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ORIGINAL RESEARCH

Pell polynomial solution of the fractional differential equations in the Caputo–Fabrizio sense

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Abstract In this paper, linear differential equations involving fractional and integer order derivatives are considered. Here fractional derivatives are defined in the Caputo–Fabrizio sense. A solution in the form of the truncated Pell series of the fractional differential equation is investigated. Firstly, the truncated Pell series solution is substituted into the fractional differential equation. Then, the collocation process leads to a system of linear equations. Finally, the unknown coefficients of the truncated Pell series are obtained by solving the linear system. The error and convergence analysis of the method is also presented. Additionally, the accuracy of the method is shown by numerical examples.

Keywords Pell polynomial solution · The Caputo–Fabrizio fractional derivative · Fractional differential equation

1 Introduction

The Caputo–Fabrizio fractional derivative is a new type of the fractional derivative which has been defined in 2015 [1]. The characteristic property of this derivative is that it has a nonsingular kernel. Fractional differential equations involving the Caputo–Fabrizio derivative have also been used to model many phenomena in physics, engineering and medicine such as the RLC circuit, the tuberculosis infection, the COVID-19 epidemic (see, for example, [2–5]).

Nowadays, solution methods of the fractional differential equations with the Caputo–Fabrizio derivative have been investigated by researchers [6–21]. The operational matrix of Caputo–Fabrizio operator based on the shifted Legendre polynomials has been obtained. The new operational matrix method has been applied to the fractional differential equations in Caputo–Fabrizio sense [6]. Fractional differential equations containing one-term derivative have been converted into fractional integral equations and the Simpson's 1/3 rule has been applied to the obtained integral equations [7]. The polynomial solutions of the fractional Painleve and Bagley–Torvik equations have been written by using high order cubic B-spline method [8]. The Haar wavelet collocation method has been extended to solve fractional differential equations containing one-term derivative [9]. A fractional logistic ordinary differential equation has been solved by converting into the classical ordinary differential equation [10]. Time fractional Fisher's reaction–diffusion equation has been solved by combining the Sumudu transform and an iteration method [11]. Adomian decomposition method has been applied to the a coupled system of linear fractional differential equations [12]. The Fokker–Planck equation with Caputo–Fabrizio fractional derivative has been solved by using the Ritz method [13]. The method based on the finite difference method

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and the Legendre-spectral method has been applied to the time-fractional diffusion equations with Caputo–Fabrizio derivative [14]. The solution of the time-fractional Cattaneo equation has been obtained by using a binary fractional reproducing kernel collocation method [15]. The continuous time random walk has been used to analyze the asymptotic behavior of the time-fractional diffusion equation [16]. The homotopy analysis and the Laplace transform methods have been applied to the time-fractional Sharma–Tasso–Olver–Burgers equation [17]. The solution of a Caputo–Fabrizio fractional order differential equations on time scales has been investigated [18]. The Caputo–Fabrizio fractional Volterra partial integro-differential equations have been studied by using the shifted Hahn polynomials and Legendre–Gauss–Lobatto quadrature method [19]. The coupled system of Caputo–Fabrizio fractional reaction advection–diffusion equations has been solved by using the orthonormal shifted discrete Legendre polynomials [20]. The Khalouta variational iteration method has been introduced for solving fractional partial differential equations [21].

Furthermore, some mathematical models including Caputo–Fabrizio derivative have also been solved by the known methods [22–26]. The Genocchi collocation method has been applied to the mathematical modeling of COVID-19 pandemic [22]. The fractional model for human liver has been solved by using homotopy analysis transform method [23]. Tumor Growth model has been investigated by using Lagrangian-piecewise interpolation method [24]. The Laplace transform and the Adomian decomposition approach have been applied to the tumor-immune model [25]. Caputo–Fabrizio fractional breast cancer model has been also investigated by using the Adams–Bashforth–Moulton method and simulation of the model in various fractional orders has been made [26].

In the recent years, methods based on the Pell polynomials have been applied to the fractional differential equations with Caputo derivative [27–30]. However, to our knowledge there is no study related with the Pell polynomial for solving fractional differential equations with the Caputo–Fabrizio derivative. The main motivation of this work is that the solutions of the multi-term fractional differential equations including the Caputo–Fabrizio derivative have been obtained by using the Pell polynomials.

In this paper, we will compute an approximate solution of the following problem

$$\sum_{i=1}^{r} a_i(x).{}^{CF}D^{\alpha_i}y(x) + \sum_{j=1}^{s} b_j(x).y^{(j)}(x) + b_0(x).y(x) = b(x),$$
$$0 \le x \le 1,\ \ \alpha_1 \ge ... \ge \alpha_r,\ \ m_k < \alpha_k < m_k + 1,\ \ k = 1, ..., r \tag{1}$$

with the mixed condition

$$a_i.y^{(i)}(0) + b_i.y^{(i)}(1) = l_i,\ \ i = 0, 1, ..., m,\ \ 0 < x < 1, \tag{2}$$

where $m_k (k = 1, ..., r)$ is a nonnegative integer number, $m = max\{m_1, s-1\}$; $a_k(x)(k = 1, ..., r)$ and $b_j(x)(j = 0, ..., s)$ are known continuous functions; l_i, a_i, $b_i (i = 0, ..., m)$ are given constants. ${}^{CF}D^{\alpha_k}(k = 1, ..., r)$ is the Caputo–Fabrizio fractional derivative of order α_k.

The paper is organized as follows: In Sect. 2, we introduce the fundamental concepts of the Caputo–Fabrizio fractional derivative and Pell polynomials. In Sect. 3, Pell collocation scheme is presented. The convergence analysis of the method is discussed in Sect. 4. Numerical examples is presented in Sect. 5. Finally, conclusion is given in Sect. 6.

2 Preliminaries and notations

2.1 The Caputo–Fabrizio fractional derivative

Definition 2.1 The Caputo–Fabrizio fractional derivative of order α of a function $f \in H^{(n+1)}(0, 1)$ is defined by (see, for example, [31])

$${}^{CF}D_x^{\alpha} f(x) = \frac{1}{1-\eta} \int_0^x f^{(n+1)}(t). \exp\left(- \frac{\eta}{1-\eta}(x-t)\right) dt$$

where $n < \alpha < n + 1$ and $\alpha = n + \eta, 0 < \eta < 1$ and n is a nonnegative integer number.

For a given integer $m \geq n+1$, the Caputo–Fabrizio derivative of order α of the t^m is given as follows [6]

$$ {}^{CF}D^{\alpha}t^{m} = \frac{1}{1-\eta}m!\left(\sum_{i=0}^{m-n-1}\frac{(-1)^{i}t^{m-n-i-1}}{(m-n-i-1)!\mu^{i+1}} + \exp(-\mu t)\frac{(-1)^{m-n}}{\mu^{m-n}}\right) \tag{3} $$

where $\mu = \frac{\eta}{1-\eta}$.

2.2 Some properties of the Pell polynomials

The recurrence form of the Pell polynomials is given by the following relation [32]:

$$ P_{j+2}(x) = 2xP_{j+1}(x) + P_{j}(x), \quad j \geq 0, \quad P_0(x) = 0, \quad P_1(x) = 1. $$

Remark 2 If m is a nonnegative integer, then the following inversion formula holds [33]:

$$ x^{m} = 2^{-m}\sum_{i=0}^{\lfloor m/2\rfloor}\frac{(-1)^{i}(m-2i+1)m!}{(m-i)!i!(m-i+1)}P_{m-2i+1}(x), \tag{4} $$

where $\lfloor j\rfloor$ is used to denote the largest integer less than or equal to j.

Theorem 2.1 *If $y(x)$ is an infinitely differentiable function, then it can be expressed in the form of the Pell polynomials* [27]

$$ y(x) = \sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{r}ky^{2r+k-1}(0)}{2^{2r+k-1}r!(r+k)!}P_{k}(x). \tag{5} $$

Lemma 2.2 *The following equalities hold for Pell polynomials for $m < \alpha < m+1$, $w \in Z^{+}$*

$$ \begin{aligned} {}^{CF}D^{\alpha}P_{n}(x) = {} & \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(n-k-1)!}{k!(1-\eta)}2^{n-2k-1}\left(\sum_{i=0}^{n-2k-2-m}\right. \\ & \frac{(-1)^{i}}{\Gamma(n-2k-1-m-i)2^{n-2k-2-m-i}\left(\frac{\eta}{1-\eta}\right)^{i+1}}\sum_{j=0}^{\lfloor\frac{n-2k-2-m-i}{2}\rfloor} \\ & \frac{(-1)^{j}(n-2k-2-m-i)!(n-2k-m-i-2j-1)}{(n-2k-2-m-i-j)!(n-2k-1-m-i-j)j!}P_{n-2k-m-i-2j-1}(x) \\ & \left.+\frac{(-1)^{n-2k-1-m}}{\left(\frac{\eta}{1-\eta}\right)^{n-2k-1-m}}\exp\left(-\frac{\eta}{1-\eta}x\right)\right), \quad n \geq m+2. \end{aligned} \tag{6} $$

$$ \begin{aligned} P_{n}^{(w)}(x) = {} & \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(n-k-1)!2^{w}}{k!}\sum_{i=0}^{\lfloor (n-2k-1-w)/2\rfloor} \\ & \frac{(-1)^{i}(n-2k-w-2i)}{i!(n-2k-w-i)!}P_{n-2k-w-2i}, \quad n \geq w+1. \end{aligned} \tag{7} $$

Proof The analytic form of the $P_j(x)$ is [33]

$$ P_{j}(x) = \sum_{r=0}^{\lfloor (j-1)/2\rfloor}\frac{(j-r-1)!}{(j-2r-1)!r!}2^{j-2r-1}x^{j-2r-1}. \tag{8} $$

Taking Caputo–Fabrizio fractional derivative of order α of the Eq. (8) and using Eq. (4) we have Eq. (6). Similarly, taking derivative of order w of the Eq. (8) and using Eq. (4) we have Eq. (7). □

Lemma 2.3 *The following equalities hold for Pell polynomials for* $\forall x \in [0, 1]$

$$|^{CF} D^{\alpha} P_n(x)| \leq K(\sqrt{6})^{n-m-1}, \ n \geq m+2, \ m < \alpha < m+1, \tag{9}$$

$$|P_n^{(w)}(x)| \leq M(\sqrt{6})^{n-w}, \ n \geq w+1, , \ w \in Z^+. \tag{10}$$

where K and M are positive constants.

Proof From [33], we can write

$$|P_n(x)| \leq (\sqrt{6})^{n-1}, \ n \geq 1, \ \forall x \in [0, 1]. \tag{11}$$

From Eqs. (6) and (11), we have

$$\begin{aligned} |^{CF} D^{\alpha} P_n(x)| &\leq K_1 |P_1(x)| + ... + K_{n-(m+1)} |P_{n-(m+1)}(x)| \\ &\leq K_1(\sqrt{6})^0 + ... + K_{n-(m+1)}(\sqrt{6})^{n-m-2} \\ &\leq K\frac{(\sqrt{6})^{n-m-1} - 1}{\sqrt{6} - 1} \\ &\leq K(\sqrt{6})^{n-m-1}, \end{aligned}$$

where $K_1, K_2, ..., K_{n-(m+1)}$ are positive constants and $K = max\{K_1, K_2, ..., K_{n-(m+1)}\}$.

From Eqs. (7) and (11), we have

$$\begin{aligned} |P_n^{(w)}(x)| &\leq M_1 |P_1(x)| + ... + M_{n-w} |P_{n-w}(x)| \\ &\leq M_1(\sqrt{6})^0 + ... + M_{n-w}(\sqrt{6})^{n-w-1} \\ &\leq M\frac{(\sqrt{6})^{n-w} - 1}{\sqrt{6} - 1} \\ &\leq M(\sqrt{6})^{n-w}, \end{aligned}$$

where $M_1, M_2, ..., M_{n-w}$ are positive constants and $M = max\{M_1, M_2, ..., M_{n-w}\}$. □

Theorem 2.4 *If* $y(x)$ *is defined on* $[0, 1]$ *and* $|y^{(i)}(x)| \leq \lambda^i, \ i \geq 0$, *where* λ *is a positive constant and if* $y(x)$ *has the expansion* $\sum_{k=1}^{\infty} c_k.P_k(x)$, *then*

$$|c_k| \leq \frac{\lambda^{k-1}}{2^{k-1}(k-1)!} cosh(\lambda), \tag{12}$$

and the series is absolutely convergent.

Proof The proof of the inequality (12) can be seen from [27]. From (11) and (12), we can write

$$|\sum_{k=1}^{\infty} c_k.P_k(x)| \leq cosh\lambda.\sum_{k=1}^{\infty} \frac{(\lambda\sqrt{6}/2)^{k-1}}{(k-1)!} = cosh\lambda.e^{\lambda\sqrt{6}/2}. \tag{13}$$

□

3 Numerical scheme

In this section, an approximate solution of the problem (1)–(2) is investigated in the form of the Pell polynomials. New matrices forms for the Caputo–Fabrizio and integer order derivatives of the unknown function which lead to the linear algebraic system are obtained.

Assume that the approximate solution of the problem (1)–(2) can be written in the form of

$$y_N(x) = \sum_{k=1}^{N} c_k P_k(x). \tag{14}$$

Substituting Eq. (14) into Eq. (1), collocating the obtained equation at points $x_p (p = 1, ..., N - (m + 1))$, we have

$$\sum_{i=1}^{r} a_i(x_p).{}^{CF}D^{\alpha_i} y_N(x_p) + \sum_{j=0}^{s} b_j(x_p).(y_N(x_p))^{(j)} = b(x_p). \tag{15}$$

First, let us consider the first term in the Eq. (15). Using the collocation points $x_p (p = 1, ..., N - (m + 1))$, the first term in the Eq. (15) can be written as follows

$$\sum_{i=1}^{r} \begin{bmatrix} a_i(x_1) & 0 & ... & 0 \\ 0 & a_i(x_2) & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & a_i(x_{N-(m+1)}) \end{bmatrix} . \begin{bmatrix} {}^{CF}D^{\alpha_i} y_N(x_1) \\ {}^{CF}D^{\alpha_i} y_N(x_2) \\ . \\ {}^{CF}D^{\alpha_i} y_N(x_{N-(m+1)}) \end{bmatrix}$$

and

$$\begin{bmatrix} {}^{CF}D^{\alpha_i} y_N(x_1) \\ {}^{CF}D^{\alpha_i} y_N(x_2) \\ . \\ {}^{CF}D^{\alpha_i} y_N(x_{N-(m+1)}) \end{bmatrix} = \begin{bmatrix} {}^{CF}D^{\alpha_i} P_1(x_1) & {}^{CF}D^{\alpha_i} P_2(x_1) & ... & {}^{CF}D^{\alpha_i} P_N(x_1) \\ {}^{CF}D^{\alpha_i} P_1(x_2) & {}^{CF}D^{\alpha_i} P_2(x_2) & ... & {}^{CF}D^{\alpha_i} P_N(x_2) \\ . & . & ... & . \\ {}^{CF}D^{\alpha_i} P_1(x_{N-(m+1)}) & {}^{CF}D^{\alpha_i} P_2(x_{N-(m+1)}) & ... & {}^{CF}D^{\alpha_i} P_N(x_{N-(m+1)}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ . \\ c_N \end{bmatrix}.$$

Since ${}^{CF}D^{\alpha_i} P_n(x) = 0$ for $n < m_i + 2$, $(m_i < \alpha_i < m_i + 1)$ from Lemma 2.2, we have the following matrix representation for the first term in the Eq. (15)

$$\sum_{i=1}^{r} \mathcal{A}_i \mathbf{P}_{\alpha_i} \mathbf{Y}, \tag{16}$$

where $\mathbf{Y} = [c_1, c_2, ..., c_N]^*$, $*$ denotes the sign of the transposition,

$$\mathcal{A}_i = \begin{bmatrix} a_i(x_1) & 0 & ... & 0 \\ 0 & a_i(x_2) & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & a_i(x_{N-(m+1)}) \end{bmatrix}_{(N-(m+1))\times(N-(m+1))},$$

$$\mathbf{P}_{\alpha_i} = \begin{bmatrix} 0 \ ... \ 0 & D^{\alpha_i} P_{m_i+2}(x_1) & ... & D^{\alpha_i} P_N(x_1) \\ 0 \ ... \ 0 & D^{\alpha_i} P_{m_i+2}(x_2) & ... & D^{\alpha_i} P_N(x_2) \\ 0 \ ... \ 0 & . & ... & . \\ 0 \ ... \ 0 & D^{\alpha_i} P_{m_i+2}(x_{N-(m+1)}) & ... & D^{\alpha_i} P_N(x_{N-(m+1)}) \end{bmatrix}_{(N-(m+1))\times N}.$$

Now, let us consider the second term in the Eq. (15). After the collocation process at the $x_p (p = 1, ..., N-(m+1))$, the second term in the Eq. (15) can be written as follows

$$\sum_{j=0}^{s} \begin{bmatrix} b_j(x_1) & 0 & ... & 0 \\ 0 & b_j(x_2) & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & b_j(x_{N-(m+1)}) \end{bmatrix} . \begin{bmatrix} (y_N)^{(j)}(x_1) \\ (y_N)^{(j)}(x_2) \\ . \\ (y_N)^{(j)}(x_{N-(m+1)}) \end{bmatrix}$$

and

$$\begin{bmatrix} (y_N)^{(j)} x_1) \\ (y_N)^{(j)}(x_2) \\ . \\ (y_N)^{(j)}(x_{N-(m+1)}) \end{bmatrix} =$$

$$\begin{bmatrix} (P_1)^{(j)}(x_1) & (P_2)^{(j)}(x_1) & ... & (P_N)^{(j)}(x_1) \\ (P_1)^{(j)}(x_2) & (P_2)^{(j)}(x_2) & ... & (P_N)^{(j)}(x_2) \\ . & . & ... & . \\ (P_1)^{(j)}(x_{N-(m+1)}) & (P_2)^{(j)}(x_{N-(m+1)}) & ... & (P_N)^{(j)}(x_{N-(m+1)}) \end{bmatrix} \mathbf{Y}.$$

Since $(P_n)^{(j)}(x) = 0$ for $n < j+1$ from Lemma 2.2, we have the following matrix representation for the second term in the Eq. (15)

$$\sum_{j=0}^{s} \mathcal{B}_j \mathbf{P}_j \mathbf{Y}, \tag{17}$$

where

$$\mathcal{B}_j = \begin{bmatrix} b_j(x_1) & 0 & ... & 0 \\ 0 & b_j(x_2) & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & b_j(x_{N-(m+1)}) \end{bmatrix}_{(N-(m+1))\times(N-(m+1))},$$

$$\mathbf{P}_j = \begin{bmatrix} 0 \ ... \ 0 & (P_{j+1})^{(j)}(x_1) & ... & (P_N)^{(j)}(x_1) \\ 0 \ ... \ 0 & (P_{j+1})^{(j)}(x_2) & ... & (P_N)^{(j)}(x_2) \\ 0 \ ... \ 0 & . & ... & . \\ 0 \ ... \ 0 & (P_{j+1})^{(j)}(x_{N-(m+1)}) & ... & (P_N)^{(j)}(x_{N-(m+1)}) \end{bmatrix}_{(N-(m+1))\times N},$$

$$(P_j)^{(k)}(x) = \sum_{i=0}^{\lfloor (j-1)/2 \rfloor} \frac{(j-i-1)!2^{j-2i-1}}{(j-2i-1-k)!(i)!} x^{j-2i-1-k}, \ \ j \geq k. \tag{18}$$

From (16) and (17), the matrix representation of Eq. (15) is

$$\Big(\sum_{i=1}^{r} \mathcal{A}_i \mathbf{P}_{\alpha_i} + \sum_{j=0}^{s} \mathcal{B}_j \mathbf{P}_j \Big) \mathbf{Y} = \mathbf{B}, \tag{19}$$

where $\mathbf{B} = [b(x_1), b(x_2), ..., b(x_{N-(m+1)})]^*$.

Finally, let us consider the mixed conditions (2). Substituting Eq. (14) into the mixed conditions (2), we have the following matrix representation

$$\mathcal{C}.\mathbf{Y} = \mathbf{D}, \tag{20}$$

where

$$\mathbf{D} = [l_0, l_1, ..., l_m]^*,$$

$$\mathcal{C} = \begin{bmatrix} a_0 P_1(0) & a_0 P_2(0) & ... & a_0 P_N(0) \\ a_1 (P_1)'(0) & a_1 (P_2)'(0) & ... & a_1 (P_N)'(0) \\ . & . & ... & . \\ a_m (P_1)^{(m)}(0) & a_m (P_2)^{(m)}(0) & ... & a_m (P_N)^{(m)}(0) \end{bmatrix}_{(m+1)\times N}$$

$$+ \begin{bmatrix} b_0 P_1(1) & b_0 P_2(1) & ... & b_0 P_N(1) \\ b_1 (P_1)'(1) & b_1 (P_2)'(1) & ... & b_1 (P_N)'(1) \\ . & . & ... & . \\ b_m (P_1)^{(m)}(1) & b_m (P_2)^{(m)}(1) & ... & b_m (P_N)^{(m)}(1) \end{bmatrix}_{(m+1)\times N}.$$

Combining Eqs. (19) and (20), we have the following matrix representation as follows

$$\mathcal{A}.\mathbf{Y} = \mathbf{F} \tag{21}$$

where

$$\mathcal{A} = \begin{bmatrix} \Big(\sum_{i=1}^{r} \mathcal{A}_i \mathbf{P}_{\alpha_i} + \sum_{j=0}^{s} \mathcal{B}_j \mathbf{P}_j \Big) \\ \mathcal{C} \end{bmatrix}_{(N\times N)}, \mathbf{F} = \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix}_{(N\times 1)}. \tag{22}$$

Solving the linear equations of system, the unknown terms $c_1, ..., c_N$ in Eq. (14) are obtained.

4 Error analysis

In this section, we investigate the convergence of the presented method. Assume that $y(x) = \sum_{k=1}^{\infty} c_k P_k(x)$ is the exact solution of the Eq. (1). If $y_N(x)$ is the approximate solution of Eq. (1), $y_N(x)$ satisfies the following equation

$$\sum_{i=1}^{r} a_i(x).{}^{CF}D^{\alpha_i} y_N(x) + \sum_{j=0}^{s} b_j(x).(y_N(x))^{(j)} = b(x) + R_N(x), \tag{23}$$

where $R_N(x)$ is the residual function. Let us take $e_N(x) = y(x) - y_N(x)$. Then following inequality is obtained

$$\mid R_N(x) \mid \leq \sum_{i=1}^{r} \mid a_i(x) \mid\mid {}^{CF}D^{\alpha_i} e_N(x) \mid + \sum_{j=0}^{s} \mid b_j(x) \mid\mid e_N^{(j)}(x) \mid . \tag{24}$$

Using (11) and (12), we can obtain the following inequality

$$\mid e_N(x) \mid \leq cosh\lambda e^{\lambda\sqrt{6}/2} \frac{(\lambda\sqrt{6}/2)^N}{(N)!}. \tag{25}$$

The proof of the inequality (25) can similarly be seen from [27].

For $l = 1, 2, ..., r$ using (9) and (12) we have

$$\begin{aligned} \mid e_N^{(\alpha_l)}(x) \mid &\leq \sum_{k=N+1}^{\infty} \mid c_k.{}^{CF}D^{\alpha_l} P_k(x) \mid \\ &\leq \frac{cosh\lambda.K}{(\sqrt{6})^{m_l}} \sum_{k=N+1}^{\infty} \frac{(\lambda\sqrt{6}/2)^{k-1}}{(k-1)!} \\ &\leq \frac{cosh\lambda.K}{(\sqrt{6})^{m_l}} e^{\lambda\sqrt{6}/2} \frac{(\lambda\sqrt{6}/2)^N}{(N)!}. \end{aligned} \tag{26}$$

Similarly for $w = 1, 2, ..., s$ using (10) and (12) we have

$$\begin{aligned} \mid e_N^{(w)}(x) \mid &\leq \sum_{k=N+1}^{\infty} \mid c_k.P_k^{(w)}(x) \mid \\ &\leq \frac{cosh\lambda.M}{(\sqrt{6})^{w-1}} \sum_{k=N+1}^{\infty} \frac{(\lambda\sqrt{6}/2)^{k-1}}{(k-1)!} \\ &\leq \frac{cosh\lambda.M}{(\sqrt{6})^{w-1}} e^{\lambda\sqrt{6}/2} \frac{(\lambda\sqrt{6}/2)^N}{(N)!}. \end{aligned} \tag{27}$$

Substituting inequalities (25), (26) and (27) into (24), we have

$$\begin{aligned} \mid R_N(x) \mid &\leq cosh\lambda.e^{\lambda\sqrt{6}/2} \frac{(\lambda\sqrt{6}/2)^N}{(N)!} \left[\sum_{i=1}^{r} A_i.\frac{K}{(\sqrt{6})^{m_i}} + \sum_{j=1}^{s} B_j.\frac{M}{(\sqrt{6})^{j-1}} + B_0 \right] \\ &\leq C.e^{\lambda\sqrt{6}/2} \frac{(\lambda\sqrt{6}/2)^N}{(N)!}, \end{aligned} \tag{28}$$

where $A_i (i = 1, ..., r)$ and $B_j (j = 0, 1, ..., s)$ are positive constants such that for $x \in [0, 1]$, $\mid a_i(x) \mid \leq A_i (i = 1, ..., r)$, $\mid b_j(x) \mid \leq B_j (j = 0, 1, ..., s)$ and $C = \sum_{i=1}^{r} A_i.\frac{K}{(\sqrt{6})^{m_i}} + \sum_{j=1}^{s} B_j.\frac{M}{(\sqrt{6})^{j-1}} + B_0$. Therefore, if N approaches to the ∞ in (25), $R_N(x)$ goes to the zero.

Table 1 The absolute errors for $y_N(x)$, $N = 6, 8, 9, 10$ in Example 4

x	$N = 6$	$N = 8$	$N = 9$	$N = 10$	$N = 6$ [6]
0.1	2.10^{-8}	2.10^{-10}	3.10^{-11}	2.10^{-12}	2.10^{-7}
0.2	10^{-8}	10^{-10}	2.10^{-11}	2.10^{-12}	2.10^{-7}
0.3	5.10^{-8}	3.10^{-10}	2.10^{-11}	2.10^{-12}	2.10^{-7}
0.4	10^{-8}	6.10^{-10}	8.10^{-12}	2.10^{-12}	5.10^{-7}
0.5	5.10^{-8}	3.10^{-10}	10^{-11}	2.10^{-12}	4.10^{-7}
0.6	6.10^{-8}	5.10^{-11}	10^{-11}	2.10^{-12}	2.10^{-7}
0.7	3.10^{-7}	5.10^{-10}	4.10^{-11}	2.10^{-12}	7.10^{-7}
0.8	6.10^{-7}	10^{-9}	3.10^{-11}	10^{-12}	5.10^{-7}
0.9	6.10^{-6}	6.10^{-10}	6.10^{-11}	10^{-13}	4.10^{-7}
1	2.10^{-5}	9.10^{-9}	7.10^{-10}	10^{-11}	2.10^{-13}

Table 2 The absolute errors for $y_N(x)$, $N = 9, 10, 11$ in Example 5

x	$N = 9$	$N = 10$	$N = 11$
0.1	2.10^{-12}	9.10^{-13}	3.10^{-16}
0.2	10^{-11}	2.10^{-12}	4.10^{-14}
0.3	10^{-11}	4.10^{-12}	6.10^{-14}
0.4	10^{-11}	5.10^{-12}	7.10^{-14}
0.5	2.10^{-11}	7.10^{-12}	6.10^{-14}
0.6	2.10^{-11}	7.10^{-12}	2.10^{-14}
0.7	3.10^{-11}	4.10^{-12}	9.10^{-14}
0.8	3.10^{-11}	4.10^{-12}	2.10^{-13}
0.9	2.10^{-9}	4.10^{-11}	3.10^{-13}
1	10^{-8}	2.10^{-11}	5.10^{-13}

Table 3 The absolute errors for $y_N(x)$, $N = 8, 9, 10, 11$ in Example 6

x	$N = 8$	$N = 9$	$N = 10$	$N = 11$
0.1	2.10^{-9}	9.10^{-10}	2.10^{-11}	2.10^{-11}
0.2	2.10^{-9}	4.10^{-10}	3.10^{-11}	10^{-11}
0.3	3.10^{-9}	2.10^{-10}	3.10^{-11}	4.10^{-13}
0.4	3.10^{-9}	2.10^{-10}	5.10^{-11}	3.10^{-12}
0.5	5.10^{-9}	2.10^{-10}	6.10^{-11}	4.10^{-12}
0.6	10^{-8}	3.10^{-10}	6.10^{-11}	10^{-11}
0.7	2.10^{-7}	2.10^{-10}	5.10^{-11}	3.10^{-12}
0.8	6.10^{-7}	10^{-9}	7.10^{-11}	2.10^{-11}
0.9	10^{-6}	5.10^{-9}	2.10^{-11}	8.10^{-11}
1	6.10^{-8}	2.10^{-8}	2.10^{-9}	6.10^{-12}

5 Applications

Example 1 Let us consider the following initial value problem

$$
\begin{aligned}
&{}^{CF}D^{3.8}y - \exp(x)y''' + x.{}^{CF}D^{1.5}y - x^2.{}^{CF}D^{0.3}y + y = f(x),\\
&y(0) = 0,\ \ y'(0) = 1,\ \ y''(0) = -6,\ \ y'''(0) = 0,\\
&f(x) = \exp(-4x)\frac{75}{2} - 101x + 252x\exp(-x) + \frac{964450x^2}{81}\exp(-3x/7)\\
&\qquad -\frac{945253x^2}{81} + \frac{134500x^3}{27} - \frac{9440x^4}{4} + \frac{1409x^5}{9} - \frac{50x^6}{3} - \frac{75}{2} - 60x^2\exp(x).
\end{aligned}
$$

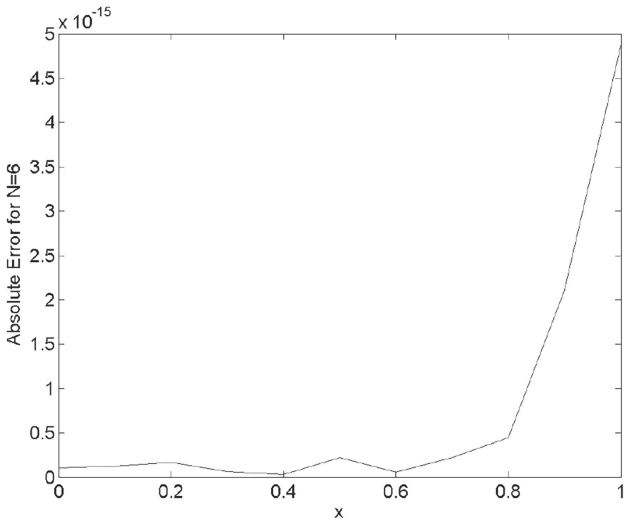


Fig. 1 The absolute error for $y_6(x)$ in Example 1

The exact solution is $y_e(x) = x^5 - 3x^2 + x$. Taking the collocation points as $x_1 = 0.3$, $x_2 = 0.5$ for $N = 6$, the matrixes and vectors in Eq. (22) are computed as follows

$$\mathcal{A}_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.5 \end{bmatrix}, \mathcal{A}_3 = \begin{bmatrix} -0.09 & 0 \\ 0 & -0.25 \end{bmatrix}, \mathcal{B}_3 = \begin{bmatrix} -\exp(0.3) & 0 \\ 0 & -\exp(0.5) \end{bmatrix},$$

$$\mathbf{B} = [10.624605, 15.626565]^*, \ \mathbf{D} = [0, 1, -6, 0]^*,$$

$$\mathbf{P}_{\alpha_1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 335.426778 & 601.433054 \\ 0 & 0 & 0 & 0 & 415.039064 & 1362.402340 \end{bmatrix},$$
$$\mathbf{P}_{\alpha_2} = \begin{bmatrix} 0 & 0 & 1.244073 & 1.175565 & 4.6956100 & 5.435439 \\ 0 & 0 & 3.147755 & 5.113472 & 16.535491 & 29.531620 \end{bmatrix},$$
$$\mathbf{P}_{\alpha_3} = \begin{bmatrix} 0 & 0.072390 & 0.044364 & 0.171680 & 0.149336 & 0.334561 \\ 0 & 0.321470 & 0.332943 & 0.981743 & 1.341195 & 2.664386 \end{bmatrix},$$
$$\mathbf{P}_0 = \begin{bmatrix} 1 & 0.6 & 1.36 & 1.416 & 2.2096 & 2.74176 \\ 1 & 1 & 2 & 3 & 5 & 8 \end{bmatrix},$$
$$\mathbf{P}_3 = \begin{bmatrix} 0 & 0 & 0 & 64.7932 & 155.5037 & 492.4285 \\ 0 & 0 & 0 & 79.1386 & 316.5545 & 1107.9407 \end{bmatrix}, \mathcal{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 & 0 & 6 \\ 0 & 0 & 8 & 0 & 24 & 0 \\ 0 & 0 & 0 & 48 & 0 & 192 \end{bmatrix},$$

$\mathcal{A}_1$ and $\mathcal{B}_0$ are unit matrices, $\mathcal{B}_1$ and $\mathcal{B}_2$ are zero matrices. The obtained absolute errors for $N = 6$ are presented in Fig. 1. Generally, the absolute errors are computed as 10^{-15} and 10^{-17}. These results show that the presented method has high accuracy for even small values of the N.

Example 2 Consider fractional Bagley–Torvik equation

$$y'' + {}^{CF}D^{0.5}y + y = -4 + 3x + x^2 + 6\exp(-x), \ \ 0 < x < 1,$$
$$y(1) = 0, \ \ y'(0) = -1. \tag{29}$$

By using the presented method for $N = 3$, the series coefficients c_k, $k = 1, 2, 3$, are computed as follows

$$c_1 = \frac{-1}{4}, \ \ c_2 = \frac{-1}{2}, \ \ c_3 = \frac{1}{4}.$$

Substituting the coefficients c_k, $(k = 1, 2, 3)$ into the (14), we have

$$y_3(x) = x^2 - x$$

which is the exact solution. The initial value problem (29) has been solved by using the high order cubic B-spline method for $N = 3$ [8] and the maximum absolute error has been obtained as 10^{-12}. Therefore, we can conclude that our method has higher accuracy than the higher order cubic B-spline method.

Example 3 Consider the following initial value problem

$$\begin{aligned} {}^{CF}D^{0.5}y(x) &= 6x^2 - 18x + 19 - 19\exp(-x), \quad 0 < x < 1, \\ y(0) &= 0. \end{aligned} \tag{30}$$

The exact solution is $y_e(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$. By using the presented method for $N = 4$, the series coefficients $c_k, k = 1, ..., 4$ are computed as follows

$$c_1 = \frac{375}{1000}, \quad c_2 = 0, \quad c_3 = -\frac{375}{1000}, \quad c_4 = \frac{125}{1000}.$$

Substituting the coefficients c_k, $(k = 1, 2, 3, 4)$, into the (14), we have the exact solution of the problem (30). The initial value problem (30) has been solved by using the Legendre operational matrix method for $N = 6$ [6] and the maximum absolute error has been computed as 10^{-23}. This means that our method has higher accuracy than the Legendre operational matrix method even for small values of N.

Example 4 Let us consider the following problem

$$\begin{aligned} {}^{CF}D^{1.2}y &= 20x^3 - 240x^2 + 1920x - 7680 + \frac{130544}{17}\exp(-x/4) + \frac{16}{17}\cos(x) \\ &\quad -\frac{4}{17}\sin(x), \quad 0 < x < 1, \\ y(0) &= 0, \quad y'(0) = 0. \end{aligned} \tag{31}$$

The exact solution is $y_e(x) = \frac{x^5}{5} + \frac{4}{5}\sin(x)$. By using the presented method, the absolute errors for $N = 6, 8, 9, 10$ are computed. The same problem has been solved by using the Legendre operational matrix method for $N = 6$ in [6]. In Table 1, the obtained absolute errors from the presented method and the Legendre operational matrix method are given. The Table shows that the solution of the Pell collocation method has better accuracy than the solution of the Legendre operational matrix method for $x \neq 1$. Furthermore, increasing the value of the N leads to the increasing accuracy.

Example 5 Let us consider

$$\begin{aligned} &{}^{CF}D^{2.5}y + x.{}^{CF}D^{0.8}y + \sin(x)y' + \cos(x)y = \exp(-x) - \cos(x) + \sin(x) \\ &-\frac{5}{17}x\exp(-4x) + 2\cos^2(x) + \frac{5}{17}x\cos(x) - \frac{20}{17}x\sin(x) - 1, \\ &y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1. \end{aligned}$$

The exact solution is $y_e(x) = \cos(x)$. By using the presented method, the absolute errors for $N = 9, 10, 11$ are given in Table 2. As the value of the N increases, the approximate solution gets closer to the exact solution.

Example 6 Let us consider

$$\begin{aligned} &{}^{CF}D^{1.5}y - \exp(x).{}^{CF}D^{0.4}y + x.{}^{CF}D^{0.2}y + x^2y = f(x), \\ &y(0) = 0, \quad y'(0) = 1, \\ &f(x) = \frac{1}{10}[15\exp(x) - 4\exp(2x) + 4\exp(x/3) + 12x\exp(x) - 10x\exp(2x) \\ &\quad -2x\exp(-x/4) + 10x^2\exp(x) + 10x^3\exp(x) - 15\exp(-x)] \end{aligned} \tag{32}$$

The exact solution is $y_e(x) = x\exp(x)$. By using the presented method for $N = 8, 9, 10, 11$, the absolute errors are presented in Table 3. These results show that the approximate solution goes to the exact solution when the value of the N increases.

6 Conclusion

In this paper, for the first time, the Pell collocation method has been applied to the multi-term, linear, fractional differential equations including Caputo–Fabrizio and integer order derivatives with mixed conditions. After the collocation procedure, new matrix representations for the Caputo–Fabrizio and integer order derivatives of the unknown function in the form of the truncated Pell series have been written and the problem has been transformed into an algebraic system of linear equations. Thus, these procedure lead to the greatly simplified problem and computer programming has been easily applied to the problem. Thanks to the computer programming, more accurate solution has been obtained in a short time. The convergence analysis of the method has been investigated in Sect. 4. In Sect. 5, numerical examples show that the method can be easily applied to the problem (1)–(2) and has high accuracy. In addition, when compared with existing methods in the literature such as the high-order cubic B-spline method, the Legendre operational matrix method, the accuracy of the presented method is higher than other methods.

There are many separately written articles in the literature investigating the solution of linear differential equations with Caputo–Fabrizio derivative for special case such as [8,9,31]. Since problem (1) is the most general form of the problems studied in many articles, its applicability is higher. The method can also be extended to fractional partial differential equations with Caputo–Fabrizio derivative.

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