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ORIGINAL RESEARCH

# On generalized degenerate harmonic numbers

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**Abstract** In this paper, we give various properties including binomial coefficients, degenerate Daehee numbers, degenerate Stirling numbers of the first kind and some types of generalized degenerate harmonic numbers. Particularly, we define generalized degenerate harmonic numbers of rank and give some interesting identities and results.

**Keywords** Generalized degenerate harmonic numbers · Degenerate Stirling numbers of the first kind · Generating functions

**Mathematics Subject Classification** 05A15 · 15A23 · 11B83

## 1 Introduction

The recent examination of degenerate iterations of various special numbers and polynomials has yielded intriguing and fruitful outcomes. This inquiry was instigated by Carlitz's contributions in [1,2], serving as the lead for many authors' investigations into this multifaceted mathematical domain. These investigations encompass a diverse array of methodologies, including the utilization of generating functions, combinatorial techniques, special functions, $p-$adic analysis, analytic number theory, differential equations, umbral calculus, and probability theory. Our goal in this paper is to get some new identities and results using degenerate Daehee numbers degenerate Stirling numbers of the first kind, families of generalized degenerate harmonic numbers, and binomial coefficients with the help of some combinatorial techniques. Firstly, we recall some generating functions, the degenerate exponential, the degenerate logarithms and some special numbers. Secondly, we give some new identities involving these numbers. Also, we present the generalized degenerate harmonic numbers of rank $r$, denoted by $H_\lambda(m, r, \alpha)$. Lastly, by using these numbers, we examine some identities and properties of special matrix in this paper.

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There are many properties involving binomial coefficients. These are useful for get some new identities. For example, let $s$, $a$, and $b$ be non-negative integers. Then the following identity is known as Vandermonde's identity:

$$\sum_{\ell=0}^{s}\binom{a}{\ell}\binom{b}{s-\ell}=\binom{a+b}{s}. \tag{1}$$

Also, for non-negative integer $r$, there is a generating function as

$$\frac{1}{(1-z)^{r+1}}=\sum_{m=0}^{\infty}\binom{m+r}{m}z^m. \tag{2}$$

The extended binomial coefficients for $x\in\mathbb{C}$ and $\ell=1,2,\cdots$, are defined by

$$\binom{x}{\ell}=\frac{(x)_\ell}{\ell!},$$

and $\binom{x}{0}=1$, where $(x)_\ell=x(x-1)(x-2)\cdots(x-\ell+1)$ for $\ell=1,2,\cdots$, and $(x)_0=1$.

Let $\lambda$ be non-zero real number. The degenerate exponential function, denoted by $e_\lambda^x(z)$, are defined by

$$e_\lambda^x(z)=(1+\lambda z)^{x/\lambda}=\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{z^m}{m!}, \tag{3}$$

where $(x)_{\ell,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(\ell-1)\lambda)$ with $(x)_{0,\lambda}=1$ [3]. When $x=1$, we denote $e_\lambda^1(z)$ as $e_\lambda(z)$. Let $\log_\lambda z$ denote the compositional inverse of the degenerate exponential function i.e. $\log_\lambda(e_\lambda(z))=z=e_\lambda(\log_\lambda(z))$. This function known as the degenerate logarithm function. Also, the degenerate logarithm function can be given by

$$\log_\lambda z=\frac{z^\lambda-1}{\lambda}=\sum_{m=1}^{\infty}\frac{(1)_{m,1/\lambda}\lambda^{m-1}}{m!}(z-1)^m=\sum_{m=1}^{\infty}\binom{\lambda}{m}\frac{(z-1)^m}{\lambda}. \tag{4}$$

The Stirling numbers of the first kind, denoted by $S_1(m,r)$, are defined by

$$(x)_m=\sum_{r=0}^{m}S_1(m,r)x^r. \tag{5}$$

Recently, the degenerate Stirling numbers of the first kind are given by [4]

$$\frac{(\log_\lambda(1+z))^r}{r!}=\sum_{m=k}^{\infty}S_{1,\lambda}(m,r)\frac{z^m}{m!}, \tag{6}$$

where $r$ is a non-negative integer.

The Daehee polynomials are given by [5,6],

$$\frac{(1+z)^t\log(1+z)}{z}=\sum_{m=0}^{\infty}D_m(t)\frac{z^m}{m!}.$$

When $t=0$, $D_m(0)=D_m$ are called the Daehee numbers. Recently, there are some works on degenerate Daehee polynomials that are the generalization of Daehee polynomials. Degenerate Daehee polynomials are given by [7]

$$\frac{(1+z)^t\log_\lambda(1+z)}{z}=\sum_{m=0}^{\infty}D_{m,\lambda}(t)\frac{z^m}{m!}. \tag{7}$$

When $t=0$, $D_{m,\lambda}(0)=D_{m,\lambda}$ are called the degenerate Daehee numbers.

Harmonic numbers and their generalizations are important in combinatorics, number theory, and there are a some works involving these numbers (see [8–16]). The harmonic numbers, denoted by $H_m$, are defined by

$$H_m = \sum_{\ell=1}^{m} \frac{1}{\ell}$$

for $m = 1, 2, \cdots$ and $H_0 = 0$. Recently, in [17], the degenerate harmonic numbers $H_{m,\lambda}$ are defined by

$$H_{m,\lambda} = \sum_{\ell=1}^{m} \frac{(-1)^{\ell-1}}{\lambda} \binom{\lambda}{\ell}$$

for $m = 1, 2, \cdots$ and $H_{0,\lambda} = 0$. These numbers are generated by

$$\frac{-\log_\lambda(1-z)}{1-z} = \sum_{m=1}^{\infty} H_{m,\lambda} z^m. \tag{8}$$

Note that $H_{m,\lambda} \to H_m$ when $\lambda \to 0$.

In [12], for positive real numbers $\alpha$, the generalized harmonic numbers $H_m(\alpha)$ are defined by

$$H_m(\alpha) = \sum_{\ell=1}^{m} \frac{1}{\ell \alpha^\ell}$$

for $m = 1, 2, \cdots$ and $H_0(\alpha) = 0$. Clearly, when $\alpha = 1$, $H_m(1) = H_m$. These numbers are generated by

$$\frac{-\log(1-\frac{z}{\alpha})}{1-z} = \sum_{m=0}^{\infty} H_m(\alpha) z^m.$$

In [15], Ömür and Bilgin introduced the generalized hyperharmonic numbers of order $r$, $H_m^r(\alpha)$ as

$$H_m^r(\alpha) = \sum_{\ell=1}^{m} H_\ell^{r-1}(\alpha)$$

for $r \geq 1$, with $H_m^0(\alpha) = \frac{1}{m\alpha^m}$. These numbers are generated by

$$\frac{-\log(1-\frac{z}{\alpha})}{(1-z)^r} = \sum_{m=0}^{\infty} H_m^r(\alpha) z^m.$$

In [14], authors presented the degenerate hyperharmonic numbers, denoted by $H_{m,\lambda}^r$, which are defined by

$$H_{m,\lambda}^r = \sum_{\ell=1}^{m} H_{\ell,\lambda}^{r-1}, \quad r \geq 2,$$

with $H_{0,\lambda}^r = 0$, $H_{m,\lambda}^1 = H_{m,\lambda}$ and $H_{m,\lambda}^0 = (-1)^{m-1}(1)_{m,1/\lambda}\lambda^{m-1}/m!$. Also, the authors gave generating function of these numbers as follows:

$$\frac{-\log_\lambda(1-z)}{(1-z)^r} = \sum_{m=1}^{\infty} H_{m,\lambda}^r z^m. \tag{9}$$

In [9], Dağlı defined the generalized degenerate harmonic numbers, denoted by $H_{m,\lambda}(\alpha)$, by

$$H_{m,\lambda}(\alpha) = \sum_{\ell=1}^{m} \frac{(-1)^{\ell-1}}{\lambda \alpha^\ell} \binom{\lambda}{\ell}$$

for $m > 0$ and $H_{0,\lambda}(\alpha) = 0$, and the generalized degenerate hyperharmonic numbers of order $r$, denoted by $H^r_{m,\lambda}(\alpha)$ , by

$$H^r_{m,\lambda}(\alpha) = \sum_{\ell=1}^{m} H^{r-1}_{\ell,\lambda}(\alpha), \quad m, r \geq 1,$$

with $H^r_{0,\lambda}(\alpha) = 0$, $H^1_{m,\lambda}(\alpha) = H_{m,\lambda}(\alpha)$ and $H^0_{m,\lambda}(\alpha) = \frac{(-1)^{n-1}}{\lambda\alpha^m}\binom{\lambda}{m}$ for $\alpha \in \mathbb{R}^+$. The generating functions of these numbers are

$$\frac{-\log_\lambda\left(1-\frac{z}{\alpha}\right)}{1-z} = \sum_{m=0}^{\infty} H_{m,\lambda}(\alpha) z^m, \tag{10}$$

and

$$\frac{-\log_\lambda(1-\frac{z}{\alpha})}{(1-z)^r} = \sum_{m=0}^{\infty} H^r_{m,\lambda}(\alpha) z^m, \tag{11}$$

respectively.

For non-negative integers $m$ and $r$, the generalized harmonic numbers of rank $r$, denoted by $H(m, r, \alpha)$, are given by [10]

$$H(m, r, \alpha) = \sum_{1 \leq k_0+k_1+\cdots+k_r \leq m} \frac{1}{k_0 k_1 \cdots k_r \alpha^{k_0+k_1+\cdots k_r}},$$

for $m = 1, 2, \cdots$ and $H(0, r, \alpha) = 0$. Notice that, when $r = 0$, $H(m, 0, \alpha) = H_m(\alpha)$. Also, $H(m, r, 1) = H(m, r)$ are called the harmonic numbers of rank $r$ (see [13]).

## 2 Main results

In this section, firstly, we will provide several identities and outcomes between special numbers given in Section 1 and closed form of degenerate hyperharmonic numbers, degenerate Daehee numbers, and degenerate Stirling numbers of the first kind by using some combinatorial techniques. Secondly, we will define the generalized degenerate harmonic numbers of rank $r$, denoted by $H_\lambda(m, r, \alpha)$, and give some identities including these numbers. Lastly, we will examine special matrices with the terms $H^r_{m,\lambda}(\alpha)$ and give some new results with the help of matrix properties.

**Theorem 1** *Let $m$ and $r$ be integers with $m \geq 0$ and $r \geq 1$. Then, following identities hold:*

$$H_{m,\lambda}(\alpha) = \sum_{\ell=0}^{m} (-1)^{m-\ell} \binom{r}{m-\ell} H^{r+1}_{\ell,\lambda}(\alpha), \tag{12}$$

*and*

$$H^r_{m,\lambda}(\alpha) = \sum_{\ell=1}^{m} \binom{m-\ell+r-1}{r-1} \frac{(-1)^{\ell-1}\lambda^{\ell-1}(1)_{\ell,1/\lambda}}{\ell!\alpha^\ell}. \tag{13}$$

*Proof* From (10), (11), and the binomial theorem, we have

$$\begin{aligned}
\sum_{m=0}^{\infty} H_{m,\lambda}(\alpha) z^m &= \frac{-\log_\lambda\left(1-\frac{z}{\alpha}\right)}{(1-z)^{r+1}} (1-z)^r \\
&= \sum_{m=0}^{\infty} H^{r+1}_{m,\lambda}(\alpha) z^m \sum_{\ell=0}^{\infty} (-1)^\ell \binom{r}{\ell} z^\ell \\
&= \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} (-1)^{m-\ell} \binom{r}{m-\ell} H^{r+1}_{\ell,\lambda}(\alpha) z^m.
\end{aligned} \tag{14}$$

So, when we compare the coefficients on both sides of (14), we attain (12). Similarly, for (13), using (2), (4), and (9), we get

$$\begin{aligned}\sum_{m=1}^{\infty} H_{m,\lambda}^{r}(\alpha)z^m &= -\log_\lambda\left(1-\frac{z}{\alpha}\right)\frac{1}{(1-z)^r}\\ &= \sum_{m=1}^{\infty}\frac{(-1)^{m-1}\lambda^{m-1}(1)_{m,1/\lambda}}{m!\alpha^m}z^m\sum_{m=0}^{\infty}\binom{m+r-1}{r-1}z^m\\ &= \sum_{m=1}^{\infty}\sum_{\ell=1}^{m}\binom{m-\ell+r-1}{r-1}\frac{(-1)^{\ell-1}\lambda^{\ell-1}(1)_{\ell,1/\lambda}}{\ell!\alpha^\ell}z^m,\end{aligned}$$

as claimed. □

**Theorem 2** *Let $m$ and $r$ be integers with $m, r \geq 0$. Then, following identities hold:*

$$S_{1,\lambda}(m,r) = \frac{(-1)^r m!}{\lambda^r r!}\sum_{\ell=0}^{r}(-1)^\ell\binom{r}{\ell}\binom{\lambda\ell}{m},$$

*and*

$$D_{m,\lambda} = \frac{m!}{\lambda}\binom{\lambda}{m+1}.$$

*Proof* For first identity, from (6), and the binomial theorem, we have

$$\begin{aligned}\sum_{m=r}^{\infty} S_{1,\lambda}(m,r)\frac{z^m}{m!} &= \frac{(\log_\lambda(1+z))^r}{r!} = \frac{1}{r!}\left(\frac{(1+z)^\lambda-1}{\lambda}\right)^r\\ &= \frac{(-1)^r}{r!\lambda^r}\sum_{\ell=0}^{r}(-1)^\ell\binom{r}{\ell}(1+z)^{\lambda\ell}\\ &= \frac{(-1)^r}{r!\lambda^r}\sum_{\ell=0}^{r}(-1)^\ell\binom{r}{\ell}\sum_{m=0}^{\infty}\binom{\lambda\ell}{m}z^m,\end{aligned}\tag{15}$$

and for second identity, (7), (11), and the binomial theorem, we have

$$\sum_{m=0}^{\infty} D_{m,\lambda}\frac{z^m}{m!} = \frac{\log_\lambda(1+z)}{z} = \frac{\frac{1}{\lambda}((1+z)^\lambda-1)}{z} = \frac{1}{\lambda}\sum_{m=0}^{\infty}\binom{\lambda}{m+1}z^m.\tag{16}$$

So, when we compare the coefficients on both sides of (15) and (16), the proof of the theorem is completed. □

**Theorem 3** *Let $m$ and $r$ be integers with $m, r \geq 1$. Then, following identity holds:*

$$\begin{aligned}\binom{m+r}{m}H_{m+r,\lambda}(\alpha) - H_{m,\lambda}^{r+1}(\alpha) &= \sum_{\ell=0}^{m}\sum_{i=1}^{r}\binom{m-\ell+r-i}{r-i}\binom{\lambda-i}{\ell}\\ &\quad\times\frac{(-1)^{\ell+i-1}(\lambda-1)_{i-1}}{\alpha^{\ell+i}i!}.\end{aligned}$$

*Proof* For $s \in \mathbb{R}\setminus\{0\}$, $r \in \mathbb{Z}^+$ and $\alpha \in \mathbb{R}^+$, by the binomial theorem and (2), we have

$$\begin{aligned}\frac{(1-\frac{z}{\alpha})^s}{(1-z)^r} &= \sum_{m=0}^{\infty}\binom{s}{m}\frac{(-1)^m}{\alpha^m}z^m\sum_{m=0}^{\infty}\binom{m+r-1}{r-1}z^m\\ &= \sum_{m=0}^{\infty}\sum_{\ell=0}^{m}\binom{m-\ell+r-1}{r-1}\binom{s}{\ell}\frac{(-1)^\ell}{\alpha^\ell}z^m.\end{aligned}\tag{17}$$

By taking derivative of (10) with respect to $z$, we write

$$\begin{aligned}
\frac{d}{dz}\left(\frac{-\log_\lambda(1-\frac{z}{\alpha})}{1-z}\right) &= \frac{d}{dz}\left(\frac{1-\left(1-\frac{z}{\alpha}\right)^\lambda}{\lambda\,(1-z)}\right)\\
&= \frac{1-\left(1-\frac{z}{\alpha}\right)^\lambda}{\lambda\,(1-z)^2} + \frac{1}{\alpha}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-1}}{1-z},\\
\frac{d^2}{dz^2}\left(\frac{-\log_\lambda(1-\frac{z}{\alpha})}{1-z}\right) &= \frac{d}{dz}\left(\frac{1-\left(1-\frac{z}{\alpha}\right)^\lambda}{\lambda}\frac{1}{(1-z)^2} + \frac{1}{\alpha}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-1}}{1-z}\right)\\
&= 2\frac{1-\left(1-\frac{z}{\alpha}\right)^\lambda}{\lambda\,(1-z)^3} + \frac{2}{\alpha}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-1}}{(1-z)^2} + \frac{1-\lambda}{\alpha^2}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-2}}{1-z},\\
\frac{d^3}{dz^3}\left(\frac{-\log_\lambda(1-\frac{z}{\alpha})}{1-z}\right) &= \frac{d}{dz}\left(2\frac{1-\left(1-\frac{z}{\alpha}\right)^\lambda}{\lambda\,(1-z)^3} + \frac{2}{\alpha}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-1}}{(1-z)^2} + \frac{1-\lambda}{\alpha^2}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-2}}{1-z}\right)\\
&= 6\frac{1-\left(1-\frac{z}{\alpha}\right)^\lambda}{\lambda\,(1-z)^4} + \frac{6}{\alpha}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-1}}{(1-z)^3}\\
&\quad - \frac{3(\lambda-1)}{\alpha^2}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-2}}{(1-z)^2}\\
&\quad + \frac{(2-\lambda)\,(1-\lambda)}{\alpha^3}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-3}}{1-z},
\end{aligned}$$

and if we continue similar way, by (9) and (17), we get

$$\begin{aligned}
\frac{d^r}{dz^r}\left(\frac{-\log_\lambda(1-\frac{z}{\alpha})}{1-z}\right) &= r!\frac{1-\left(1-\frac{z}{\alpha}\right)^\lambda}{\lambda\,(1-z)^{r+1}} + \frac{r!}{1}\frac{1}{\alpha}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-1}}{(1-z)^r}\\
&\quad - \frac{r!}{2!}\frac{(\lambda-1)}{\alpha^2}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-2}}{(1-z)^{r-1}}\\
&\quad + \frac{r!}{3!}\frac{(\lambda-2)\,(\lambda-1)}{\alpha^3}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-3}}{(1-z)^{r-2}} + \cdots\\
&\quad + (-1)^{r-1}\frac{r!}{r!}\frac{(\lambda-r+1)\cdots(\lambda-1)}{\alpha^r}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-r}}{1-z}\\
&= r!\frac{1-\left(1-\frac{z}{\alpha}\right)^\lambda}{\lambda\,(1-z)^{r+1}}\\
&\quad + r!\sum_{i=1}^{r}\frac{(-1)^{i-1}(\lambda-1)_{i-1}}{i!\alpha^i}\frac{\left(1-\frac{z}{\alpha}\right)^{\lambda-i}}{(1-z)^{r+1-i}}\\
&= r!\sum_{m=0}^{\infty}H_{m,\lambda}^{r+1}(\alpha)z^m + r!\sum_{i=1}^{r}\frac{(-1)^{i-1}(\lambda-1)_{i-1}}{i!\alpha^i}\\
&\quad \times\sum_{m=0}^{\infty}\sum_{\ell=0}^{m}\binom{m+r-i-\ell}{r-i}\binom{\lambda-i}{\ell}\frac{(-1)^\ell}{\alpha^\ell}z^m. \qquad (18)
\end{aligned}$$

On the other hand, since $H_{n,\lambda}(\alpha)=0$, it can be clearly seen that

$$\frac{d^r}{dz^r}\frac{-\log_\lambda(1-\frac{z}{\alpha})}{1-z} = \frac{d^r}{dz^r}\sum_{m=0}^{\infty}H_{m,\lambda}(\alpha)z^m = r!\sum_{m=0}^{\infty}\binom{m+r}{r}H_{m+r,\lambda}(\alpha)z^m. \qquad (19)$$

By equality of (18) and (19), multiply $\frac{1}{r!}$ both sides, we have the proof. □

**Definition 1** Let $m$ and $r$ be integers with $m, r \geq 0$. Then the generalized degenerate harmonic numbers of rank $r$ are defined by

$$\frac{\left(-\log_\lambda(1-\frac{z}{\alpha})\right)^{r+1}}{1-z} = \sum_{m=0}^{\infty} H_\lambda(m,r,\alpha) z^m. \tag{20}$$

In the case of $\alpha = 1$, $H_\lambda(m,r,1) = H_\lambda(m,r)$ are called degenerate harmonic numbers of rank $r$.

**Theorem 4** *For non-negative integer $r$, we have*

$$H_\lambda(m,r,\alpha) = \sum_{1\leq k_0+k_1+\cdots+k_r\leq m} \frac{(-1)^{k_0+k_1+\cdots k_r+r+1}}{\lambda^{r+1}\alpha^{k_0+k_1+\cdots k_r}} \binom{\lambda}{k_0}\binom{\lambda}{k_1}\cdots\binom{\lambda}{k_r}.$$

*Proof* From left hand side of (20), by using (4), we can write

$$\begin{aligned}
\sum_{m=0}^{\infty} H_\lambda(m,r,\alpha) z^m =& \frac{\left(-\log_\lambda(1-\frac{z}{\alpha})\right)^{r+1}}{1-z} \\
=& \frac{1}{1-z} \underbrace{\left(-\log_\lambda\left(1-\frac{z}{\alpha}\right)\right)\cdots\left(-\log_\lambda\left(1-\frac{z}{\alpha}\right)\right)}_{r+1 \text{ times}} \\
=& \sum_{m=0}^{\infty} z^m \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\lambda\alpha^m}\binom{\lambda}{m} z^m \cdots \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\lambda\alpha^m}\binom{\lambda}{m} z^m \\
=& \sum_{m=1}^{\infty} \sum_{1\leq k_0+\cdots+k_r\leq m} \frac{(-1)^{k_0+k_1+\cdots k_r+r+1}}{\lambda^{r+1}\alpha^{k_0+k_1+\cdots k_r}} \binom{\lambda}{k_0}\binom{\lambda}{k_1}\cdots\binom{\lambda}{k_r} z^m,
\end{aligned}$$

as claimed. □

**Theorem 5** *Let $m$ and $r$ be integers with $m, r \geq 0$. following identity holds:*

$$H_\lambda(m,r,\alpha) = \frac{1}{\lambda^{r+1}} \sum_{i=0}^{r+1} \sum_{\ell=0}^{m} \binom{r+1}{i}\binom{\lambda i}{\ell} \frac{(-1)^{\ell+i}}{\alpha^\ell}. \tag{21}$$

*Proof* From (4), (17), (20), and the binomial theorem, we write

$$\begin{aligned}
\sum_{m=0}^{\infty} H_\lambda(m,r,\alpha) z^m =& \frac{\left(-\log_\lambda(1-\frac{z}{\alpha})\right)^{r+1}}{1-z} = \frac{1}{1-z}\left(\frac{1-(1-\frac{z}{\alpha})^\lambda}{\lambda}\right)^{r+1} \\
=& \frac{1}{\lambda^{r+1}} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^i \frac{(1-\frac{z}{\alpha})^{\lambda i}}{1-z} \\
=& \frac{1}{\lambda^{r+1}} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^i \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} \binom{\lambda i}{\ell} \frac{(-1)^\ell}{\alpha^\ell} z^m \\
=& \sum_{m=0}^{\infty} \left( \frac{1}{\lambda^{r+1}} \sum_{i=0}^{r+1} \sum_{\ell=0}^{m} \binom{r+1}{i}\binom{\lambda i}{\ell} \frac{(-1)^{\ell+i}}{\alpha^\ell} \right) z^m.
\end{aligned} \tag{22}$$

By (22), we have the proof. □

**Corollary 6** *Let $m$ and $r$ be integers with $m, r \geq 0$. We have*

$$H_\lambda(m,r) = \frac{1}{\lambda^{r+1}} \sum_{\ell=0}^{r+1} (-1)^{m+\ell} \binom{r+1}{\ell}\binom{\lambda\ell-1}{m}. \tag{23}$$

*Proof* Substituting $\alpha = 1$ in (21) and using the identity

$$\sum_{i=0}^{k}(-1)^i\binom{r+1}{i} = (-1)^k\binom{r}{k},$$

we have (23), immediately. □

**Lemma 7** *Let $n$ and $m$ be integers with $n, m \geq 1$. We have*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^m = \begin{cases} 0, & \text{if } 1 \leq m < n \\ (-1)^n n!, & \text{if } m = n. \end{cases} \tag{24}$$

*Proof* For $1 \leq m < n$, from the binomial theorem, we have

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^m = \left(x\frac{d}{dx}\right)^m\left(\sum_{i=0}^{n}\binom{n}{i}x^i\right)\Bigg|_{x=-1} = \left(x\frac{d}{dx}\right)^m(1+x)^n\Big|_{x=-1} = 0,$$

and similarly, for $m = n$,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^n = \left(x\frac{d}{dx}\right)^n\left(\sum_{i=0}^{n}\binom{n}{i}x^i\right)\Bigg|_{x=-1} = \left(x\frac{d}{dx}\right)^n(1+x)^n\Big|_{x=-1} = (-1)^n n!,$$

as claimed. □

**Theorem 8** *Let $m$ and $r$ be integers with $m, r \geq 0$. We have*

$$m!H(m,r) = (-1)^{m+r+1}(r+1)!S_1(m+1, r+2).$$

*Proof* We know that $H(m,r) = 0$ when $r \leq m$. Assume that $m > r$. By (5), we get

$$\begin{aligned}
\lim_{\lambda\to 0}\frac{1}{\lambda^{r+1}}\binom{\lambda i-1}{n} &= \lim_{\lambda\to 0}\frac{1}{\lambda^{r+1}}\sum_{j=0}^{n} i^j\lambda^j\frac{S_1(n+1,j+1)}{n!} \\
&= \lim_{\lambda\to 0}\frac{1}{\lambda^{r+1}}\sum_{j=0}^{r} i^j\lambda^j\frac{S_1(n+1,j+1)}{n!} \\
&\quad + \lim_{\lambda\to 0}\frac{1}{\lambda^{r+1}} i^{r+1}\lambda^{r+1}\frac{S_1(n+1,r+1+1)}{n!} \\
&\quad + \lim_{\lambda\to 0}\frac{1}{\lambda^{r+1}}\sum_{j=r+2}^{n} i^j\lambda^j\frac{S_1(n+1,j+1)}{n!} \\
&= \lim_{\lambda\to 0}\frac{1}{\lambda^{r+1}}\sum_{j=0}^{r} i^j\lambda^j\frac{S_1(n+1,j+1)}{n!} \\
&\quad + \frac{S_1(n+1,r+2)i^{r+1}}{n!}.
\end{aligned} \tag{25}$$

By $\lim_{\lambda\to 0} H_\lambda(m,r) = H(m,r)$ and (23)–(25), we write

$$\begin{aligned}
H(m,r) &= \lim_{\lambda\to 0}\frac{1}{\lambda^{r+1}}\sum_{i=0}^{r+1}(-1)^{m+i}\binom{r+1}{i}\binom{\lambda i-1}{m} \\
&= \sum_{i=0}^{r+1}(-1)^{m+i}\binom{r+1}{i}\left(\lim_{\lambda\to 0}\frac{1}{\lambda^{r+1}}\sum_{j=0}^{r} i^j\lambda^j\frac{S_1(m+1,j+1)}{m!}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{S_1(m+1,r+2)i^{r+1}}{m!}\Big)\\
&=\lim_{\lambda\to 0}\frac{1}{\lambda^{r+1}}\sum_{j=0}^{r}\frac{S_1\left(m+1,j+1\right)\lambda^j}{m!}\sum_{i=0}^{r+1}(-1)^{m+i}\binom{r+1}{i}i^j\\
&\quad+\sum_{i=0}^{r+1}(-1)^{m+i}\binom{r+1}{i}\frac{S_1\left(m+1,r+2\right)i^{r+1}}{m!}\\
&=\frac{(-1)^m S_1\left(m+1,r+2\right)}{m!}\sum_{i=0}^{r+1}(-1)^i\binom{r+1}{i}i^{r+1}\\
&=\frac{(-1)^{m+r+1}(r+1)!S_1(m+1,r+2)}{m!},
\end{aligned}
$$

as claimed. □

**Theorem 9** *Let $n, m$ and $r$ be integers with $n, m, r \geq 0$. We have*

$$\sum_{k=0}^{n}H_\lambda(k,r,\alpha)H_\lambda(n-k,m,\alpha)=\sum_{k=0}^{n}H_{n-k,\lambda}(\alpha)H_\lambda(k,r+m,\alpha),$$

*and*

$$
\begin{aligned}
H_\lambda(n,r+1,\alpha)&=\sum_{k=0}^{n}\sum_{i=0}^{n-k}(-1)^{n-k-i}\binom{m}{n-k-i}H_{i,\lambda}^m(\alpha)H_\lambda(k,r,\alpha)\\
&=\sum_{k=1}^{n}H_\lambda(k-1,r,\alpha)\frac{(-1)^{n-k}D_{n-k,\lambda}}{\alpha^{n-k+1}(n-k)!}.
\end{aligned}
$$

*Proof* From (10) and (20), we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{n}H_\lambda(k,r,\alpha)H_\lambda(n-k,m,\alpha)z^n\\
&\quad=\frac{-\log_\lambda(1-\frac{z}{\alpha})}{1-z}\frac{\left(-\log_\lambda(1-\frac{z}{\alpha})\right)^{r+m+1}}{1-z}\\
&\quad=\sum_{n=0}^{\infty}H_{n,\lambda}(\alpha)z^n\sum_{n=0}^{\infty}H_\lambda(n,r+m,\alpha)z^n\\
&\quad=\sum_{n=0}^{\infty}\sum_{k=0}^{n}H_{n-k,\lambda}(\alpha)H_\lambda(k,r+m,\alpha)z^n,
\end{aligned}
$$

and from (11), (20), and the binomial theorem, we have

$$
\begin{aligned}
\frac{\left(-\log_\lambda(1-\frac{z}{\alpha})\right)^{r+2}}{1-z}&=\frac{\left(-\log_\lambda(1-\frac{z}{\alpha})\right)^{r+1}}{1-z}\frac{-\log_\lambda(1-\frac{z}{\alpha})}{(1-z)^m}(1-z)^m\\
&=\sum_{n=0}^{\infty}H_\lambda(n,r,\alpha)z^n\sum_{n=0}^{\infty}H_{n,\lambda}^m(\alpha)z^n\sum_{n=0}^{\infty}(-1)^n\binom{m}{n}z^n\\
&=\sum_{n=0}^{\infty}H_\lambda(n,r,\alpha)z^n\sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^{n-i}\binom{m}{n-i}H_{i,\lambda}^m(\alpha)z^n\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{n-k}(-1)^{n-k-i}\binom{m}{n-k-i}H_{i,\lambda}^m(\alpha)H_\lambda(k,r,\alpha)z^n,
\end{aligned}
$$

and

$$\begin{aligned}\frac{\left(-\log_\lambda(1-\frac{z}{\alpha})\right)^{r+2}}{1-z} &= \frac{\left(-\log_\lambda(1-\frac{z}{\alpha})\right)^{r+1}}{1-z}\frac{z}{\alpha}\frac{-\log_\lambda(1-\frac{z}{\alpha})}{\frac{z}{\alpha}}\\ &= \sum_{n=1}^{\infty} H_\lambda(n-1,r,\alpha)z^n \sum_{n=0}^{\infty}\frac{(-1)^n D_{n,\lambda}}{\alpha^{n+1}}\frac{z^n}{n!}\\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} H_\lambda(k-1,r,\alpha)\frac{(-1)^{n-k}D_{n-k,\lambda}}{\alpha^{n-k+1}(n-k)!}z^n.\end{aligned}$$

Therefore, by comparing the coefficients of the above equalities, we get the desired results. □

**Theorem 10** *Let $m$ and $r$ be integers with $m \geq r \geq 0$. We have*

$$H_\lambda(m,r,\alpha) = \sum_{\ell=r+1}^{m}\frac{(-1)^{\ell+r+1}S_{1,\lambda}(\ell,r+1)(r+1)!}{\ell!\alpha^\ell}.$$

*Proof* By (6), (10), and (20), we can write

$$\begin{aligned}\sum_{m=0}^{\infty} H_\lambda(m,r,\alpha)z^m &= \frac{-\log_\lambda(1-\frac{z}{\alpha})}{1-z}\left(-\log_\lambda\left(1-\frac{z}{\alpha}\right)\right)^r\\ &= \sum_{m=0}^{\infty} H_{m,\lambda}(\alpha)z^m \sum_{m=0}^{\infty}\frac{(-1)^{m+r}S_{1,\lambda}(m,r)r!}{\alpha^m}\frac{z^m}{m!}\\ &= \sum_{m=0}^{\infty}\sum_{\ell=0}^{m}\frac{(-1)^{\ell+r}S_{1,\lambda}(\ell,r)r!}{\ell!\alpha^\ell}H_{m-\ell,\lambda}(\alpha)z^m,\end{aligned}$$

From equality of two power series, we have the proof. □

Now, let define $n \times n$ type matrices as $A_n = [a_{i,j}]$, and $B_n = [b_{i,j}]$ with $a_{i,j} = H^j_{i,\lambda}(\alpha)$, and $b_{i,j} = H^j_{i-j+1,\lambda}(\alpha)$, respectively.

**Theorem 11** *For positive integer $n$, we have*

$$A_n = B_n P_n, \tag{26}$$

*where $P_n = [p_{i,j}]$ is $n \times n$ type matrix with $p_{i,j} = \binom{j-1}{i-1}$.*

*Proof* It is clearly seen that $b_{1,1}p_{1,1} = H^1_{1,\lambda}(\alpha) = a_{1,1}$ for $i = 1$ and $j = 1$. In next step,

$$\sum_{k=1}^{n} b_{1,k}p_{k,j} = \sum_{k=1}^{n} H^k_{2-k,\lambda}(\alpha)\binom{j-1}{k-1} = H^j_{1,\lambda}(\alpha) = a_{1,j}$$

for $i = 1$ and $j > 1$, and

$$\sum_{k=1}^{n} b_{i,k}p_{k,1} = \sum_{k=1}^{n} H^k_{i-k+1,\lambda}(\alpha)\binom{0}{k-1} = H^1_{i,\lambda}(\alpha) = a_{i,1}$$

for $i > 1$ and $j = 1$. Finally, by (13), we can write

$$\begin{aligned}\sum_{k=1}^{n} b_{i,k}p_{k,j} &= \sum_{k=1}^{j} H^k_{i-k+1,\lambda}(\alpha)\binom{j-1}{k-1} = \sum_{k=1}^{i} H^k_{i-k+1,\lambda}(\alpha)\binom{j-1}{k-1}\\ &= \sum_{k=1}^{i}\sum_{t=1}^{i-k+1}\binom{i-t}{k-1}\binom{j-1}{k-1}\frac{(-1)^{t-1}\lambda^{t-1}(1)_{t,1/\lambda}}{t!\alpha^t}\end{aligned}$$

$$= \sum_{k=1}^{i} \sum_{t=1}^{k} \binom{i-t}{i-k} \binom{j-1}{i-k} \frac{(-1)^{t-1} \lambda^{t-1} (1)_{t,1/\lambda}}{t! \alpha^t}$$
$$= \sum_{t=1}^{i} \frac{(-1)^{t-1} \lambda^{t-1} (1)_{t,1/\lambda}}{t! \alpha^t} \sum_{k=t}^{i} \binom{i-t}{k-t} \binom{j-1}{i-k}$$

for $i > 1$ and $j > 1$. From here, substituting $x = i - t$, $y = j - 1$, and $n = i$ in (1), we get

$$\sum_{k=1}^{n} b_{i,k} p_{k,j} = \sum_{t=1}^{i} \frac{(-1)^{t-1} \lambda^{t-1} (1)_{t,1/\lambda}}{t! \alpha^t} \binom{i+j-t-1}{j-1} = H_{i,\lambda}^{j}(\alpha),$$

as claimed. □

Now, we can give the following results of Theorem 11:

**Corollary 12** *Let n and m be integers with $n, m \geq 1$. Then, we have*

$$H_{n,\lambda}^{m}(\alpha) = \sum_{k=1}^{\min\{n,m\}} \binom{m-1}{k-1} H_{n-k+1,\lambda}^{k}(\alpha).$$

*Proof* From the equality of the terms $(n, m)$ of (26), the proof is clearly given. □

**Corollary 13** *For the $n \times n$ matrix $A_n$, we have*

$$\det(A_n) = \frac{1}{\alpha^n}.$$

*Proof* By (26), using determinant properties, we have

$$\det(A_n) = \det(B_n) \underbrace{\det(P_n)}_{=1} = \prod_{k=1}^{n} H_{1,\lambda}^{k}(\alpha) = \frac{1}{\alpha^n},$$

as claimed. □

Now, we define $C_n = [c_{i,j}]$ be an $n \times n$ matrix with

$$c_{i,j} = \begin{cases} \frac{(-1)^{i-j} \lambda^{i-j} (1)_{i-j+1,1/\lambda}}{\alpha^{i-j+1} (i-j+1)!} - \frac{(-1)^{i-j-1} \lambda^{i-j-1} (1)_{i-j,1/\lambda}}{(i-j)! \alpha^{i-j}}, & i > j, \\ \frac{1}{\alpha}, & i = j, \\ 0, & i < j. \end{cases}$$

**Theorem 14** *For positive integer n, we have*

$$A_n = C_n Q_n, \tag{27}$$

*where, the $n \times n$ Pascal matrix $Q_n = [q_{i,j}]$ with $q_{i,j} = \binom{i+j-1}{j}$.*

*Proof* It is clearly seen that $c_{1,1} q_{1,1} = H_{1,\lambda}^{1}(\alpha) = a_{1,1}$ for $i = 1$ and $j = 1$. In next step, since $c_{i,j} = 0$ when $i < j$, we have

$$\sum_{k=1}^{n} c_{1,k} q_{k,j} = c_{1,1} q_{1,j} = \frac{1}{\alpha} = H_{1,\lambda}^{j}(\alpha) = a_{1,j}$$

for $i = 1$ and $j > 1$ and

$$\sum_{k=1}^{n} c_{i,k} q_{k,1} = \left( \frac{(-1)^{i-1} \lambda^{i-1} (1)_{i,1/\lambda}}{\alpha^i i!} - \frac{(-1)^{i-2} \lambda^{i-2} (1)_{i-1,1/\lambda}}{\alpha^{i-1} (i-1)!} \right) \binom{1}{1}$$
$$+ \left( \frac{(-1)^{i-2} \lambda^{i-2} (1)_{i-1,1/\lambda}}{\alpha^{i-1} (i-1)!} - \frac{(-1)^{i-3} \lambda^{i-3} (1)_{i-2,1/\lambda}}{\alpha^{i-2} (i-2)!} \right) \binom{2}{1}$$

$$
\begin{aligned}
&+\left(\frac{(-1)^{i-3}\lambda^{i-3}\,(1)_{i-2,1/\lambda}}{\alpha^{i-2}(i-2)!}-\frac{(-1)^{i-4}\lambda^{i-4}\,(1)_{i-3,1/\lambda}}{\alpha^{i-3}\,(i-3)!}\right)\binom{3}{1}+\cdots\\
&+\left(\frac{(-1)^{1}\lambda^{1}\,(1)_{2,1/\lambda}}{\alpha^{2}2!}-\frac{(1)_{1,1/\lambda}}{\alpha}\right)\binom{i-1}{1}+\frac{1}{\alpha}\binom{i}{1}\\
=&\frac{(-1)^{i-1}\lambda^{i-1}\,(1)_{i,1/\lambda}}{\alpha^{i}i!}+\frac{(-1)^{i-2}\lambda^{i-2}\,(1)_{i-1,1/\lambda}}{\alpha^{i-1}\,(i-1)!}\\
&+\frac{(-1)^{i-3}\lambda^{i-3}\,(1)_{i-2,1/\lambda}}{\alpha^{i-2}(i-2)!}+\cdots+\frac{1}{\alpha}\\
=&\sum_{t=1}^{i}\frac{(-1)^{t-1}\lambda^{t-1}\,(1)_{t,1/\lambda}}{\alpha^{t}t!}=H_{i,\lambda}(\alpha)=H_{i,\lambda}^{1}(\alpha)=a_{i,1}
\end{aligned}
$$

for $i>1$ and $j=1$. By (13), and $\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$, we have

$$
\begin{aligned}
\sum_{k=1}^{n}c_{i,k}q_{k,j}=&\left(\frac{(-1)^{i-1}\lambda^{i-1}\,(1)_{i,1/\lambda}}{\alpha^{i}i!}-\frac{(-1)^{i-2}\lambda^{i-2}\,(1)_{i-1,1/\lambda}}{\alpha^{i-1}\,(i-1)!}\right)\binom{j}{j}\\
&+\left(\frac{(-1)^{i-2}\lambda^{i-2}\,(1)_{i-1,1/\lambda}}{\alpha^{i-1}\,(i-1)!}-\frac{(-1)^{i-3}\lambda^{i-3}\,(1)_{i-2,1/\lambda}}{\alpha^{i-2}(i-2)!}\right)\binom{j+1}{j}\\
&+\left(\frac{(-1)^{i-3}\lambda^{i-3}\,(1)_{i-2,1/\lambda}}{\alpha^{i-2}(i-2)!}-\frac{(-1)^{i-4}\lambda^{i-4}\,(1)_{i-3,1/\lambda}}{\alpha^{i-3}\,(i-3)!}\right)\binom{j+2}{j}\\
&+\cdots+\left(\frac{(-1)^{1}\lambda^{1}\,(1)_{2,1/\lambda}}{\alpha^{2}2!}-\frac{(1)_{1,1/\lambda}}{\alpha}\right)\binom{j+i-2}{j}\\
&+\frac{1}{\alpha}\binom{j+i-1}{j}\\
=&\frac{(-1)^{i-1}\lambda^{i-1}\,(1)_{i,1/\lambda}}{\alpha^{i}i!}\\
&+\frac{(-1)^{i-1}\lambda^{i-1}\,(1)_{i,1/\lambda}}{\alpha^{i}i!}+\binom{j}{j-1}\frac{(-1)^{i-2}\lambda^{i-2}\,(1)_{i-1,1/\lambda}}{\alpha^{i-1}\,(i-1)!}\\
&+\binom{j+1}{j-1}\frac{(-1)^{i-3}\lambda^{i-3}\,(1)_{i-2,1/\lambda}}{\alpha^{i-2}(i-2)!}+\cdots+\binom{j+i-2}{j-1}\frac{1}{\alpha}\\
=&\sum_{t=1}^{i}\binom{i+j-t-1}{j-1}\frac{(-1)^{t-1}\lambda^{t-1}\,(1)_{t,1/\lambda}}{\alpha^{t}t!}=H_{i,\lambda}^{j}(\alpha)=a_{i,j}
\end{aligned}
$$

for $i>1$ and $j>1$. So, the proof of the theorem is completed. □

**Corollary 15** *Let $n$ and $m$ be integers with $n,m\geq 1$, we have*

$$
\begin{aligned}
&\sum_{k=1}^{n-1}\binom{k+m-1}{m}\left(\frac{(-1)^{n-k}\,(1)_{n-k+1,1/\lambda}\,\lambda^{n-k}}{\alpha^{n-k+1}(n-k+1)!}-\frac{(-1)^{n-k-1}\,(1)_{n-k,1/\lambda}\,\lambda^{n-k-1}}{(n-k)!\alpha^{n-k}}\right)\\
&=H_{n,\lambda}^{m}(\alpha)-\binom{n+m-1}{m}\frac{1}{\alpha}.
\end{aligned}
$$

*Proof* From the equality of the terms $(n,m)$ of (27), the proof is clearly given. □

**Corollary 16** *Let n and m be integers with $n, m \geq 1$, we have*

$$\sum_{l=1}^{n-1}\sum_{k=1}^{m}(-1)^{k+m}\binom{l+k-1}{k}\binom{m-1}{k-1}$$
$$\left(\frac{(-1)^{n-l}\,\lambda^{n-l}\,(1)_{n-l+1,1/\lambda}}{\alpha^{n-l+1}(n-l+1)!}-\frac{(-1)^{n-l-1}\,\lambda^{n-l-1}\,(1)_{n-l,1/\lambda}}{(n-l)!\alpha^{n-l}}\right)$$
$$=H^{m}_{n-m+1,\lambda}(\alpha)-\frac{1}{\alpha}\binom{n}{m}.$$

*Proof* We can see $B_n = C_n Q_n P_n^{-1}$ from (26) and (27). Clearly, we calculate matrix $P_n^{-1} = \left[(-1)^{i+j}\binom{j-1}{i-1}\right]$ since

$$\sum_{k=1}^{n}(-1)^{k+j}\binom{k-1}{i-1}\binom{j-1}{k-1}=\delta_{i,j},$$

where $\delta_{i,j}$ is Kronecker delta. Thus, using matrix multiplication, we have the proof. □

## 3 Conclusions

Recently, some authors have been working on degenerate type numbers and some special numbers, such as degenerate Daehee numbers and degenerate Stirling numbers of the first kind, thanks to the degenerate exponential function. In this paper, we define the generalized degenerate harmonic numbers of rank $r$ , denoted by $H_\lambda(m, r, \alpha)$. Using combinatorial techniques, we examine various identities and sums involving binomial coefficients, degenerate Daehee numbers, degenerate Stirling numbers of the first kind, generalized degenerate harmonic numbers, generalized degenerate hyperharmonic numbers of order $r$, and generalized harmonic numbers of rank $r$. Additionally, we introduce matrices with terms derived from generalized hyperharmonic numbers and provide matrix equalities along with their results.

In conclusion, this study highlights promising directions for future research in applied mathematics, combinatorics, and number theory. We believe that the results presented in this paper lay the groundwork for guiding researchers toward new discoveries and innovative approaches in these vital fields of study.

## References


1. Carlitz, L.: Degenerate Stirling, Bernoulli and Eulerian numbers. Utilitas Math. 15, 51–88 (1979)
2. Carlitz, L.: A degenerate Staudt-Clausen theorem. Archiv der Mathematik 7, 28–33 (1956)
3. Kim, T., Kim, D., Kim, H., Jang, L.: Degenerate poly-bernoulli numbers and polynomials. Informatica 31(3), 2–8 (2020)
4. Kim, T., Kim, D.S.: Some identities on degenerate hyperharmonic numbers. Georgian Mathematical Journal 30(2), 255–262 (2023)
5. Kim, T., Kim, D.S.: Extended Stirling numbers of the first kind associated with Daehee numbers and polynomials. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 113(2), 1159–1171 (2019)
6. Park, J.W., Rim, S.H., Kwon, J.: The twisted Daehee numbers and polynomials. Advances in Difference Equations 2014(1), 1–9 (2014)
7. Kim, T., Kim, D.S., Kim, H.Y., Kwon, J.: Some results on degenerate Daehee and Bernoulli numbers and polynomials. Advances in Difference Equations 2020(1), 1–13 (2020)
8. Dağlı, M.C., Qi, F.: Several recurrence relations and identities on generalized derangement numbers. Results in Nonlinear Analysis 5(2), 1–6 (2022)
9. Dağlı, M.C.: A new generalization of hyperharmonic numbers. Annales Polonici Mathematici 129, 17–24 (2022)
10. Duran, Ö., Ömür, N., Koparal, S.: On sums with generalized harmonic, hyperharmonic and special numbers. Miskolc Mathematical Notes 21(2), 791–803 (2020)
11. Cheon, G.S., El-Mikkawy, M.E.A.: Generalized harmonic numbers with Riordan arrays. Journal of Number Theory 128, 413–425 (2008)
12. Genčev, M.: Binomial sums involving harmonic numbers. Mathematica Slovaca 61(2), 215–226 (2011)
13. Gertsch, A.: Nombres harmoniques généralisés and Generalized harmonic numbers. Academie des Sciences Paris Comptes Rendus Serie Sciences Mathematiques 324(1), 7–10 (1997)
14. Kim, T., Kim, D.S.: Some identities on degenerate hyperharmonic numbers (2022). arXiv:2205.10010
15. Ömür, N., Bilgin, G.: Some applications of the generalized hyperharmonic numbers of order $r$, $H_n^r(\alpha)$. Advances and Applications in Mathematical Sciences 17(9), 617–627 (2018)

16. Wang, W.S.: Sums of involving the harmonic numbers and the binomial coefficients. American Journal of Computational Mathematics 5, 96–105 (2015)
17. Kim, T., Kim, D.S.: On some degenerate differential and degenerate difference operators. Russian Journal of Mathematical Physics 29(1), 37–46 (2022)

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