

Theorem 5.3 *For any finite group G, both $P^*(G)$ is self-complementary in the sense of bipartiteness if and only if $G \cong C_2$ or C_3.*

Proof Let G be a finite group, and let $P^*(G)$ be complementary in the sense of bipartiteness. By Theorem 5.2, either $P^*(G)$ or its complement is an induced subgraph of P_4 or $2K_2$. So $P^*(G)$ is isomorphic to P_3, K_2, $\overline{K_2}$, K_1, or $K_1 \cup K_2$. Clearly, neither $P^*(G)$ nor $\overline{P^*(G)}$ is isomorphic to P_3. Again, neither $P^*(G)$ nor $\overline{P^*(G)}$ is isomorphic to $K_1 \cup K_2$.

Now $P^*(G)$ is K_2 if and only if $G \cong C_3$, and in that case, $\overline{P^*(G)}$ is $\overline{K_2}$. On the other hand, both $P^*(G)$ and $\overline{P^*(G)}$ are K_1 if and only if $G \cong C_2$. The converse part is obvious. This completes the proof. □

A graph is called balanced if the length of each induced cycle is a multiple of 4. Balanced graphs are bipartite. Our next theorem will show when $P^*(G)$ and $\overline{P^*(G)}$ are balanced.

Proposition 5.1 *Let G be a finite group. Then $\overline{P^*(G)}$ is balanced if and only if $\overline{P^*(G)}$ is bipartite.*

Proof First suppose $\overline{P^*(G)}$ is balanced. Then $\overline{P^*(G)}$ is bipartite.

Conversely, let $\overline{P^*(G)}$ be bipartite. Then, by Corollary 5.1, G is either cyclic p-group or C_{p^aq}.

If G is a cyclic p-group, then $\overline{P^*(G)}$ is a null graph, so it is balanced.

Let $G \cong C_{p^aq}$. We show that $\overline{P^*(G)}$ contains every induced cycle of length exactly 4. If possible, let $\overline{P^*(G)}$ have an even induced cycle of length at least 6. Then the order of at least one vertex in that cycle is a power of p. Let a, b, c, and d be four consecutive vertices such that $o(a) = p^k$. Then $o(b) = q$ or p^mq. If $o(b) = q$, then $o(c)$ must be a power of p, and hence $o(d) = q$ or p^nq, then $a \sim d$. Similarly, if $o(b) = p^mq$, then we get $a \sim d$. Thus, $\overline{P^*(G)}$ does not contain any induced cycle of length greater than 5. Hence, $\overline{P^*(G)}$ is balanced. □

Using Theorems 5.3 and 5.1, we can prove the following result.

Theorem 5.4 *If G is a finite group, then both $P^*(G)$ and $\overline{P^*(G)}$ are balanced if and only if $G \cong C_2$ or C_3.*

We now discuss about P_3-free proper power graphs. We start with the following lemma.

Lemma 5.1 *Let G be a finite group. Then $\overline{P^*(G)}$ is P_3-free if and only if $G \cong C_{p^k}$ or $(C_2)^r$.*

Proof Let $\overline{P^*(G)}$ be P_3-free. Let p, q be two distinct prime divisors of $|G|$ and a, b be elements of order p, q respectively. Then $a \sim b \sim a^2 \nsim a$ is a path of length 2 in $\overline{P^*(G)}$. Therefore G must be a p-group.

First we assume that p be an odd prime. Now if G is non-cyclic, then G has at least two elements say a, b of order ≥ 3 such that they form distinct cyclic subgroups. Therefore $\overline{P^*(G)}$ has a path (a, b, a^2). So G must be a cyclic p-group and in that case $\overline{P^*(G)}$ is P_3-free.

Now consider $p = 2$. If G is cyclic, then $\overline{P^*(G)}$ is P_3-free. If G is non-cyclic, then clearly G can never have an element of order ≥ 4. So $G \cong (C_2)^r$ and in that case $\overline{P^*(G)}$ does not contain P_3. □

Lemma 5.2 *For any finite group G, the proper power graph of G is P_3-free if and only if G is either a cyclic p-group or an EPPO group whose all p-subgroups are either cyclic or of prime exponent.*

Proof Let $P^*(G)$ be P_3-free. If G be a cyclic p-group, then $P^*(G)$ is P_3-free. Next, suppose that G be a non-cyclic p-group. If G has an element of order p^2, then G must contain two cyclic subgroups of order p^2 (by Kulakoff's theorem). Let $H = \langle a \rangle$ and $K = \langle b \rangle$ be two distinct cyclic subgroups of G of order p^2. Then $H \cap K \neq \phi$. So in that case, $a \sim a^p = b^p \sim b$ is a path of length 2. So G is a non-cyclic p-group in which every element has order p.

Next, suppose that $|G|$ has two or more distinct prime divisors. Let p and q be two distinct prime divisors of $|G|$. If G contains an element of order pq, say a, then $P^*(G)$ contains a 3 vertex path (a^p, a, a^q). Thus, G must be an EPPO group. Now if G is an EPPO group, then $P^*(G)$ is a disjoint union of proper power graphs of groups of prime power order. So, from the previous argument, we get the required conclusion.

The converse part is clear. □

Theorem 5.5 *If G is any finite group, then both $P^*(G)$ and $\overline{P^*(G)}$ are P_3-free if and only if $G \cong C_{p^k}, (C_2)^r$.*

Proof The theorem follows by Lemmas 5.1 and 5.2. □

The complete bipartite graph $K_{1,3}$ is called the claw, and the four-vertex graph obtained by joining an isolated vertex to one vertex of K_3 is called the paw. Let Γ be a graph with a claw, i.e., Γ has an induced subgraph isomorphic to $K_{1,3}$. Then clearly, the four vertices of the claw form $K_3 \cup \{v\}$ in $\overline{\Gamma}$. Thus, a graph is claw-free if and only if its complement is claw-free. By a similar argument, we conclude that a graph is paw-free if and only if its complement is $K_{1,2} \cup \{v\}$-free. In the next few theorems, we search for finite groups G for which both $P^*(G)$ and $\overline{P^*(G)}$ are either claw-free or paw-free.

Theorem 5.6 *If G is any finite group, then $\overline{P^*(G)}$ is claw-free if and only if G is one of the following group:*

(a) *C_{p^k}, C_6, elementary abelian 2-group, quaternion group Q_8;*
(b) *a 3-group of exponent 3;*
(c) *an EPO group (a group whose each non-identity elements are of prime order) whose order is $2^r 3^s$, where r, s are positive integers.*

Proof First, we consider that $\overline{P^*(G)}$ is claw-free. Then $P^*(G)$ is $K_3 \cup \{v\}$-free. Now if $|\pi(G)| \geq 3$ and p, q, and r be distinct elements of $\pi(G)$. Then one of p, q, or r must be of at least 5. Let $r \geq 5$, and there exists $a \in G$ such that $o(a) = r$. Now G contains an element, say b, of order p. Then $P^*(G)$ contains $K_3 \cup \{v\}$ with vertices a, a^2, a^3 as K_3 together with an isolated vertex b. So, $|\pi(G)| \leq 2$.

If $|\pi(G)| = 1$, then G is a p-group. If G is cyclic, then the conclusion is obvious. But if G is non-cyclic, then p cannot be more than 3; elsewhere, there exist two non-adjacent elements of order p such that $P^*(G)$ contains $K_3 \cup \{v\}$. Thus, p is either 2 or 3. If G is a 3-group, then, using a similar argument, we can conclude that G must be a 3-group of exponent 3.

Next, consider G as a 2-group such that G has an element of order 4; if G has two or more elements of order 2, then $P^*(G)$ will comprise the structure $K_3 \cup \{v\}$. So, the minimal cyclic subgroup must be unique in this case, and hence, as G is non-cyclic, G is possibly the generalized quaternion group. Note that G cannot have an element of order more than 4 because such an element generates K_3 and there exists another element $b \nsim a$ in $P^*(G)$, so b together with K_3 is contained in $P^*(G)$. Thus, $G \cong Q_8$.

On the other hand if G has all elements of order 2, then G is elementary abelian 2-group.

Next, let $|\pi(G)| = 2$ and $|G| = p^r q^s$. Now, if any one of p or q is 5 and more, then by a similar argument as above, we can show that $P^*(G)$ contains $K_3 \cup \{v\}$. Thus, $|G| = 2^r 3^s$. We observe that if G is cyclic, then the only possibility for G is C_6. Next, suppose that G is not cyclic. In that case, if G contains an element of order ≥ 4, then $P^*(G)$ encloses $K_3 \cup \{v\}$. Hence, G is an EPPO group of order $2^r 3^s$, having all elements of order either 2 or 3.

For the converse, if G is any one of the groups in (a), (b), or (c), then $P^*(G)$ is K_3-free. Thus, $P^*(G)$ is $K_3 \cup \{v\}$-free, and hence $\overline{P^*(G)}$ is claw-free. □

Corollary 5.3 *For any finite group G, both $P^*(G)$ and $\overline{P^*(G)}$ are claw-free if and only if G is any one of the following:*

(a) *C_{p^k}, elementary abelian 2-group;*
(b) *a 3-group of exponent 3;*
(c) *an EPO group of order is $2^r 3^s$ where r, s are positive integers.*

Proof Let both $P^*(G)$ and $\overline{P^*(G)}$ be claw-free. Since $\overline{P^*(G)}$ is claw-free, G must be one of the three forms of Theorem 5.6.

If G is the quaternion group $Q_8 = \langle a, b | a^4 = 1, a^2 = b^2, ab = ba^{-1} \rangle$, then $P^*(G)$ contains $K_{1,3}$ with the vertices a, b, a^2, ab. So $G \ncong Q_8$.

If G is either C_{p^k} or an elementary abelian 2-group, then the proof is obvious. So statement (a) holds.

For a 3 group of exponent 3, as the elements belong to the same cyclic subgroup C_3, they will be adjacent vertex in $P^*(G)$, and hence $P^*(G)$ never contains a claw. So (b) holds.

Similarly for (c), if $P^*(G)$ contains a claw, then as the group is EPO, so all the vertices of a claw belong to the same cyclic subgroup. But the cyclic subgroups of G are either C_2 or C_3. Therefore, $P^*(G)$ fails to enclose a claw. This completes the proof. □

Theorem 5.7 *Let G be a finite group. Then $\overline{P^*(G)}$ is paw-free if and only if G is either $C_{p^m q}$ or $C_{p^2q^2}$ or C_{p^k} or an EPO group or a generalized quaternion group.*

Proof Let $\overline{P^*(G)}$ be paw-free. Then $P^*(G)$ is $K_{1,2} \cup \{v\}$-free. Here we consider two cases:

Case I. First, suppose that $|G|$ has only one prime divisor, say p. If G is cyclic, then clearly $P^*(G)$ is $K_{1,2}$-free and hence $K_{1,2} \cup \{v\}$-free.

Next, suppose G is non-cyclic. We observe that G cannot have an element of order power of p. If G has an element say a of order p^2, then G has at least two subgroups of order p^2 say $H = \langle a \rangle$ and $K = \langle b \rangle$ with $a^p = b^p$. If G is other than generalized quaternion group, then there exists another element c of order p such that c neither belongs to $\langle a^p \rangle$ nor $\langle b^p \rangle$. Therefore, $P^*(G)$ contains $K_{1,2} \cup \{c\}$, where $K_{1,2}$ forms by the vertices $a, a^p = b^p, b$. Therefore, G is a p-group of exponent p, or an EPO group. If G is generalized quaternion group, then there exists at least three distinct cyclic subgroups of order 4, say $H = \langle a \rangle$, $K = \langle ab \rangle$, and $L = \langle a^2 b \rangle$, such that the unique involution element is the only common element among all the subgroups. Now if $P^*(G)$ contains $K_{1,2} \cup \{c\}$, then as the involution element must be present in $K_{1,2}$, this means c must be adjacent to the involution element. Thus, $P^*(G)$ is $K_{1,2} \cup \{c\}$-free.

Case II. Next, let $|G|$ have two or more distinct prime divisors. In this case, if G is an EPPO group, just as in argument in Case I, we can say that G must be an EPO-group.

On the other hand, if G is not EPPO, we observe that $|G|$ cannot have three distinct prime divisors, say p, q, r; elsewhere, there exist elements a, b, c, d such that $o(a) = pq$, $o(b) = p$, $o(c) = q$, and $o(d) = r$, then $P^*(G)$ encloses $K_{1,2} \cup \{v\}$ where $v = d$. Thus, $|G|$ can have at most 2 distinct prime divisors.

Let $|G|$ have two distinct prime divisors, say p, q. Let both p, q be odd. Now if any of the Sylow subgroups is non-cyclic, then $P^*(G)$ contains $K_{1,2} \cup \{v\}$ as an induced subgraph, so both Sylow subgroups must be cyclic. For similar reasons, they are also normal. Thus, $G \cong C_{p^m q^n}$. Clearly, both m, n cannot be greater than 2, then $P^*(G)$ contains $K_{1,2} \cup \{v\}$ where $o(v) = p^3$ and the vertices of $K_{1,2}$ are of orders p^2q, pq^2 and p^2q^2, respectively. Thus, in this case, G is either $C_{p^m q}$ or $C_{p^2q^2}$. Let $p = 2$ and q odd. Clearly the Sylow q-subgroup is cyclic and normal. But if the Sylow 2 subgroup is neither cyclic nor generalized quaternion then one can check that $P^*(G)$ contains $K_{1,2} \cup \{v\}$. Again if the Sylow 2 subgroup is generalized quaternion then it has two distinct elements of order 4 that are non-adjacent in $P^*(G)$ and as G is non-EPPO group so $P^*(G)$ contains an induced subgraph $K_{1,2} \cup \{v\}$. Hence $G \cong C_{2^m q^n}$. By following the similar argument as done in p, q odd case we have G is one of the groups $C_{2^m q}$ or C_{4q^2}.

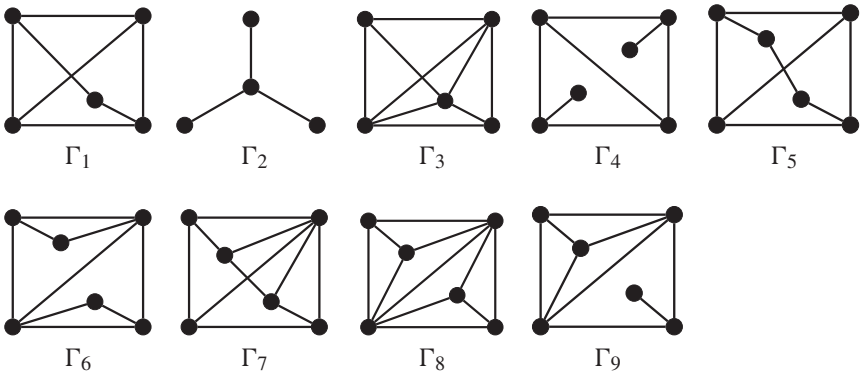
Conversely, if G is C_{p^k} or an EPO group except generalized quaternion group, then $P^*(G)$ is $K_{1,2}$-free; if $G \cong C_{p^m q}, C_{p^2q^2}$ or a generalized quaternion group, then $P^*(G)$ is $K_{1,2} \cup v\}$-free, so by Lemma 5.2, $\overline{P^*(G)}$ is paw-free. □

Corollary 5.4 *For any finite group G, both $P^*(G)$ and $\overline{P^*(G)}$ are paw-free if and only if G is either an EPPO group or $C_{p^m q}$ or $C_{p^2q^2}$ or C_{p^k} or a non-cyclic p-group.*

Proof Proof is similar as Corollary 5.3. □

Let Γ be a graph. Then the line graph of Γ, denoted by $L(\Gamma)$, is another graph whose vertices are the edges of Γ and two vertices in $L(\Gamma)$ are adjacent if and only if the corresponding edges in Γ have a vertex in common. Here we try to identify when the proper power graph of a finite group is complementary in the sense of line graph of some other graph. First we recall a well known theorem.

Theorem 5.8 [15] *A graph* Γ *is the line graph of some graph if and only if if* Γ *does not have any one of the following* 9 *Beineke graphs as its induced subgraph.*



Theorem 5.9 *If G is any finite group, then* $\overline{P^*(G)}$ *is a line graph of some graph if and only if G is one of the group*

(a) C_{p^k};
(b) C_6;
(c) *an elementary abelian* 2*-group.*

Moreover, $\overline{P^*(C_{p^k})} = L((p^k - 1)K_2)$, $\overline{P^*(C_6)} = L(P_4 \cup 2K_2)$ *and* $\overline{P^*((C_2)^r)} = L(K_{1,2^r-1})$.

Proof Let $\overline{P^*(G)}$ be a line graph of a graph Γ. Since line graphs forbid claws, first we discard those groups G such that $\overline{P^*(G)}$ contains a claw. By Theorem 5.6, the possibilities for G are: C_{p^k}, C_6, $(C_2)^r$, Q_8, a 3 group of exponent 3 and a non-nilpotent group of order $2^m 3^n$ such that each non-identity element is of order 2 or 3.

If $G \cong S_3$, choosing elements $a = (123), b = (12), c = (132), d = (23), e = (13)$, we get that $\overline{P^*(G)}$ contains the graph Γ_3. Similarly, we can discard the case when $G \cong Q_8$.

Now, consider that G is a non-cyclic 3-group of exponent 3; then, by Theorem 2.2, G has at least 4 subgroups of order 3. Again, if we choose such 4 elements, namely x, y, z, w, from these 4 subgroups, then the vertices x, y, x^2, z, w form an induced Γ_3 in $\overline{P^*(G)}$.

Next, let us consider that G is a group of order $2^r 3^s$, which is non-nilpotent, and every non-identity element is of order 2 or 3. Here we claim that G contains a unique subgroup of order 3. If possible, let G have two distinct subgroups of order 3, and let x, y be the generators of such cyclic subgroups. Let z be an element of order 2 in G. Then the elements x, z, x^2, y, y^2 form an induced Γ_1 in $\overline{P^*(G)}$. Again, we observe that the number of 2-subgroups is at most two; otherwise, $\overline{P^*(G)}$ contains Γ_2. But by Theorem 2.2, G has at least 3 such subgroups as G is non-cyclic. So in this case also, we get a contradiction that $\overline{P^*(G)}$ is a line graph. Then the remaining possibilities for G are: C_{p^k}, C_6, and an elementary abelian 2-group. To complete the proof of the theorem, we show that for each of these groups, $\overline{P^*(G)}$ is a line graph.

If $G \cong C_{p^k}$, then $\overline{P^*(G)}$ is a null graph, and if $G \cong (C_2)^r$, then $\overline{P^*(G)}$ is a complete graph. Hence, it does not contain any one of the nine Beineke graphs as an induced subgraph. In that case, $\overline{P^*(G)}$ is the line graph of the disjoint union of $p^k - 1$ copies of K_2.

If G is C_6, then $\overline{P^*(G)}$ is the path P_3 union two isolated vertices, so this graph forbids the Beineke graphs as its induced subgraph. Therefore, in this case, $\overline{P^*(G)}$ is a line graph. The structure of $\overline{P^*(G)}$ is $P_3 \cup \{u\} \cup \{v\}$. So $\overline{P^*(G)} = L(P_4 \cup 2K_2)$.

Finally, if G is the elementary abelian 2-group of order 2^r, then $\overline{P^*(G)}$ is the complete graph with $2^r - 1$ vertices. Since the complete graph is the line graph of the star graph, in that case, $\overline{P^*(G)} = L(K_{1,2^r-1})$. □

Corollary 5.5 *For any finite group G, both* $P^*(G)$ *and* $\overline{P^*(G)}$ *are line graphs if and only if G is either* C_{p^k} *or* C_6 *or an elementary abelian* 2*-group.*

Proof Firstly, let both $P^*(G)$ and $\overline{P^*(G)}$ be line graphs of some graph Γ. Then G is either C_{p^k} or C_6 or an elementary abelian 2-group by Theorem 5.9.

Note that if $G \cong C_{p^k}$, then $P^*(G)$ is complete and $P^*(G)$ is the line graph of a star graph. Whereas if we take $G \cong C_6$, then $P^*(G) = L(\Gamma)$, where Γ is the 5 vertex pineapple graph obtained by joining two isolated

Table 1 Complementary proper power graphs

Complementary in sense of	Groups
Self-complementary	no groups except possibly $G \cong C_n \rtimes C_{p_k^{\alpha_k}}$ with $p_k \nmid n$ and $C_n \rtimes Q_{2^r}$ with $2 \nmid n$
Unicyclicity	no group exists
2-connectedness(for nilpotent groups)	G is non-cyclic having 2 or more distinct prime divisors
Bipartiteness and Balancedness	C_2, C_3
P_3-free	C_{p^k} or $(C_2)^r$
Claw-free	C_{p^k}, $(C_2)^r$, a 3-group of exponent 3, an EPO group of order $2^r.3^s$
Paw-free	$C_{p^m q}, C_{p^2q^2}, C_{p^k}$ or a non-cyclic p-group
Line Graph	$C_{p^k}, C_6, (C_2)^r$
Diamond-free	$C_{p^k}, (C_2)^r$

vertices to a vertex of a triangle. Finally, if G is an elementary abelian 2-group, then $P^*(G)$ is a null graph, and hence it is the line graph of the disjoint union of K_2. □

A 4-vertex graph obtained by removing an edge from K_4 is called a diamond. A graph Γ is called diamond-free if it does not contain any induced subgraph isomorphic to diamond. We observe that a graph is diamond-free if and only if its complement is $K_2 \cup \{K_1\} \cup \{K_1\}$-free. We now characterize finite groups G such that both $P^*(G)$ and $\overline{P^*(G)}$ are diamond-free.

Theorem 5.10 *Let G be a finite group. Then $\overline{P^*(G)}$ is diamond-free if and only if G is either the cyclic group C_{p^r} or $C_{p^r q}$ or C_{pq^r} for $r \geq 1$ or the elementary abelian 2-group.*

Proof First, suppose that $\overline{P^*(G)}$ is diamond-free. This implies that $P^*(G)$ does not enclose the structure $K_2 \cup \{u\} \cup \{v\}$ as an induced subgraph. If possible, let p, q, and r be three distinct prime divisors of $|G|$, and let $p > 2$. Then clearly, $P^*(G)$ contains $K_2 \cup \{u\} \cup \{v\}$ by taking an induced subgraph with two adjacent elements of order p, one element of order q, and one element of order r. Thus, G is either a p-group or $|G|$ has exactly two prime divisors.

First, we assume that G is a p-group. If p is odd, then, as $P^*(G)$ forbids $K_2 \cup K_1 \cup K_1$, the group G can have at most two distinct subgroups of order a power of p. But by Theorem 2.2, the number of such subgroups is 1 (mod p). Therefore, G must have a unique subgroup of order every power of p, and so G is a cyclic p-group. Next, let $p = 2$. If G contains an element of order 4, then G cannot have more than 2 distinct subgroups of order either 2 or a higher power of 2. Then, by Theorem 2.2, G must be a cyclic 2-group. Otherwise, G is an elementary abelian 2-group.

Next, suppose that $|G|$ has exactly two divisors and $|G| = p^r q^s$. Let both p, q be odd. In this case, we claim that both the Sylow subgroups of G must be cyclic and normal. If possible, let one of the Sylow subgroups of G (say, the Sylow p-subgroup) be either non-cyclic or not normal. Then there exist two elements a, b of order p such that $a \nsim b$ in $P^*(G)$ and $c \in G$ of order q. Then $P^*(G)$ contains $K_2 \cup \{u\} \cup \{v\}$ with $u = a, v = c$. Thus, $G \cong C_{p^r q^s}$. Another observation is that one of r, s must be 1; otherwise, we can choose 4 vertices as a, b, c, d of orders pq, p^2, pq, q^2, respectively, and $P^*(G)$ contains $K_2 \cup \{u\} \cup \{v\}$ with $u = b, v = d$. Thus, without loss of generality, $G \cong C_{p^r q}$ where $r \geq 1$.
Let $p = 2$ and q be odd. Then clearly Sylow q-subgroup of $G$is cyclic and normal. If the Sylow 2-subgroup is generalized quaternion then G has two elements of order 4 that are non-adjacent in $P^*(G)$ contains $K_2 \cup K_1 \cup K_1$. Similarly if the Sylow 2-subgroup of G is either non-cyclic or not normal then $K_2 \cup K_1 \cup K_1$ contained in $P^*(G)$. Hence we get both the Sylow subgroups are cyclic as well as normal which implies that $G \cong C_{2^r q^s}$. Following the similar argument as done in p, q both odd case we obtain that $G \cong C_{2^r q}$ or C_{2q^r}, where $r \geq 1$.

To complete the proof, we need to show that if G is either a cyclic p-group, an elementary abelian 2-group, or a cyclic group of order $p^r q$, then $P^*(G)$ is $K_1 \cup K_1 \cup K_1$-free. If G is either C_{p^k} or $(C_2)^r$, the proof is obvious. Again, if $G \cong C_{p^r q}$, then every non-identity element is of order either a power of p or of the form $p^m q$, where $0 \leq m \leq r$. Now any two elements of the order power of p are adjacent. Also, any two elements of order of the form $p^m q$ are adjacent. Thus, no four elements can make the structure $K_2 \cup K_1 \cup K_1$ in $P^*(G)$. Thus, $\overline{P^*(G)}$ is diamond-free. □

Corollary 5.6 *For any finite group G, both $P^*(G)$ and $\overline{P^*(G)}$ are diamond-free if and only if $G \cong C_{p^k}$ or an elementary abelian group.*

We have considered complementary proper power graphs of finite groups in various aspects. We now conclude by summarizing results of the paper. The following table gives a characterization of complementary proper power graphs in different aspects.

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Declarations

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