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ORIGINAL RESEARCH

Group embedding into wreath products: Cartesian decomposition approach

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Abstract In this paper, we proved that a permutation representation of a group acting regularly and two cases of diagonal groups are embedded into wreath products in there product action using the Cartesian Decomposition.

Keywords Permutation group · Embedding · Wreath products · Cartesian decomposition · Product action

Mathematics subject classification 20B05 · 20E22 · 20B35 · 20B30 · 20B99

1 Introduction

Diagonal groups are a class of permutation groups which first appeared in the ONan-Scott Theorem describing the structure of primitive permutation groups. Bailey et al [7] gave the geometry of such diagonal groups.

Praeger and Schneider [1] proved that if for a given set X, a subgroup that is transitive of a given wreath product $Sym\Gamma wr Sym\Delta$ on Δ, then X is shown to be isomorphic to a subgroup of the wreath product of the permutation group prompted by its stabilizer X_δ on the set Γ and also a given group prompted from a set X on Δ.

Praeger and Schneider [2] also proved that quasiprimitive permutation groups that is of simple diagonal type is not in any way isomorphic to a subgroup of wreath products that is acting on the same point set.

Many other people have made some progress on embedment of Groups into wreath products [3,5,6,9]. In this work, we were able to show that a permutation representation of a group acting regularly and two cases of diagonal groups were embedded into the wreath products in there product actions using the Cartesian Decomposition method.

2 Notion of Cartesian decomposition and wreath products

Definition 2.1 [8] Cartesian decomposition is define given a set ϵ that is finite of partitions of Ω, $\epsilon = \{\Gamma_1, \Gamma_2, \cdots, \Gamma_k\}$, with $|\Gamma_i| \geq 2$, for all $i \geq 1$ and $|\gamma_1 \cap \gamma_2 \cap \cdots \cap \gamma_k| = 1$ for all $\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2, \cdots, \gamma_k \in \Gamma_k$.

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Any Cartesian decomposition is called a *trivial Cartesian decomposition* if it comprises of just a single partition, that is partition into singletons. Cartesian decomposition is called *homogeneous* if it has the property that all the Γ_i have the equal number of elements. If $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_k\}$, is a particular cartesian decomposition of a given set Ω, then the defining property yields a well-defined one-to-one correspondence between Ω and $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$ given by $\omega \mapsto (\gamma_1, \gamma_2, \cdots, \gamma_k)$ where, for a given $i = 1, 2, \cdots, k$, the block $\gamma_i \in \Gamma_i$ is the unique block of Γ_i which contains ω. Thus the set Ω can be obviously recognized with the cartesian product $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$.

Example 2.2 Let $\Omega = (1, 2)^3$, we have the subsequent partitions of Ω as.

$$\Gamma_1 = \{\{(1,1,1),(1,1,2),(1,2,1),(1,2,2)\},\{(2,1,1),(2,1,2),(2,2,1),(2,2,2)\}\}$$
$$\Gamma_2 = \{\{(1,1,1),(1,1,2),(2,1,1),(2,1,2)\},\{(1,2,1),(1,2,2),(2,2,1),(2,2,2)\}\}$$
$$\Gamma_3 = \{\{(1,1,1),(1,2,1),(2,1,1),(2,2,1)\},\{(1,1,2),(1,2,2),(2,1,2),(2,2,2)\}\}$$

With $|\gamma_1 \cap \gamma_2 \cap \gamma_3| = 1$ for all $\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2, \gamma_3 \in \Gamma_3$ and $|\gamma_1| = |\gamma_2| = |\gamma_3| = 2$.

Definition 2.3 (Wreath products of groups [1,9]) Suppose G and H are two given groups with ϕ a homomorphism from H to the automorphism group $Aut\,(G)$. The semidirect product of G and H, denoted by $G \rtimes_\phi H$ or simply by $G \rtimes H$, is given as follows. The underlying set of the group $G \rtimes H$ is the direct product $G \times H$ of sets and the multiplication of two elements (g_1, h_1) and (g_2, h_2) is defined as

$$(g_1, h_1)\,(g_2, h_2) := \left(g_1\left(g_2\left(h_1^{-1}\phi\right)\right), h_1 h_2\right)$$

It is routine to check that the semidirect product $G \rtimes H$ is a group. It can easily be showned that $\overline{G} = \{(g, 1)\,|g \in G\}$ is a normal subgroup of $G \rtimes H$ which is isomorphic to G, and $\overline{H} = \{(1, h)\,|h \in H\}$ is also a subgroup of $G \rtimes H$ which is isomorphic to H. Further, $\overline{G} \cap \overline{H} = 1$ and $\overline{G}\,\overline{H} = G \rtimes H$. Identifying $\overline{G}$ with G and H with $\overline{H}$, one may view G and H as subgroups of $G \rtimes H$, and we will often write the element (g, h) of $G \rtimes H$ as gh.

We are going to utilize the 'function notation' in describing wreath product with its particular product action. Suppose G is a group, Δ be a set, and let H be a subgroup of $Sym\Delta$. Since our focus will be on Cartesian decomposition, which, by definition, are finite, we shall throughout assume that the set Δ is finite. Take $B := Func\,(\Delta, G)$, be a collection of all functions defined from Δ into G. Since B being a given group defined based on the pointwise multiplication of elements of B. It has subgroups G_δ, for $\delta \in \Delta$, defined

$$G_\delta = \left\{f \in Func\,(\Delta, G)\ |\ \delta' f = 1\ for\ all\ \delta' \in \Delta \backslash \{\delta\}\right\}$$

each G_δ is isomorphic to G. Additionally, B is known to be isomorphic with the direct product of these $|\Delta|$ duplicates of the group G, and the mapping $\sigma_\delta : f \mapsto f_\delta$ where $\delta' f_\delta = \begin{cases} \delta f\ if\ \ \delta' = \delta \\ 1\ \ if\ \delta' \neq \delta \end{cases}$ is defined as the natural projection mapping $B \to G_\delta$.

We then give a definition of a group homomorphism, namely τ, defined from the group H to $Aut\,(B)$: Let $h \in H$ and $f \in B$. We take $f\,(h\tau)$ to be a function that is mapping $f\,(h\tau) : \delta \mapsto \delta h^{-1} f$.

It is routine to check that τ is indeed a homomorphism. The wreath product $GwrH$ of G by H is known in a general sense as semidirect product of B by H, i.e. $B \rtimes H$ with respect to the given homomorphism defined as τ, and the subgroup B of $GwrH$ is known as the *base group* of the $GwrH$, and group H known as the *top group*. As the two components of a semidirect product are considered subgroups of the semidirect product, the base group B and then the top group H can also be considered subgroups of the wreath product and, in this way, B becomes a normal subgroup of $GwrH$. Considering H as a known subgroup of $GwrH$, the conjugation action of H on B will be induced by τ in $f\,(h\tau) : \delta \mapsto \delta h^{-1} f$, and so we obtained:

$$\left(\delta h^{-1}\right) f = \delta\left(f^h\right)\ \ for\ all\ h \in H,\ \ f \in Func\,(\Delta, G),\ \ for\ all\ \delta \in \Delta.$$

The wreath product $GwrH$ has a natural action by conjugation on the set of subgroups G_δ of its base group.

Remark 2.4 [8] Let us have a closer look at the special case $\Delta = \{1, 2, \cdots, k\}$. Then, the wreath product $GwrH$ can be described using tuples, instead of functions. For $i \in \Delta$, the image if can be written as f_i, and f can be given by the k-tuple $(f_1, f_2, \cdots, f_k)$, If $h \in H$ and $(g_1, g_2, \cdots, g_k) \in B$ then the conjugate action of h on $(g_1, g_2, \cdots, g_k)$ is known as

$$(g_1, g_2, \cdots, g_k)^h = \left(g_{1h^{-1}}, g_{2h^{-1}}, \cdots, g_{kh^{-1}}\right)$$

Hence H permutes the coordinates of the elements of B.

Definition 2.5 (Product action of Wreath Products [1,8]) Given a product action of a wreath product $GwrH$ defined on the set of functions $\Pi = Func(\Delta, \Gamma)$ to be: For $G \leq Sym\Gamma$ and $H \leq Sym\Delta$, let $f \in Func(\Delta, G)$ and with h being an element of H and fix $g = fh$. Let $\phi \in \Pi$, then we define the function ϕg that maps the element $\delta \in \Delta$ to

$$\delta(\phi g) = \left(\delta h^{-1}\phi\right)\left(\delta h^{-1} f\right)$$

Observe the element $\delta h^{-1}\phi \in \Gamma$, with $\delta h^{-1} f \in Sym\Gamma$ giving $\left(\delta h^{-1}\phi\right)\left(\delta h^{-1} f\right) \in \Gamma$, and $\phi g \in Func(\Delta, \Gamma) = \Pi$, as expected.

Since Δ is finite, it is good to express the product action of it's given wreath product in the form of a coordinate notation. Suppose that $\Delta = \{1, 2, \ldots, k\}$, and view $Func(\Delta, G)$ and $Func(\Delta, \Gamma)$ as G^k and Γ^k, respectively, as in Remark 2.4. Then for $(\gamma_1, \gamma_2, \cdots, \gamma_k) \in \Gamma^k$, $(g_1, g_2, \cdots, g_k)\, h \in GwrH$ we have

$$(\gamma_1, \gamma_2, \cdots, \gamma_k)\left((g_1, g_2, \cdots, g_k)\, h\right) = \left(\gamma_{1h^{-1}} g_{1h^{-1}}, \gamma_{2h^{-1}} g_{2h^{-1}}, \cdots, \gamma_{kh^{-1}} g_{kh^{-1}}\right)$$

To have a deeper understanding of the subgroup(s) of wreath products of groups we entreat the concept of Cartesian decomposition.

Looking at the set $\Pi = Func(\Delta, \Gamma)$, and $\forall \delta \in \Delta$, we give the definition of a partition Γ_δ of the set Π to be: Take

$$\Gamma_\delta = \{\gamma_\delta |\ \forall \gamma \in \Gamma\},\ where\ \gamma_\delta := \{\psi \in \Pi \mid \delta\psi = \gamma\}.$$

We can easily check that Γ_δ is certainly a partition of the set Π. The representation shows two significant facts.

First, the mapping $\delta \mapsto \Gamma_\delta$ is defined as a one-to-one correspondence between Δ and $\{\Gamma_\delta | \delta \in \Delta\}$.

Second, let $\delta \in \Delta$, be a fixed element, the mapping $\gamma \mapsto \gamma_\delta$ defined is a given one-to-one correspondence between Γ and Γ_δ. Let $\gamma \in \Gamma$ and $\delta \in \Delta$, then the element $\gamma_\delta \in \Gamma_\delta$ is well thought-out to be 'a copy' of γ in the set Γ_δ, and it is known as the γ-part of Γ_δ.

The Cartesian product $\prod_{\delta\in\Delta} \Gamma_\delta$ is given to be a one-to-one correspondence with the original set Π: taking $\gamma_\delta \in \Gamma_\delta$, one for all element $\delta \in \Delta$, the intersection $\bigcap_{\delta\in\Delta} \gamma_\delta$ comprises of single element of Π, namely the map that takes each δ to the element $\gamma \in \Gamma$ that corresponds to γ_δ. This gives a one-to-one corespondence from the Cartesian product $\prod_{\delta\in\Delta} \Gamma_\delta$ to the set of functions Π. Then, the set

$$\epsilon = \{\Gamma_\delta \mid \forall\, \delta \in \Delta\}$$

is a Cartesian decomposition of the set Π. Precisely, is a set of partitions seen as the sets of natural Cartesian decomposition of the set Π. Since $Sym\Gamma wrSym\Delta$ is also a group that acts on the set Π, and the given action of $Sym\Gamma wrSym\Delta$ is being stretched to subsets of the set Π, subsets of subsets, etc. To be specific, we look at the action of the group $Sym\Gamma wrSym\Delta$ on the particular sets of partitions of the set Π. We will observe that $\{\Gamma_\delta | \delta \in \Delta\}$ is invariant under the action. The normal product action of the group $Sym\Gamma wrSym\Delta$ on $\prod_{\delta\in\Delta} \Gamma_\delta$ is known to be permutationally isomorphic to the action that is defined on the set Π, and so the stabiliser in the permutation group $Sym\Pi$ of the Cartesian decomposition is the permutation group $Sym\Gamma wrSym\Delta$.

If X is a given subgroup of $Sym\Gamma wrSym\Delta$ in its product action on the set of functions $\Pi = Func(\Delta, \Gamma)$, and $\Delta = \{1, \cdots, k\}$, then we identify Π with the set Γ^k of ordered k-tuples of the elements of the set Γ, and in this situation, subgroups of $Sym\Gamma wrSym\Delta$ ascend as automorphism groups of different types of graph products (see (Praeger and Schneider, [8])), as groups of automorphism of some given codes of length say k on the alphabet Γ, seen as subsets of Γ^k.

Then following theorem and its proof can be found in Praeger and Schneider [8], Theorem 5.13.

Theorem 2.6 (Wreath Embedding Theorem [8]) *Suppose that X is a given permutation group on a set Ω preserving a homogeneous Cartesian decomposition $\epsilon = \{\Gamma_\delta | \delta \in \Delta\}$ of Ω, and let $\Gamma \in \epsilon$. Then there is a permutational isomorphism that maps X to a subgroup of $Sym\Gamma wr Sym\Delta$ with its product action on $Func(\Delta, \Gamma)$, and maps ϵ to the natural Cartesian decomposition of ϵ defined above.*

3 Main results

Now we are in a better position to prove our results. Embedding a wreath product in it's product action is equivalent to proving that a Cartesian decomposition is preserved.

Definition 3.1 [3] The G-action is said to be transitive if Ω is a G-orbit; that is, for all $\alpha, \beta \in \Omega$ there is a given element $g \in G$ such that $\alpha g = \beta$. If G is not transitive, then it is known as *intransitive*. A permutation group is known as *semiregular* if all its point stabilisers are trivial. A permutation group is *regular* if it is transitive and semiregular.

Proposition 3.2 *Let $T = S^k$, for some group S, let S act regularly on Γ such that $|\Gamma| = |S|$, then $Sym\Gamma wr S_k$ in its natural action acts on $\Omega = \Gamma^k$, and the permutation representation of T is embeddable in $Sym\Gamma wr S_k$ acting regularly on $\Omega = \Gamma^k$.*

Proof We suppose that S act regularly on Γ such that $|\Gamma| = |S|$, $T = S^k$, for some group S acts regularly on $\Omega = \Gamma^k$. Let $\Omega = \Gamma^k$ and suppose there exist a given subgroup W of $Sym\Omega$ that is permutationally isomorphic to $Sym\Gamma wr S_k$, with $|\Gamma| \geq 2$ and $k \geq 2$, then $W = Sym\Gamma wr S_k$.

The normal subgroup $N = (Sym\Gamma)^k$ is known as the base group of W and $H \cong S_k$ is known as the top group. The product action of W on $\Omega = \Gamma^k$ is defined by

$$(\gamma_1, \cdots, \gamma_k)^{xh} = \left(\gamma_{1h^{-1}}^{x_{1h^{-1}}}, \cdots, \gamma_{kh^{-1}}^{x_{kh^{-1}}}\right)$$

for all $(\gamma_1, \cdots, \gamma_k) \in \Omega, x = (x_1, \cdots, x_k) \in N$ and $h \in H$, where the image of $\gamma \in \Gamma$ under $y \in Sym\Gamma$ is γ^y. W is obviously transitive on $\Omega = \Gamma^k$.

The Cartesian decomposition corresponding to the identity map on Ω is

$$\epsilon = \{\Gamma_1, \cdots, \Gamma_k\}$$

Where Γ_i is the partition of $\Omega = \Gamma^k$ into disjoint subsets according to the ith coordinate of a point in $\Omega = \Gamma^k$, that is to say, the parts of Γ_i are indexed by Γ and the γ-part is the set of all points $(\gamma_1, \cdots, \gamma_k)$ with $\gamma_i = \gamma$. Thus $|\Gamma_i| = |\Gamma|$ for all i.

Thus ϵ is homogenous. Also each element $xh \in W$ maps the partition Γ_i to the partition Γ_{ih}. Thus W preserves the Cartesian decomposition ϵ.

Also W permutes the partitions Γ_i transitively. Thus the permutation representation of T is isomorphic to a subgroup of $Sym\Gamma wr S_k$. □

Definition 3.3 (Diagonal group $D(G, n)$ [3]) Suppose that G is a group with order $|G| > 1$, and $n \geq 0$ positive integer. Let $\delta(G, n+1)$ be the diagonal subgroup $\{(g, g, \cdots, g) | g \in G\}$ of G^{n+1}. We select coset representatives for the element $\delta(G, n+1)$ in G^{n+1}. A suitable selection is to figure out the direct factors of G^{n+1} as $G_0; G_1; \cdots; G_n$, and employ the representatives of the form $(1, g_1, g_2, \cdots, g_n)$ where $g_i \in G_i$. and let Ω denote the collection of all such symbols. Then, Ω is bijective by means of G^n.

We are now going to designate the action of $D(G, n)$ as:

(a) Let $1 \leq i \leq n$, the factor G_i acts by right multiplication on symbols in the ith position in the elements of the set Ω.
(b) G_0 is acting by simultaneous left multiplication of all the coordinates by the inverse. Since, for $x \in G_0$, x maps the coset containing $(1, g_1, g_2, \cdots, g_n)$ to the coset containing $(x, g_1, g_2, \cdots, g_n)$, which is equal to the coset containing $\left(1, x^{-1}g_1, x^{-1}g_2, \cdots, x^{-1}g_n\right)$. Automorphisms of G also acts simultaneously on each of the coordinates; nevertheless the inner automorphisms are recognized with the action of elements in the diagonal subgroup $\delta(G, n+1)$ (the element $((x, x, x, \cdots, x)$ maps the coset containing $(1, g_1, g_2, \cdots, g_n)$ to the coset containing $(x, g_1x, g_2x, \cdots, g_nx)$, which is equal to the coset containing $\left(1, x^{-1}g_1x, x^{-1}g_2x, \cdots, x^{-1}g_nx\right)$.

(c) Elements of the symmetric group S_n (fixing coordinate 0) also acts by permuting the coordinates in elements of Ω.
(d) Look at the element of S_{n+1} which transposes the coordinates 0 and 1. It is mapping the coset containing $(1, g_1, g_2, \cdots, g_n)$ to the coset containing $(g_1, 1, g_2, \cdots, g_n)$,that also contains $\left(1, g_1^{-1}, g_1^{-1}g_2, \cdots, g_1^{-1}g_n\right)$. So the action of the given transposition is

$$(1, g_1, g_2, \cdots, g_n) \mapsto \left(1, g_1^{-1}, g_1^{-1}g_2, \cdots, g_1^{-1}g_n\right).$$

(e) Now S_n and the transposition generates S_{n+1}.

Proposition 3.4 *Let $D(G,k)$ be a diagonal group where G is a group such that $G \leq Sym\Gamma$ and positive integer $k \geq 2$, then $D(G,k)$ is embeddable in the wreath product $Sym\Gamma wr S_k$ where the wreath product acts on $\Omega = \Gamma^k$ and $\Gamma \geq 2$ is a non-empty set.*

Proof Suppose that $D(G,k)$ is a diagonal group where G is a group and there is a positive integer $k \geq 2$.

Now, $Sym\Gamma wr S_k$ acts naturally on the set $\Omega = \Gamma^k$ in its product action and is defined by

$$(\gamma_1, \cdots, \gamma_k)^{xh} = \left(\gamma_{1h^{-1}}^{x_{1h^{-1}}}, \cdots, \gamma_{kh^{-1}}^{x_{kh^{-1}}}\right)$$

for all $(\gamma_1, \cdots, \gamma_k) \in \Omega$, $x = (x_1, \cdots, x_k) \in (Sym\Gamma)^k$ and $h \in S_k$ and the image of $\gamma \in \Gamma$ under $y \in Sym\Gamma$ is γ^y. $Sym\Gamma wr S_k$ is obviously transitive on $\Omega = \Gamma^k$.

Now for each coset of the diagonal group $\delta(G,k) = \{(g, g, \cdots, g) \,|g \in G\}$ of G^k, there is a unique representative of the form $(1, g_1, g_2, \cdots, g_{k-1})$ and define η as

$$\eta\left((1, g_1, g_2, \cdots, g_{k-1}), ((g, g, ..., g), (g, g, \cdots, g))\right) = \left(1, g^{-1}g_1g, g^{-1}g_2g, \cdots, g^{-1}g_{k-1}g\right)$$

which is the action on $\delta(G,k)$.

The Cartesian decomposition corresponding to the identity map on Ω is

$$\epsilon = \{\Gamma_1, \cdots, \Gamma_k\}$$

Where Γ_i is the partition of $\Omega = \Gamma^k$ into disjoint subsets according to the ith coordinate of a point in $\Omega = \Gamma^k$, that is to say, the parts of Γ_i are indexed by Γ and the γ-part is the set of all points $(\gamma_1, \cdots, \gamma_k)$ with $\gamma_1 = \gamma$. Thus $|\Gamma_i| = |\Gamma|$ for all i.

Thus ϵ is homogenous. Also each element of $Sym\Gamma wr S_k$ maps the partition Γ_i to the partition Γ_{ik}. Thus W preserves the Cartesian decomposition ϵ. Thus $D(G,k)$ is isomorphic to a subgroup of $Sym\Gamma wr S_k$.

Definition 3.5 Suppose that G is a given group and n a positive. Let $\Omega = G^n$; this will be the domain of a permutation, and its elements are written as $[x_1, x_2, \cdots, x_n]$, where $x_1, x_2, \cdots, x_n \in G$. The diagonal group $D(G,n)$ is generated by the following five types of permutations on Ω:

(a) The group G^n acting by right multiplication; so the element $(g_1, g_2, ..., g_n)$ maps $[x_1, x_2, \cdots, x_n]$ to $[x_1g_1, x_2g_2, \cdots, x_ng_n]$. We will let G_i be the ith factor of G^n, so that G_i acts on the ith coordinate of elements of Ω.
(b) The group G, acting by simultaneous left multiplication; so g maps $[x_1, x_2, \cdots, x_n]$ to $\left[g^{-1}x_1, g^{-1}x_2, \cdots, g^{-1}x_n\right]$. It will be denoted by G_0.
(c) The automorphism group of G, acting simultaneously on all coordinates.
(d) The symmetric group S_n, acting by permuting the coordinates.
(e) A permutation τ , defined by

$$\tau : [x_1, x_2, \cdots, x_n] \mapsto \left[x_1^{-1}, x_1^{-1}x_2, \cdots, x_1^{-1}x_n\right]$$

.

Proposition 3.6 *Let $D(G,n)$ be a diagonal group, where G is a finite group and n is a positive integer, and assume that $\Gamma = G$ with G acting regularly by right multiplication, and identify G with the corresponding subgroup of $Sym\Gamma$. Suppose also that in $Aut(G)$ there is a subgroup $Out(G)$, which complements the group of inner automorphisms. Then $D(G,n)$ is embeddable into the wreath product $Sym\Gamma wr S_n$, acting naturally in product action on the set $\Omega = \Gamma^n$.*

Proof Let $D(G, n)$ be a diagonal group for a finite group G and n a positive integer, and let $\Gamma = G$ with G acting by right multiplication. Now, $W = Sym\Gamma wr S_n$ acts naturally on the set $\Omega = \Gamma^n$ in its product action and is defined by

$$(\gamma_1, \cdots, \gamma_n)^{xh} = \left(\gamma_{1h^{-1}}^{x_{1h^{-1}}}, \cdots, \gamma_{nh^{-1}}^{x_{nh^{-1}}}\right)$$

for all $(\gamma_1, \cdots, \gamma_n) \in \Omega$, $x = (x_1, \cdots, x_n) \in (Sym\Gamma)^n$ and $h \in S_n$ and the image of $\gamma \in \Gamma$ under $y \in Sym\Gamma$ is γ^y. $W = Sym\Gamma wr S_n$ is obviously transitive on $\Omega = \Gamma^n$. Ω is bijective with G^n. □

Now,

1.We identified G with a subgroup of $Sym\Gamma$. So we have $M := G^n$ as a subgroup of the base group $(Sym\Gamma)^n$ of W.

2. From the definition of the action of W it is clear that the top group S_n normalizes M.

3. Also, $N_{Sym\Gamma}(G)$ is the holomorph of G, this is known as a semidirect product of G and a group $A = Aut(G)$ and A acts on $\Gamma = G$ naturally as automorphisms.

4. By assumption A has a subgroup $O \subseteq Out(G)$ which complements the inner automorphism group of G.

5. Hence the normalizer of M in the base group of W contains a semidirect product MO^n, and we define D as the diagonal subgroup of O^n, namely $D = \{(x, x, \cdots, x) \mid x \in O\}$.

6. Finally consider the group generated by M, D and S_n, this is a copy of $D(G, n)$ in W.

The Cartesian decomposition corresponding to the identity map on $\Omega = \Gamma^n$ is

$$\epsilon = \{\Gamma_1, \cdots, \Gamma_n\}$$

Where Γ_i is the partition of $\Omega = \Gamma^n$ into disjoint subsets according to the ith coordinate of a point in $\Omega = \Gamma^n$, that is to say, the parts of Γ_i are indexed by Γ and the γ-part is the set of all points $(\gamma_1, \cdots, \gamma_n)$ with $\gamma_i = \gamma$. Thus $|\Gamma_i| = |\Gamma|$ for all i.

Thus ϵ is homogenous. Also each element of $Sym\Gamma wr S_n$ maps the partition Γ_i to the partition Γ_{ik}. Thus W preserves the Cartesian decomposition ϵ. Now since the automorphism group of G acts simultaneously on all coordinates, thus $W = Sym\Gamma wr S_n$ preserves the Cartesian decomposition ϵ and we conclude that $D(G, n)$ is embeddable into the wreath product $Sym\Gamma wr S_n$.

4 Conclusion

We proved that a group that a permutation representation of a group acting regularly on a set and two cases of diagonal groups acting on a set in product action are embedable into wreath products in such actions using the Cartesian decomposition.

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