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ORIGINAL RESEARCH

Some remarks on the stability of approximate pseudospectrum

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Abstract In this note, we study the stability of the approximate pseudospectrum correspondences by investigating its continuity nature.

Keywords Pseudospectrum · Approximate pseudospectrum · Upper and lower hemicontinuous

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1 Introduction

Let $B(X)$ be the space of all bounded linear operators acting on a complex Banach space X. The identity operator defined on X is denoted by I. The spectrum of an element $T \in B(X)$ is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in B(X)}\}.$$

The complement of $\sigma(T)$ is known as the resolvent of T which is denoted by $\rho(T)$. One can naturally think about the stability of the spectrum of an operator. The stability of the spectrum is connected with the continuity nature of the map $T \mapsto \sigma(T)$. In Theorem 1 of [1] it is proved that the spectrum map is upper semicontinuous in any Banach algebra $\mathcal{A}$ and examples can be given to illustrate the map is not lower semicontinuous in general. There are many generalizations available on the spectrum. One among them is the concept of pseudospectrum.

Let $T \in B(X)$ and $\varepsilon > 0$. The ε-pseudospectrum of T is denoted by $\Lambda_\varepsilon(T)$ and is defined as

$$\Lambda_\varepsilon(T) = \left\{\lambda \in \mathbb{C} : \|(T - \lambda I)^{-1}\| \geq \frac{1}{\varepsilon}\right\}$$

by convention $\|(T-\lambda I)^{-1}\| = +\infty$ if, and only if, $\lambda \in \sigma(T)$. The notion of the pseudospectra is an innovative tool for learning about matrices and linear operators. Pseudospectra provides an analytical and graphical approach for studying issues with non-normal matrices and operators. One should note that, Pseudospectra have nevertheless shown to be an effective method for studying them. J. M. Varah [7] first proposed the concept of pseudospectra

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in 1967. L. N. Trefethen pioneered the research of pseudospectrum for matrices and operators during the 1990s (see [8,9] and [10]).

Like spectrum map and its stability, one can think of the pseudospectrum map and its stability. The article [5], discusses the stability and related results for pseudospectrum map of a Banach algebra element.

In our article, we focus on stability nature of the approximate pseudospectrum map. The article [11] provides the impetus for discussing stability. E. B. Davies [6], and M. P. H. Wolff [2] introduced the approximate pseudospectrum for linear operators. The following is the definition of approximate pseudospectrum.

Definition 1 (*Page 4 in* [2]) Let $T \in B(X)$ and $\varepsilon > 0$. The approximate pseudospectrum is defined by

$$\Sigma_{ap,\varepsilon}(T) = \left\{\lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T-\lambda)x\| \leq \varepsilon\right\}$$

The article [2] deals with the discrete approximation idea of the spectrum of an operator and it is shown that the sequence of ε-approximate pseudospectrum sets discretely approximate the spectrum from the below. In [3], A. Ammar, A. Jeribi and Mahfoudhi, Kamel investigated the concept of approximate pseudospectrum and the set

$$\sigma_{ap,\varepsilon}(T) = \left\{\lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T-\lambda)x\| < \varepsilon\right\} \tag{1}$$

for unbounded linear operators. The articles, [2] and [3] contain various other important and interesting properties of approximate pseudospectrum. For $\varepsilon = 0$, the above defined sets $\Sigma_{ap,\varepsilon}(T), \sigma_{ap,\varepsilon}(T)$ are nothing but the approximate spectrum of T, written as $\sigma_{ap}(T)$ and defined as

$$\sigma_{ap}(T) = \left\{\lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T-\lambda)x\| = 0\right\}.$$

Note that, for any $\varepsilon > 0$, $\sigma_{ap}(T) \subseteq \Sigma_{ap,\varepsilon}(T) \cap \sigma_{ap,\varepsilon}(T)$.

For any $\varepsilon > 0$, it is clear from the definition of $\sigma_{ap,\varepsilon}(T)$ and $\Sigma_{ap,\varepsilon}(T)$ that $\sigma_{ap,\varepsilon}(T) \subset \Sigma_{ap,\varepsilon}(T)$. Moreover, $\sigma_{ap,\varepsilon}(T)$ is a non empty bounded open subset of $\mathbb{C}$ but $\Sigma_{ap,\varepsilon}(T)$ is a non empty compact subset of $\mathbb{C}$. While proving the results related to stability of the approximate pseudospectrum, the nature of compactness plays a vital role. Due to this reason, in this article we look in to continuity nature of the map $T \mapsto \Sigma_{ap,\varepsilon}(T)$ under appropriate settings. In the following sections, there are few results, which gives relations and distinctions between nature of the sets $\Sigma_{ap,\varepsilon}(T)$ and $\sigma_{ap,\varepsilon}(T)$.

This article has the following structure: Section 1 presents basic notions, theories, and established results. To fulfill our primary goal, these are essential. Some preliminary results are obtained in Section 2. We demonstrate in Section 3 the approximate pseudospectrum mapping's upper hemicontinuity property. The lower hemicontinuity is generally impossible, and this is shown through the use of different examples.

2 Basic definitions and notions

This section deals with some basic definitions, notions, and existing results that are required to prove our main results. We start with a theorem, which is the equivalent characterization of the set defined in (1).

Theorem 1 (Theorem 3.3 in [3]) *Let $T \in B(X)$ and $\varepsilon > 0$. Then, $\lambda \in \sigma_{ap,\varepsilon}(T)$ if and only if there exists $D \in B(X)$ with $\|D\| < \varepsilon$ such that $\lambda \in \sigma_{ap}(T+D)$.*

Corollary 1 (Corollary 3.4 in [3]) *Let $T \in B(X)$ and $\varepsilon > 0$. Then, $\sigma_{ap,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{ap}(T+D)$*

In [3], example 3.7, illustrates that the above theorem does not hold for $\Sigma_{ap,\varepsilon}(T)$. The following result is true for $\Sigma_{ap,\varepsilon}(T)$

Theorem 2 (Theorem 3.6 in [3]) *Let $T \in B(X)$ and $\varepsilon > 0$.*

1. *If there exists $D \in B(X)$ with $\|D\| \leq \varepsilon$ such that $\lambda \in \sigma_{ap}(T+D)$ then, $\lambda \in \Sigma_{ap,\varepsilon}(T)$.*
2. $\bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D) \subseteq \Sigma_{ap,\varepsilon}(T)$.

One of our results describes Hermitian operators through the approximate pseudospectrum. A Hermitian operator is defined as follows:

Definition 2 (*Definition 3.1 in* [5]) Let $T \in B(X)$. T is said to be Hermitian if

$$\left\{f(T) : f \in B(X)', f(I) = 1, \|f\| = 1\right\} \subseteq \mathbb{R}$$

where $B(X)'$ denotes the dual of $B(X)$.

The following two theorems will be used to prove the results related to the Hermitian operator.

Theorem 3 (Theorem 3.4 in [5]) *Let $T \in B(X)$ and $\varepsilon > 0$. If $\lambda \in \sigma(T)$ then $B(\lambda, \varepsilon) \subseteq \Lambda_\varepsilon(T)$ where $B(\lambda, \varepsilon) = \{\mu \in \mathbb{C} : |\lambda - \mu| \leq \varepsilon\}$.*

Theorem 4 (Theorem 1.10.17 in [14]) *Let $T \in B(X)$. If T is Hermitian then $r(T) = \|T\|$ where $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$.*

The preliminary results section discusses set valued maps and their continuity, so we recall few ideas about it. The definition of a correspondence is described as follows:

Definition 3 (*Definition 17.1 in* [12]) A correspondence ϕ from a set X to a set Y assigns to each x in X a subset $\phi(x)$ of Y. We write the correspondence ϕ between X and Y by $\phi : X \twoheadrightarrow Y$.

The upper and lower hemicontinuous ideas are discussed in the following definition, we use them to demonstrate the continuity property of the approximation pseudo spectrum.

Definition 4 (*Definition 17.2 in* [12]) A correspondence $\phi : X \twoheadrightarrow Y$ between topological spaces is:

1. upper hemicontinuous at the point $x \in X$ if for every neighborhood U of $\phi(x)$, there is a neighborhood V of x such that $z \in V$ implies $\phi(z) \subseteq U$
2. lower hemicontinuous at $x \in X$ if for every open set U with $U \cap \phi(x) \neq \emptyset$, there is a neighborhood V of x such that $z \in V$ implies $\phi(z) \cap U \neq \emptyset$.
3. continuous at $x \in X$ if it is both upper and lower hemicontinuous at x.

It is known that using the equivalent characterizations of limit inferior of the sequence of associated sets, one can obtain the upper hemicontinuity of a set valued map. So the definition of the limit inferior of a sequence of sets is discussed below:

Definition 5 (*see page 2 in* [13]) Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of subsets of a metric space X. We say that the subset

$$\limsup_{n\to\infty} K_n := \{\lambda \in \mathbb{C} : \text{ for every } \varepsilon > 0, \exists J \subseteq \mathbb{N} \text{ infinite such that } B(\lambda, \varepsilon) \cap K_n \neq \emptyset \quad \forall n \in J\}$$

is the upper limit of the sequence K_n and that the subset

$$\liminf_{n\to\infty} K_n := \{\lambda \in \mathbb{C} : \text{ for every } \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } B(\lambda, \varepsilon) \cap K_n \neq \emptyset \quad \forall n \geq N\}$$

is its lower limit. A subset K is said to be the limit or the set limit of the sequence K_n if

$$K = \limsup_{n\to\infty} K_n = \liminf_{n\to\infty} K_n =: \lim_{n\to\infty} K_n.$$

The following theorem is the equivalent characterization of the limit inferior of a sequence of sets.

Theorem 5 (see page 2 in [13]) *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of subsets of a metric space X. $\lambda \in \liminf_{n\to\infty} K_n$ if and only if there exists $\{\lambda_n\} \subseteq \mathbb{C}$ such that $\lim_{n\to\infty} \lambda_n = \lambda$ and $\lambda_n \in K_n$ for all $n \in \mathbb{N}$.*

The following theorem is obtained in [3]

Theorem 6 *Let $T \in B(X)$ and $\varepsilon_0 > 0$. If $\lim_{n\to\infty} \varepsilon_n = \varepsilon_0$ then $\liminf_{n\to\infty} \Sigma_{ap,\varepsilon_n}(T) = \Sigma_{ap,\varepsilon_0}(T)$ and if $\lim_{n\to\infty} \varepsilon_n = 0$ then $\lim_{n\to\infty} \Sigma_{ap,\varepsilon_n}(T) = \sigma_{ap}(T)$*

To demonstrate the upper, lower hemicontinuity of a specific correspondence, the following set inclusion is to be used.

Theorem 7 (Theorem 3.5 in [3]) *Let $T \in B(X)$ and $\varepsilon > 0$. Let $E \in B(X)$ such that $\|E\| < \varepsilon$. Then,*

$$\sigma_{ap,\varepsilon-\|E\|}(T) \subseteq \sigma_{ap,\varepsilon}(T+E) \subseteq \sigma_{ap,\varepsilon+\|E\|}(T)$$

Note 1 1. *Note that Theorem 7 is true for $\|E\| \leq \varepsilon$. Suppose $\|E\| = \varepsilon$ then $\varepsilon - \|E\| = 0$, consequently, $\sigma_{ap,\varepsilon-\|E\|}(T) = \sigma_{ap}(T) \subseteq \sigma_{ap,\varepsilon}(T+E)$. Further, if $\lambda \in \sigma_{ap,\varepsilon}(T+E)$ then by Theorem 1, there exists $D \in B(X)$ with $\|D\| < \varepsilon$ such that $\lambda \in \sigma_{ap}(T+E+D)$. Since $\|E+D\| < 2\varepsilon$, by Theorem 1, $\lambda \in \sigma_{ap,2\varepsilon}(T) = \sigma_{ap,\varepsilon+\|E\|}(T)$.*
2. *In Theorem 13, the set inclusion relationship for $\Sigma_{ap,\varepsilon}(T)$ is proved.*

3 Preliminary results

This section focuses on determining various topological characteristics related to approximate pseudospectrum. To start, we prove the following few fundamental properties of the approximate pseudospectrum.

Proposition 8 *If $T \in B(X)$ and $\varepsilon > 0$, then the following holds*

1. *$\Sigma_{ap,\varepsilon}(T)$ is a non empty compact subset of $\mathbb{C}$.*
2. *$\sigma_{ap}(T) = \bigcap_{\varepsilon>0} \Sigma_{ap,\varepsilon}(T)$*
3. *If $\varepsilon_1 < \varepsilon_2$ then $\Sigma_{ap,\varepsilon_1}(T) \subseteq \Sigma_{ap,\varepsilon_2}(T)$.*
4. *If $\lambda \in \Sigma_{ap,\varepsilon}(T)$ then $|\lambda| \leq \varepsilon + \|T\|$.*

Proof We first prove (4). Let $\lambda \in \Sigma_{ap,\varepsilon}(T)$. If $\lambda \in \sigma(T)$ then, $|\lambda| \leq \|T\| < \varepsilon + \|T\|$. Suppose $\lambda \notin \sigma(T)$, then $\inf_{\|x\|=1} \|(T-\lambda)x\| = \dfrac{1}{\|(T-\lambda)^{-1}\|}$. It follows that,

$$\frac{1}{\|(T-\lambda)^{-1}\|} = \inf_{\|x\|=1} \|(T-\lambda)x\| \leq \varepsilon. \tag{2}$$

Moreover if $|\lambda| > \|T\|$ then, we know that

$$\|(T-\lambda)^{-1}\| \leq \frac{1}{|\lambda| - \|T\|} \tag{3}$$

On combining the above equations (2) and (3), we get $|\lambda| \leq \varepsilon + \|T\|$.

Next, we prove (1). In [3], Lemma 2.4 yield us $\Sigma_{ap,\varepsilon}(T)$ is a closed set. The fact that $\sigma_{ap}(T)$ is a nonempty subset of $\Sigma_{ap,\varepsilon}(T)$ and (4), proves that $\Sigma_{ap,\varepsilon}(T)$ is a nonempty compact subset of $\mathbb{C}$.

The proof (2) and (3) follows directly from the definition of $\Sigma_{ap,\varepsilon}(T)$. □

On next, we establish that each point in the approximate spectrum set is an interior point of the approximate pseudospectrum.

Theorem 9 *Let $T \in B(X)$ and $\varepsilon > 0$. Then for every $\lambda \in \sigma_{ap}(T)$, $B(\lambda, \epsilon) \subseteq \sigma_{ap,\varepsilon}(T)$.*

Proof Let $\mu \in B(\lambda, \varepsilon)$ where $\lambda \in \sigma_{ap}(T)$. Consider the operator $S = (\mu - \lambda)I$. Clearly $\|S\| < \varepsilon$ and $T + S - \mu = T - \lambda$. By using Theorem 1, we have $\mu \in \sigma_{ap,\varepsilon}(T)$. □

In the next two results, we characterize the identity operator and Hermitian idempotent operator through the approximate pseudospectrum.

Theorem 10 *Let $T \in B(X)$ and $\varepsilon > 0$. $T = \lambda I$ if and only if $\Sigma_{ap,\varepsilon}(T) = \{\mu \in \mathbb{C} : |\lambda - \mu| \leq \varepsilon\}$.*

Proof If $T = \lambda I$ then $\Sigma_{ap,\varepsilon}(T) = \{\mu \in \mathbb{C} : |\lambda - \mu| \leq \varepsilon\}$. Conversely, let $\Sigma_{ap,\varepsilon}(T) = \{\mu \in \mathbb{C} : |\lambda - \mu| \leq \varepsilon\}$. We show that $\sigma_{ap}(T) = \{\lambda\}$. It is known that $\sigma_{ap}(T) \neq \emptyset$. Consider $\lambda_0 \in \sigma_{ap}(T)$ then, by Theorem 9, $B(\lambda_0, \varepsilon) \subseteq \Sigma_{ap,\varepsilon}(T) = B(\lambda, \varepsilon)$. Thus, $\sigma_{ap}(T) = \{\lambda\}$. The fact boundary of $\sigma(T)$ is contained in $\sigma_{ap}(T)$ implies that $\sigma(T) = \{\lambda\}$. It follows that $\Lambda_\varepsilon(T) = \{\mu \in \mathbb{C} : |\lambda - \mu| \leq \varepsilon\}$. By Corollary 3.17 of [5], we have $T = \lambda I$. □

Theorem 11 *Let $T \in B(X)$, $T \neq 0$, I and $\varepsilon > 0$. Then, $\Sigma_{ap,\varepsilon}(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon\} \cup \{\lambda \in \mathbb{C} : |\lambda - 1| \leq \varepsilon\}$ if and only if T is a nontrivial Hermitian idempotent operator and $\|T\| = 1$.*

Proof Suppose,

$$\Sigma_{ap,\varepsilon}(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon\} \cup \{\lambda \in \mathbb{C} : |\lambda - 1| \leq \varepsilon\}$$

then by Theorem 9, $\sigma_{ap}(T) = \{0, 1\}$. Consequently, $\sigma(T) = \{0, 1\}$ and

$$\Lambda_{\varepsilon}(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon\} \cup \{\lambda \in \mathbb{C} : |\lambda - 1| \leq \varepsilon\}.$$

By Corollary 3.18 of [5], T is a nontrivial Hermitian idempotent operator and $\|T\| = 1$. Conversely, let T be a nontrivial Hermitian idempotent operator and $\|T\| = 1$, then by Corollary 3.18 of [5],

$$\Lambda_{\varepsilon}(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon\} \cup \{\lambda \in \mathbb{C} : |\lambda - 1| \leq \varepsilon\}.$$

We know that $\sigma(T) \subseteq \Lambda_{\varepsilon}(T)$. We show that $\sigma(T) = \{0, 1\}$. Let $\lambda_0 \in \sigma(T)$, by Theorem 3,

$$B(\lambda_0, \varepsilon) \subseteq \Lambda_{\varepsilon}(T) = B(0, \varepsilon) \cup B(1, \varepsilon).$$

It gives us $\sigma(T) \subseteq \{0, 1\}$. If $\sigma(T) = \{0\}$, then Theorem 3 implies $T = 0$. Suppose $\sigma(T) = \{1\}$, then $\sigma(T - I) = \{0\}$. By Theorem 3 and the fact $T - I$ is hermitian, we have $T = I$ which is a contradiction. Thus $\sigma(T) = \{0, 1\}$. Since $\sigma(T) = \{0, 1\}$, we have $\sigma_{ap}(T) = \{0, 1\}$ and

$$\Sigma_{ap,\varepsilon}(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon\} \cup \{\lambda \in \mathbb{C} : |\lambda - 1| \leq \varepsilon\}.$$

□

The nature of the boundary of the approximate pseudospectrum is briefly described in the following theorem.

Theorem 12 *Let $T \in B(X)$ and $\varepsilon > 0$. Consider the following set,*

$$M = \left\{\lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T - \lambda)x\| = \epsilon\right\}.$$

Suppose $\partial\Sigma_{ap,\varepsilon}(T)$ denotes the all boundary points of $\Sigma_{ap,\varepsilon}(T)$ the $\partial\Sigma_{ap,\varepsilon}(T) \subseteq M$.

Proof Let $\mu \in \partial\Sigma_{ap,\varepsilon}(T)$. Consider a sequence $\lambda_n \in \Sigma^c_{ap,\varepsilon}(T)$ such that $\lambda_n \to \lambda$. Since

$$\inf_{\|x\|=1} \|(T - \lambda_n)x\| > \varepsilon,$$

there exits $x_n \in X$ with $\|x_n\| = 1$ such that

$$\|(T - \lambda_n)x_n\| > \varepsilon.$$

We observe that,

$$\|(T - \lambda)x_n\| \geq \|(T - \lambda_n)x_n\| - |\lambda_n - \lambda| > \varepsilon - |\lambda_n - \lambda|.$$

From the last inequality, $\inf_{\|x\|=1} \|(T - \lambda)x\| \geq \varepsilon$. Thus $\lambda \in M$ □

The example given below demonstrates that strict inclusion in the aforementioned theorem may be possible. It should be noted that the authors of [3] mentioned that the sets M and $\partial\Sigma_{ap,\varepsilon}(T)$ are equal without providing any proof.

Example 1 Consider the Banach space $\ell^\infty(\mathbb{Z})$ with the norm

$$\|x\|_* = |x_0| + \sup_{n\neq 0} |x_n| \text{ where } x = (\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \ldots).$$

Here the box represents the zeroth coordinate of an element in $\ell^\infty(\mathbb{Z})$. Let $T \in B(\ell^\infty(\mathbb{Z}))$ be defined by

$$T\left(\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \ldots\right) = \left(\ldots, x_{-2}, x_{-1}, x_0, \boxed{\frac{x_1}{3}}, x_2, \ldots\right).$$

It is proved in Theorem 3.1 of [4], that

$$\left\|(T-\lambda)^{-1}\right\| = \frac{1}{3} \text{ for all } |\lambda| < \frac{1}{6} \tag{4}$$

Since $\inf_{\|x\|=1} \|(T-\lambda)x\| = \dfrac{1}{\left\|(T-\lambda)^{-1}\right\|}$ for all $|\lambda| < \frac{1}{6}$, the point $0 \in \left\{\lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T-\lambda)x\| = 3\right\}$. We prove that, $0 \notin \partial\Sigma_{ap,3}(T)$. Suppose $0 \in \partial\Sigma_{ap,3}(T)$, then by the definition of the boundary point of a set,

$$0 \in \Sigma_{ap,3}(T) \cap \overline{\Sigma^c_{ap,3}(T)}$$

where $\Sigma^c_{ap,3}(T) = \left\{\lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T-\lambda)x\| > 3\right\}$ and $\overline{\Sigma^c_{ap,3}(T)}$ denotes the closure of $\Sigma^c_{ap,3}(T)$. But from equation (4), for any $\delta < \frac{1}{6}$, we have $B(0,\delta) \cap \Sigma^c_{ap,3}(T) = \emptyset$, which is a contradiction.

The following theorem is an analogous result of the Theorem 7, but the proof given here is different from the proof of Theorem 7.

Theorem 13 *Let $T \in B(X)$ and $\varepsilon > 0$. Let $E \in B(X)$ with $\|E\| \leq \varepsilon$. Then,*

$$\Sigma_{ap,\varepsilon-\|E\|}(T) \subseteq \Sigma_{ap,\varepsilon}(T+E) \subseteq \Sigma_{ap,\varepsilon+\|E\|}(T)$$

Proof We first prove $\Sigma_{ap,\varepsilon}(T+E) \subseteq \Sigma_{ap,\varepsilon+\|E\|}(T)$. Suppose $\lambda \notin \Sigma_{ap,\varepsilon+\|E\|}(T)$ then,

$$\inf_{\|x\|=1} \|(T-\lambda)x\| > \varepsilon + \|E\|.$$

There exists $r > 0$ such that

$$\inf_{\|x\|=1} \|(T-\lambda)x\| \geq \varepsilon + \|E\| + r \tag{5}$$

Now, for any $x \in X$ with $\|x\| = 1$, we observe that

$$\|(T+E-\lambda)x\| \geq \|(T-\lambda)x\| - \|E\|$$

By equation (5),

$$\|(T+E-\lambda)x\| \geq \varepsilon + \|E\| + r - \|E\| = \varepsilon + r.$$

Thus, $\inf_{\|x\|=1} \|(T+E-\lambda)x\| \geq \varepsilon + r > \varepsilon$ and so $\lambda \notin \Sigma_{ap,\varepsilon}(T+E)$. Hence $\Sigma_{ap,\varepsilon}(T+E) \subseteq \Sigma_{ap,\varepsilon+\|E\|}(T)$. Next, suppose $\lambda \notin \Sigma_{ap,\varepsilon}(T+E)$ then,

$$\inf_{\|x\|=1} \|(T+E-\lambda)x\| > \varepsilon.$$

There exists $r_1 > 0$ such that

$$\inf_{\|x\|=1} \|(T+E-\lambda)x\| \geq \varepsilon + r_1 \tag{6}$$

Now, for any $x \in X$ with $\|x\| = 1$, we observe that

$$\|(T-\lambda)x\| = \|(T+E-E-\lambda)x\| \geq \|(T+E-\lambda)x\| - \|E\|$$

By equation (6),

$$\|(T-\lambda)x\| \geq \varepsilon + r_1 - \|E\|.$$

Thus, $\inf_{\|x\|=1} \|(T-\lambda)x\| \geq (\varepsilon - \|E\|) + r_1 > \varepsilon - \|E\|$ and so $\lambda \notin \Sigma_{ap,\varepsilon-\|E\|}(T)$. Hence $\Sigma_{ap,\varepsilon-\|E\|}(T) \subseteq \Sigma_{ap,\varepsilon}(T+E)$. □

4 Main results

The continuity property of the approximate pseudospectrum correspondence is discussed in the following few theorems. The upper hemicontinuity of approximate pseudospectrum correspondence is derived at first.

Theorem 14 *Let $T \in B(X)$. The correspondence*

$$\mathcal{APE} : \mathbb{R}^+ \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{APE}(\varepsilon) = \Sigma_{ap,\varepsilon}(T)$$

is upper hemicontinuous where $\mathbb{R}^+ = \{\varepsilon : \varepsilon > 0\}$.

Proof Let $\varepsilon_0 \in \mathbb{R}^+$ and V be an open set such that $\Sigma_{ap,\varepsilon_0}(T) \subset V$. For any $\varepsilon \leq \varepsilon_0$, we note that

$$\Sigma_{ap,\varepsilon}(T) \subset V.$$

If there exists $\varepsilon' > \varepsilon_0$ such that $\Sigma_{ap,\varepsilon'}(T) \subset V$ then for $\delta = \frac{\varepsilon' - \varepsilon_0}{2}$, Proposition 8 (3), gives us $\Sigma_{ap,\varepsilon}(T) \subset V$ for all $\varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0 + \delta)$. Suppose for every $\varepsilon_n = \varepsilon_0 + \frac{1}{n}$, there exists $\lambda_n \in \Sigma_{ap,\varepsilon_n}(T)$ but $\lambda_n \notin V$. Since $\lambda_n \in \Sigma_{ap,\varepsilon_1}(T)$, the sequence $\{\lambda_n\}$ is bounded and it has a sub sequence, $\{\lambda_{n_k}\}$ such that $\lambda_{n_k} \to \lambda$. By Theorem 6, $\liminf \Sigma_{ap,\varepsilon_n}(T) = \Sigma_{ap,\varepsilon_0}(T)$ and Definition 5,

$$\lambda \in \Sigma_{ap,\varepsilon_0}(T).$$

Hence $\lambda \in V$. Since V is open infinitely many $\lambda'_{n_k}s$ are in V which is a contradiction. Thus $\mathcal{APE}$ is upper hemicontinuous. □

Theorem 15 *Let $\varepsilon > 0$. The correspondence*

$$\mathcal{AP} : B(X) \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{APE}(T) = \Sigma_{ap,\varepsilon}(T)$$

is upper hemicontinuous.

Proof Let $T_0 \in B(X)$ and V be an open set such that $\Sigma_{ap,\varepsilon}(T_0) \subset V$. By Theorem 14, there exists $\delta > 0$ such that for every $\varepsilon' \in (\varepsilon - \delta, \varepsilon + \delta)$, we have

$$\Sigma_{ap,\varepsilon'}(T_0) \subset V. \tag{7}$$

Consider the open ball $\mathcal{O} = \{S \in B(X) : \|T_0 - S\| < \frac{\delta}{2}\}$. For every $S \in \mathcal{O}$, there exists $E \in B(X)$ with $\|E\| < \frac{\delta}{2}$ such that $T_0 + E = S$. From equation (7), we get $\Sigma_{ap,\varepsilon+\frac{\delta}{2}}(T_0) \subset V$. Theorem 13, gives us

$$\Sigma_{ap,\varepsilon}(T_0 + E) \subseteq \Sigma_{ap,\varepsilon+\|E\|}(T_0) \subseteq \Sigma_{ap,\varepsilon+\frac{\delta}{2}}(T_0) \subseteq V.$$

Thus $\mathcal{AP}$ is upper hemicontinuous. □

Theorem 16 *Consider the product metric space $\mathbb{R}^+ \times B(X)$ with metric $d((\varepsilon_1, T_1), (\varepsilon_2, T_2)) = |\varepsilon_1 - \varepsilon_2| + \|T_1 - T_2\|$. Then The correspondence*

$$\mathcal{APS} : \mathbb{R}^+ \times B(X) \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{APS}(\varepsilon, T) = \Sigma_{ap,\varepsilon}(T)$$

is upper hemicontinuous.

Proof Let $(\varepsilon_0, T_0) \in \mathbb{R}^+ \times B(X)$ and U be an open subset of $\mathbb{C}$. Suppose $\Sigma_{ap,\varepsilon_0}(T_0) \subseteq U$. By Theorem 14, there exists $\delta_1 > 0$ such that for every $\varepsilon' \in (\varepsilon - \delta_1, \varepsilon + \delta_1)$, we have

$$\Sigma_{ap,\varepsilon'}(T_0) \subset V.$$

Consider the set

$$N = \{(\varepsilon, T) : d((\varepsilon_1, T_1), (\varepsilon_2, T_2) < \delta\}$$

where $\delta < \min\{\delta_1, \frac{\varepsilon_0}{2}\}$. Take $(\varepsilon', T') \in N$. It is clear that $T' = T_0 + E$ and $\|E\| < \delta$. Since $|\varepsilon_0 - \varepsilon'| + \|E\| < \delta$ We have $\|E\| < \delta - |\varepsilon_0 - \varepsilon'|$ and so

$$\varepsilon' + \|E\| < \varepsilon' + \delta - |\varepsilon_0 - \varepsilon'| \leq \varepsilon_0 + \delta.$$

The following is true by Theorem 13 and by the previous equation,

$$\Sigma_{ap,\varepsilon'}(T') = \Sigma_{ap,\varepsilon'}(T_0 + E) \subseteq \Sigma_{ap,\varepsilon'+\|E\|}(T_0) \subseteq \Sigma_{ap,\varepsilon_0+\delta}(T_0)$$

Next by Theorem 14, $\Sigma_{ap,\varepsilon'}(T') \subseteq V$. □

The following example illustrates that the correspondence $\varepsilon \mapsto \Sigma_{ap,\varepsilon}(T)$ is not lower hemicontinuous in general.

Example 2 Consider the operator given in example 1. For $\varepsilon_0 = 3$, we have

$$\inf_{\|x\|=1} \|(T-\lambda)x\| = 3 \text{ for all } |\lambda| < \frac{1}{6}.$$

Take the open set, $V = \{\lambda \in \mathbb{C} : |\lambda| < r\}$ where $0 < r < \frac{1}{6}$. Clearly, $\Sigma_{ap,3}(T) \cap V \neq \emptyset$. But for any $\varepsilon < 3$, $\Sigma_{ap,\varepsilon}(T) \cap V = \emptyset$.

In the next theorem, we establish an equivalent characterization of the lower hemicontinuity of the approximate pseudospectrum correspondence with the equality of the closure of the set $\sigma_{ap,\varepsilon}(T)$ and $\Sigma_{ap,\varepsilon}(T)$.

Theorem 17 *Let $T \in B(X)$ and $\varepsilon_0 > 0$. The correspondence $\mathcal{APE}$ is lower hemicontinuous at ε_0, if and only if $\overline{\sigma_{ap,\varepsilon_0}(T)} = \Sigma_{ap,\varepsilon_0}(T)$.*

Proof Assume $\overline{\sigma_{ap,\varepsilon_0}(T)} = \Sigma_{ap,\varepsilon_0}(T)$. Let V be an open set such that $\Sigma_{ap,\varepsilon_0}(T) \cap V \neq \emptyset$. For any $\varepsilon > \varepsilon_0$, by Proposition 8, property (3), we have

$$\Sigma_{ap,\varepsilon}(T) \cap V \neq \emptyset.$$

For any $\mu \in \Sigma_{ap,\varepsilon_0}(T) \cap V$ with $\inf_{\|x\|=1} \|(T-\mu)x\| = \varepsilon_0$ there exists $r > 0$ such that, $B(\mu, r) \subseteq V$. By our assumption, there exists $\lambda_0 \in \sigma_{ap,\varepsilon_0}(T)$ such that $\lambda_0 \in B(\mu, r)$. Take $\varepsilon' = \inf_{\|x\|=1} \|(T-\lambda_0)x\|$. From the definition of $\sigma_{ap,\varepsilon_0}(T)$, we have $\inf_{\|x\|=1} \|(T-\lambda_0)x\| < \varepsilon_0$ and so $\varepsilon' < \varepsilon_0$. Take $\delta = \varepsilon_0 - \varepsilon'$. It is easy to see that for any $\varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0 - \delta)$, we have

$$\Sigma_{ap,\varepsilon}(T) \cap V \neq \emptyset.$$

Thus $\mathcal{APE}$ is lower hemicontinuous at ε_0.

Conversely, assume $\mathcal{APE}$ is lower hemicontinuous at ε_0. Consider $\mu \in \Sigma_{ap,\varepsilon_0}(T)$ with $\inf_{\|x\|=1} \|(T-\mu)x\| = \varepsilon_0$. Suppose, there exists $r > 0$ such that

$$B(\mu, r) \cap \sigma_{ap,\varepsilon_0}(T) = \emptyset,$$

then, for every $\lambda \in B(\mu, r)$, we have $\inf_{\|x\|=1} \|(T-\lambda)x\| \geq \varepsilon_0$. Consequently, for any $\varepsilon < \varepsilon_0$, we have

$$\Sigma_{ap,\varepsilon}(T) \cap B(\mu, r) = \emptyset.$$

This is a contradiction. □

In the next theorem, we find a relation between the lower hemicontinuity of the $\mathcal{APS}$ and $\mathcal{AP}$ correspondences.

Theorem 18 *Let $T_0 \in B(X)$ and $\varepsilon_0 > 0$. If the correspondence $\mathcal{APE}$ is lower hemicontinuous at ε_0 then the correspondence $\mathcal{APS}$ is lower hemicontinuous at (ε_0, T_0).*

Proof Let V be an open set such that $\Sigma_{ap,\varepsilon_0}(T_0) \cap V \neq \emptyset$. By our assumption, there exists $\delta > 0$ such that for every $\varepsilon' \in (\varepsilon - \delta, \varepsilon + \delta)$, we have

$$\Sigma_{ap,\varepsilon'}(T_0) \cap V \neq \emptyset. \tag{8}$$

Consider the set

$$N = \{(\varepsilon, T) : d((\varepsilon_1, T_1), (\varepsilon_2, T_2) < \delta\}$$

where $\delta < \min\{\delta_1, \frac{\varepsilon_0}{2}\}$. Take $(\varepsilon', T') \in N$. It is clear that $T' = T_0 + E$ and $\|E\| < \delta$. We have

$$\varepsilon_0 - \delta + \|E\| < \varepsilon' - \delta + (\delta - |\varepsilon_0 - \varepsilon'|) \leq \varepsilon'.$$

From Theorem 13 and by the previous equation, we also observe the following,

$$\Sigma_{ap,\varepsilon_0-\delta}(T_0) \subseteq \Sigma_{ap,\varepsilon_0-\delta}(T' - \|E\|) \subseteq \Sigma_{ap,\varepsilon_0-\delta+\|E\|}(T') \subseteq \Sigma_{ap,\varepsilon'(T')}.$$

Hence $\Sigma_{ap,\varepsilon'}(T') \cap V \neq \emptyset$. Thus $\mathcal{APS}$ is lower hemicontinuous at T_0. □

Corollary 2 *Let $T_0 \in B(X)$ and $\varepsilon_0 > 0$. If the correspondence $\mathcal{APE}$ is lower hemicontinuous at ε_0 then the correspondence $\mathcal{AP}$ is lower hemicontinuous at T_0.*

Proof Directly follows from Theorem 18. □

In the following example, an operator has been given so that the map $\mathcal{APE}$ is lower hemicontinuous.

Example 3 Consider the Banach space $\ell^2(\mathbb{N})$. Let

$$S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \text{ by } S(e_i) = e_{i+1} \ \ \forall i \in \mathbb{N}.$$

The following holds for S:

$$\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}, \|S\| = 1, \|Sx\| = \|x\|$$

and

$$\sigma_{ap}(S) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

Moreover, for any $\lambda \in \sigma(S)$ and for any $x \in \ell^2(\mathbb{N})$ with $\|x\| = 1$, it is easy to observe that,

$$\inf_{\|x\|=1} \|(S - \lambda)x\| \geq 1 - |\lambda| \tag{9}$$

For $0 \leq \varepsilon \leq 1$, in [2], it is shown that

$$\{\lambda \in \mathbb{C} : 1 - \varepsilon \leq |\lambda| \leq 1\} \subseteq \Sigma_{ap,\varepsilon}(S). \tag{10}$$

Consider $r \in [0, 1]$. For $\varepsilon_0 = 1 - r$, from equation (10),

$$\{\lambda \in \mathbb{C} : r \leq |\lambda| \leq 1\} \subseteq \Sigma_{ap,\varepsilon_0}(S) \tag{11}$$

Thus for any $\lambda_0 \in \sigma(S)$ with $|\lambda_0| = r$,

$$\inf_{\|x\|=1} \|(S - \lambda_0)x\| \leq \varepsilon_0 = 1 - r = 1 - |\lambda_0| \tag{12}$$

From, equation (9) and (12),

$$\inf_{\|x\|=1} \|(S - \lambda_0)x\| = 1 - |\lambda_0|. \tag{13}$$

Let V be an open subset of $\mathbb{C}$ and $\varepsilon_1 \in (0, 1]$. Assume $\Sigma_{ap,\varepsilon_1}(S) \cap V \neq \emptyset$. If there exists $\mu \in \Sigma_{ap,\varepsilon_1}(S)$ such that $(S - \mu)^{-1}$ exists then there exists $r > 0$ such that $B(\mu, r) \subseteq V \cap \rho(S)$. From Theorem 4.6 of [5], it is know that the map $\epsilon \mapsto \Lambda_\varepsilon(S)$ is continuous and so there exists $\delta > 0$ such that for every $\varepsilon' \in (\varepsilon_1 - \delta, \varepsilon_1 - \delta)$, we have

$$\Lambda_{\varepsilon'}(S) \cap B(\mu, r) \neq \emptyset.$$

The facts $B(\mu, r) \subseteq \rho(S)$ and $\inf_{\|x\|=1} \|(S - \lambda_0)x\| = \dfrac{1}{\|(S - \lambda_0)^{-1}\|}$ implies that for any $\varepsilon' \in (\varepsilon_1 - \delta, \varepsilon_1 - \delta)$, we have

$$\Sigma_{ap,\varepsilon'}(S) \cap B(\mu, r) \neq \emptyset.$$

Next assume that,

$$\Sigma_{ap,\varepsilon_1}(S) \cap V \neq \emptyset \text{ and } \Sigma_{ap,\varepsilon_1}(S) \cap V \subseteq \sigma(S).$$

For any $\mu \in \Sigma_{ap,\varepsilon_1}(S) \cap V$, it is clear from equation (13) that

$$\inf_{\|x\|=1} \|(S - \mu)x\| = 1 - |\mu| = 1 - r \tag{14}$$

where $r = |\mu|$. Suppose $1 - |\mu| < \varepsilon_1$, then take $1 - |\mu| = \varepsilon_2$ and $\delta = \varepsilon_1 - \varepsilon_2$. It is easy to see that for any $\varepsilon' = (\varepsilon_1 - \delta, \varepsilon_1 + \delta)$,

$$\inf_{\|x\|=1} \|(S - \mu)x\| = 1 - |\mu| = \varepsilon_2 < \varepsilon'. \tag{15}$$

and it follows that

$$\Sigma_{ap,\varepsilon'}(S) \cap V \neq \emptyset.$$

Suppose $1 - |\mu| = \varepsilon_1$ then there exists μ' such that $\mu' \in B(\mu, r') \subseteq V \cap \sigma(S)$ for some $r' > 0$ and $|\mu'| > |\mu|$. Take $1 - |\mu'| = \varepsilon_2$ and $\delta = \varepsilon_1 - \varepsilon_2$. Now, for any $\varepsilon' = (\varepsilon_1 - \delta, \varepsilon_1 + \delta)$,

$$\inf_{\|x\|=1} \|(S - \mu')x\| = 1 - |\mu'| = \varepsilon_2 < \varepsilon'. \tag{16}$$

and it follows that

$$\Sigma_{ap,\varepsilon'}(S) \cap V \neq \emptyset.$$

Thus the correspondence $\mathcal{APE}$ is lower hemicontinuous for any $\varepsilon \in (0, 1]$.

For any $\varepsilon > 1$, it is easy to see that $\Sigma_{ap,\varepsilon}(S) = \Lambda_\varepsilon(S)$ and by Theorem 4.6 in [5], we get the lower hemicontinuity.

Question 1 *We know that the set $\sigma_{ap,\varepsilon}(T)$ is a bounded open subset of $\mathbb{C}$. In this article, we do not discuss about the stability of $\sigma_{ap,\varepsilon}(T)$ and hence we have the following questions:*

1: *Does the set valued map $\varepsilon \mapsto \sigma_{ap,\varepsilon}(T)$ continuous at a given ε_0?*
2: *Does the set valued map $T \mapsto \sigma_{ap,\varepsilon}(T)$ continuous at given a $T_0 \in B(X)$?*
3: *Does the set valued map $(\varepsilon, T) \mapsto \sigma_{ap,\varepsilon}(T)$ continuous at a given $(\varepsilon_0, T_0) \in \mathbb{R}^+ \times B(X)$?*

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