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ORIGINAL RESEARCH

Bijections between different combinatorial models for q-Whittaker and modified Hall-Littlewood polynomials

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Abstract We consider the monomial expansion of the q-Whittaker polynomials and the modified Hall-Littlewood polynomials arising from specialization of the modified Macdonald polynomial. The two combinatorial formulas for the latter, due to Haglund, Haiman, and Loehr and Ayyer, Mandelshtam and Martin, give rise to two different parameterizing sets in each case. We produce bijections between the parameterizing sets, which preserve the content and major index statistics. We identify the major index with the charge or cocharge of appropriate words, and use descriptions of the latter due to Lascoux–Schützenberger and Killpatrick to show that our bijections have the desired properties.

Keywords q-Whittaker polynomials · Modified Hall-Littlewood polynomials · Charge and cocharge of a word, inv, quinv, and maj of a filling

1 Introduction

Let $x = (x_1, x_2, \dots,)$ and q, t denote a collection of algebraically independent indeterminates. The modified Macdonald polynomials $\widetilde{H}_\lambda(x; q, t)$ defined by Garsia and Haiman [1], are an important class of symmetric functions in the variables x_i, indexed by partitions λ. From the the well-known identity $\widetilde{H}_\lambda(x; q, t) = \widetilde{H}_{\lambda'}(x; t, q)$ [2, Prop 2.6], and by interchanging the role of t and q in [3, (3.1)], we can expand $\widetilde{H}_\lambda(x; q, t)$ in powers of q:

$$\widetilde{H}_\lambda(x; q, t) = Q_{\lambda'}(x; t)q^{n(\lambda')} + \ldots + \widetilde{H}_{\lambda'}(x; t), \tag{1}$$

where λ' denotes the conjugate of λ, and $n(\lambda) = \sum_{k=0}^{\lambda_1} \binom{\lambda_k{}'}{2}$. The leading coefficient $Q_{\lambda'}(x; t)$ is the so-called q-Whittakerpolynomial [1,3] and the constant term $\widetilde{H}_{\lambda'}(x; t)$ is the modified Hall-Littlewood polynomial [4, Section 2].

In [5], Haglund, Haiman, and Loehr provided a combinatorial interpretation of $\widetilde{H}_\lambda(x; q, t)$. They considered fillings of the Young diagram of λ. To each filling, they associated an x-weight (a monomial in the x_i) along with q- and t-weights in terms of statistics called inv and maj. In these terms, the $\widetilde{H}_\lambda(x; q, t)$ is just the generating function of all fillings counted with these weights. Later in [6] Ayyer, Mandelshtam and, Martin found another statistic called quinv and showed that the formula for $\widetilde{H}_\lambda(x; q, t)$ still holds if inv is replaced with quinv. As part of their proof, they discovered and proved some novel identities involving the coinversion statistic. In a recent work [7], Loehr has obtained bijective proofs for these identities. The discovery of the quinv statistic led the

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authors to conjecture that there should exist a "natural" bijection from the collection of all fillings to itself, which preserves both x-weight and maj and takes inv to quinv.

In other words, for each $0 \leq p \leq n(\lambda')$, one seeks to describe a bijection from the collection of all fillings having inv $= p$ to the collection of all fillings having quinv $= p$, which preserves both maj and x weight. The main results of this paper are constructions of such bijections for the extreme values of p, i.e., for $p = n(\lambda')$ and $p = 0$.

To this end, we first show that on the set of fillings which maximise (resp. minimise, i.e., make zero) inv or quinv, the maj statistic coincides with the Lascoux–Schützenberger charge (resp. cocharge) statistic on a suitable word. We use two descriptions of charge - the classical one due to Lascoux–Schützenberger [8] and a variant due to Killpatrick [9] - to show that the natural bijection we construct between the inv and quinv pictures preserves maj.

2 Preliminaries

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition, and let $\mathrm{dg}(\lambda)$ be its Young diagram, which we draw in English convention, as a left-up justified array of boxes with λ_i boxes in the i^{th} row from the top. Denote the box in the i^{th} row and j^{th} column as the cell $u = (i, j)$.

A *filling* of $\mathrm{dg}(\lambda)$ is a function $\sigma : \mathrm{dg}(\lambda) \to \mathbb{N}$, i.e., it assigns a positive integer to each cell in $\mathrm{dg}(\lambda)$. Let $\mathcal{F}(\lambda)$ be the set of fillings of $\mathrm{dg}(\lambda)$, and let $|\lambda| = \sum_i \lambda_i$.

A *descent* of a filling σ is a pair of entries $\sigma(u) > \sigma(v)$, where $u, v \in \mathrm{dg}(\lambda)$ and the cell u is immediately below v, that is if $v = (i, j)$ then $u = (i + 1, j)$. Define

$$\mathrm{Des}(\sigma) = \{u \in \mathrm{dg}(\lambda) : \sigma(u) > \sigma(v) \text{ is a descent}\} \tag{2}$$

and

$$\mathrm{maj}(\sigma) = \sum_{u \in \mathrm{Des}(\sigma)} (\mathrm{leg}(u) + 1), \tag{3}$$

where $\mathrm{leg}(u)$ is the number of cells strictly below u in its column.

Example 2.1 For $n = 8$ and $\lambda = (6, 4, 2)$, let

$$\sigma = \begin{array}{|c|c|c|c|c|c|}\hline 1&3&5&5&6&8\\\hline 2&4&2&7\\\cline{1-4} 3&1\\\cline{1-2}\end{array}$$

then $\mathrm{Des}(\sigma) = \{(2, 1), (3, 1), (2, 2), (4, 2)\}$ and $\mathrm{maj}(\sigma) = 6$.

Given σ in $\mathcal{F}(\lambda)$,

1. The *inversion type augmented filling* $\hat{\sigma}$ is obtained by adding a cell with entry 0 above the first cell in each column of σ.
2. The *q-inversion type augmented filling* $\widetilde{\sigma}$ is obtained by adding a cell with entry ∞ below the last cell in each column of σ.

We denote the shapes of the resulting diagrams by $\mathrm{dg}(\hat{\sigma})$ and $\mathrm{dg}(\widetilde{\sigma})$ respectively. An *inversion triple* in σ is a triple of cells (u, v, w) in $\mathrm{dg}(\hat{\sigma})$ such that:

1. $u, w \in \mathrm{dg}(\lambda)$, and w is in the same row as u, to its right,
2. v is immediately above u, in its column, and
3. the triple of values $(\hat{\sigma}(u), \hat{\sigma}(v), \hat{\sigma}(w))$, belongs to the set $\mathcal{I}$, where

$$\mathcal{I} = \{(x, y, z) : y < z < x,\ x < y < z,\ z < x < y,\ \text{or } x = y \neq z\}. \tag{4}$$

Let $\mathrm{inv}(\sigma)$ denote the total number of *inversion triples* in σ. Likewise, a *q-inversion triple* in σ is a triple of cells (u', v', w') in $\mathrm{dg}(\widetilde{\sigma})$ such that:

1. $v', w' \in \mathrm{dg}(\lambda)$ and w' is in the same row as v', to its right,
2. u' is immediately below v', in its column, and

3. The triple of values $(\widetilde{\sigma}(u'), \widetilde{\sigma}(v'), \widetilde{\sigma}(w'))$, belongs to the set $\mathcal{Q}$, where

$$\mathcal{Q} = \{(x, y, z) : y < z < x,\ x < y < z,\ z < x < y \text{ or } x = y \neq z\}. \tag{5}$$

We let quinv(σ) denote the total number of *q-inversion triples* in σ.

For a more detailed discussion of the inv and quinv statistics, we refer the reader to [5] and [6].

Example 2.2 For σ from Example 2.1,

$$\hat{\sigma} = \begin{array}{|c|c|c|c|c|c|c|}\hline 0&0&0&0&0&0&0\\\hline 1&3&5&2&6&8\\\cline{1-6} 2&4&1&7\\\cline{1-4} 3&1\\\cline{1-2}\end{array}$$

and

$$\widetilde{\sigma} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&3&5&2&6&8\\\hline 2&4&1&7&\infty&\infty\\\hline 3&1&\infty&\infty\\\cline{1-4}\infty&\infty\\\cline{1-2}\end{array}$$

with inv$(\sigma) = 3$, and quinv$(\sigma) = 6$.

For a filling σ, its *content* is the sequence $\mu = (\mu_1, \mu_2 \ldots,)$ of non negative integers, where $\mu_i = |\{u \in \mathrm{dg}(\lambda) : \sigma(u) = i\}|$ is the number of i's in σ. Let

$$x^{\sigma} = \prod_{i \geq 1} x^{\mu_i}.$$

The following are the main theorems of [5] and [6].

Theorem 2.3 *[5, Theorem 2.2] Let λ be a partition. The modified Macdonald polynomial can be written as*

$$\widetilde{H}_{\lambda}(x; q, t) = \sum_{\sigma \in \mathcal{F}(\lambda)} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^{\sigma}. \tag{6}$$

Theorem 2.4 *[6, Theorem 2.6] Let λ be a partition. The modified Macdonald polynomial can be written as*

$$\widetilde{H}_{\lambda}(x; q, t) = \sum_{\sigma \in \mathcal{F}(\lambda)} q^{\mathrm{quinv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^{\sigma}. \tag{7}$$

From the above theorems one can naturally ask whether there exists a bijection on $\mathcal{F}(\lambda)$, that preserves both maj and content, and takes inv to quinv. In [6], the authors conjectured an even stronger statement, which we explain below.

Definition 2.5 Let σ, τ in $\mathcal{F}(\lambda)$, we say σ and τ are *row-equivalent* if every row of σ is a permutation of the corresponding row of τ. We write $\sigma \sim \tau$ when σ and τ are row-equivalent.

It will also be convenient to define $R_i(\sigma)$ to be the multiset of entries in the i^{th} row of σ. In terms of this notation, $\sigma \sim \tau$ iff $R_i(\sigma) = R_i(\tau)$ for all i.

Conjecture 1 ([6]): Given a partition λ, there exists a bijection $\delta : \mathcal{F}(\lambda) \to \mathcal{F}(\lambda)$ such that for all σ,

$$\begin{aligned} \sigma &\sim \delta(\sigma), \\ \mathrm{maj}(\delta(\sigma)) &= \mathrm{maj}(\sigma), \\ \mathrm{quinv}(\sigma) &= \mathrm{inv}(\delta(\sigma)). \end{aligned}$$

In the following sections, we prove the conjecture for the extreme values of inv and quinv[1].

[1] The conjecture in [6] is stated for fillings with an upper bound stipulated for the entries, but the version stated here can be seen to be equivalent to that.

2.1 Two descriptions of the charge and cocharge statistics

In this subsection we will recall the classical definition of the statistics *charge* and *cocharge* on words, introduced by Lascoux and Schützenberger [8] and a new description to calculate them due to Killpatrick [9]. It will emerge that the classical definition is more attuned to the inv statistic and the one due to Killpatrick corresponds to the quinv statistic.

2.1.1 Classical description of charge

Let w be a (finite) word with letters in the alphabet $\mathbb{N}$. The content of w is the sequence $(\mu_i)_{i\geq 1}$ where μ_i is the number of occurrences of i in w. We say that w is of *partition content* if $\mu_i \geq \mu_j$ for all $i < j$. The length of a word is defined as the total number of letters in it. A word with content $(1, 1, \ldots, 1, 0, 0, \ldots)$ is called a *standard word*.

Definition 2.6 Let w be a standard word w of length k. The *charge* of w, denoted $\mathrm{ch}(w)$, is defined as $\sum_i (k-i)$, where the sum ranges over all $1 \leq i \leq k$ for which $i+1$ occurs to the right of i in w.

For instance, for the standard word $w = 1465327$ of length 7, $\mathrm{ch}(w) = 6 + 3 + 1 = 10$. For a word w with an arbitrary partition content $\mu = (\mu_1, \mu_2, \ldots)$, the charge $\mathrm{ch}(w)$ can be calculated as the sum of the charges of some particular standard subwords of w. The algorithm to choose such subwords proceeds as follows:

For the first standard subword, we begin from the right end of w, scan to the left until we come across a 1. Then starting from that point, we scan left again until we encounter a 2 and so on. Suppose for an entry j if there is no $j+1$ to the left of it, then we wrap-around and continue to scan from the right end (i.e., picking the rightmost entry $j+1$ in w). We continue this process until we reach the largest letter in w. The subset of the letters of w thus chosen is now viewed as a subword of w, denoted w_1.

Once w_1 has been picked, we remove its letters from w to obtain a new word w'. We repeat the above algorithm on w' to obtain another standard subword w_2 of w. Continuing this process, we will get μ_1 standard subwords $w_1, w_2, \ldots, w_{\mu_1}$ of w. Define the *charge* of w to be:

$$\mathrm{ch}(w) = \sum_{i=1}^{\mu_1} \mathrm{ch}(w_i)\,.$$

Example 2.7 Let $w = 121123132213$ be a word with content $(5, 4, 3)$. Then, the standard subwords of w given by the above algorithm are $w_1 = 321$, $w_2 = 213$, $w_3 = 213$, $w_4 = 12$, and $w_5 = 1$ with the respective charges $\mathrm{ch}(w_1) = 0$, $\mathrm{ch}(w_2) = 1$, $\mathrm{ch}(w_3) = 1$, $\mathrm{ch}(w_4) = 1$ and $\mathrm{ch}(w_5) = 0$. Therefore the charge of w is $\mathrm{ch}(w) = 3$.

2.1.2 New description of charge

We now describe a variant of this procedure due to Killpatrick, which will play a key role in our bijections. In this description, instead of the w_i's we choose different standard subwords v_i's as follows:

1. For the first standard subword, we begin from the left end of w, scan to the right until we come across the largest letter of w (say m). Then, starting from that point, scan right again until we encounter an $m-1$, and continue in this order.
2. If there is no j to the right of the previous entry $j+1$, then wrap-around and continue scanning from the left end (i.e., pick the leftmost entry j). We continue this process until we reach 1. The letters in w chosen as described are viewed as forming a subword v_1 of w.
3. Once v_1 is chosen, delete its entries from w to obtain a new word v'. Repeat the above algorithm on the word v' (starting with the largest letter that occurs in v') to pick the standard subword v_2 etc. Continue this process till we get μ_1 standard subwords $v_1, v_2, \ldots, v_{\mu_1}$ of w.

Define

$$\widehat{\mathrm{ch}}(w) = \sum_{i=1}^{\mu_1} \mathrm{ch}(v_i)\,,$$

where $\mathrm{ch}(v_i)$ is the charge of the standard word v_i as defined in Section 2.1.1.

Theorem 2.8 *[9, Theorem 8] The statistic* $\widehat{\text{ch}}$ *coincides with charge, that is,* $\widehat{\text{ch}} = \text{ch}$.

Example 2.9 For w from Example 2.7, the standard subwords of w given by the new description are $v_1 = 321$, $v_2 = 132$, $v_3 = 213$, $v_4 = 21$, and $v_5 = 1$ with respective charges $\text{ch}(v_1) = 0$, $\text{ch}(v_2) = 2$, $\text{ch}(v_3) = 1$, $\text{ch}(v_4) = 0$ and $\text{ch}(v_5) = 0$.

2.1.3 Cocharge of a word

Definition 2.10 For a standard word w of length k, the *cocharge* of w, denoted $\text{cch}(w)$, is defined as $\sum_i (k-i)$, where the sum ranges over all i, for which $i+1$ occurs to the left of i in w. The cocharge of an arbitrary word w with partition content $\mu = (\mu_1, \mu_2, \ldots)$, is defined as

$$\text{cch}(w) = \sum_{i=1}^{\mu_1} \text{cch}(w_i)\,,$$

where w_i's are the standard subwords of w chosen as in Section 2.1.1.

In [10, Section 2.6], an alternative definition for cocharge is provided in terms of charge. For a word w with partition content $(\mu_i)_{i\geq 1}$, its cocharge

$$\text{cch}(w) = \sum_{k=1}^{\mu_1} \binom{\mu'_k}{2} - \text{ch}(w)\,, \tag{8}$$

where μ' is the conjugate of μ. Thus, from Theorem 2.8, we have

$$\text{cch}(w) = \sum_{k=1}^{\mu_1} \left[\binom{\mu'_k}{2} - \text{ch}(v_k) \right]$$

where v_k's are chosen as in Section 2.1.2. Since the length of v_k is μ'_k, we have $\text{cch}(v_k) = \binom{\mu'_k}{2} - \text{ch}(v_k)$. This allows us to define the cocharge of the word w as the sum of cocharges of the subwords, chosen as in Section 2.1.2, that is

$$\text{cch}(w) = \sum_{k=1}^{\mu_1} \text{cch}(v_k)\,. \tag{9}$$

Throughout this paper we use (9) as the definition of cocharge.

For a more detailed discussion of the charge and cocharge statistics, we refer the interested reader to [10, Chapter 2].

2.2 Charge and cocharge words

Define the following total order on $\text{dg}(\lambda)$: for two cells $u = (i, j)$ and $v = (i', j')$, we say $u < v$, if:

1. $i < i'$, or
2. $i = i'$ and $j > j'$.

Given σ in $\mathcal{F}(\lambda)$ or in $\mathcal{F}(\lambda, n)$, arrange the cells of $\text{dg}(\lambda)$ in sequence: $u_1 = (i_1, j_1), u_2 = (i_2, j_2), \ldots, u_k = (i_k, j_k)$, where $\sigma(u_1) \geq \sigma(u_2) \geq \ldots \geq \sigma(u_k)$ and $|\lambda| = k$. In cases where there is a constant segment $\sigma(u_p) = \sigma(u_{p+1}) = \ldots = \sigma(u_q)$, the cells $u_p, u_{p+1}, \ldots, u_q$ are arranged in increasing order with respect to the total order $<$ defined above.

Definition 2.11 ([5, Section 7]) The *cocharge word* of σ, denoted $cw(\sigma)$, is the word $i_1 i_2 \ldots i_k$ of content λ, where i_p is the row index of the cell u_p in the above order. The *charge word* $w(\sigma)$ is the reverse of $cw(\sigma)$, that is $w(\sigma) = i_k i_{k-1} \ldots i_1$.

Next, instead of the total order $<$ on $\text{dg}(\lambda)$, we consider a slightly different total order $<'$ defined as follows: for two cells $u = (i, j)$ and $v = (i', j')$ we say $u <' v$, if

1. $i < i'$, or
2. $i = i'$ and $j < j'$.

For a filling σ, as above one can arrange the cells of $\mathrm{dg}(\lambda)$ in sequence with respect to the total order $<'$, and let $cw'(\sigma)$ (resp. $w'(\sigma)$) be the cocharge word (resp. charge word) of σ with respect to $<'$, defined in the same way as in Definition 2.11. Then, it is easy to see that $cw'(\sigma) = cw(\sigma)$ and $w'(\sigma) = w(\sigma)$.

Example 2.12 For σ from Example 2.1, the ordering of cells of $\mathrm{dg}(\lambda)$ with respect to σ and $<$ is: $u_1 = (1, 6)$, $u_2 = (2, 4)$, $u_3 = (1, 5)$, $u_4 = (1, 4)$, $u_5 = (1, 3)$, $u_6 = (2, 2)$, $u_7 = (1, 2)$, $u_8 = (3, 1)$, $u_9 = (2, 3)$, $u_{10} = (2, 1)$, $u_{11} = (1, 1)$ and $u_{12} = (3, 2)$.

On the other hand, the ordering with respect to $<'$ is: $v_1 = (1, 6)$, $v_2 = (2, 4)$, $v_3 = (1, 5)$, $v_4 = (1, 3)$, $v_5 = (1, 4)$, $v_6 = (2, 2)$, $v_7 = (1, 2)$, $v_8 = (3, 1)$, $v_9 = (2, 1)$, $v_{10} = (2, 3)$, $v_{11} = (1, 1)$ and $v_{12} = (3, 2)$.

The corresponding words are $cw(\sigma) = cw'(\sigma) = 121112132213$ and $w(\sigma) = w'(\sigma) = 312231211121$.

3 Bijection between the models for $q-$Whittaker polynomials

From Theorems 2.3, 2.4, and (1), we may express the q-Whittakerpolynomial as:

$$Q_{\lambda'}(x; t) = \sum_{\substack{\sigma \in \mathcal{F}(\lambda) \\ \mathrm{inv}(\sigma) = n(\lambda')}} t^{\mathrm{maj}(\sigma)} x^{\sigma},$$

$$= \sum_{\substack{\sigma \in \mathcal{F}(\lambda) \\ \mathrm{quinv}(\sigma) = n(\lambda')}} t^{\mathrm{maj}(\sigma)} x^{\sigma}.$$

The following two propositions characterise the class of inv and quinv maximum fillings of $\mathrm{dg}(\lambda)$.

3.1 inv maximisers

Proposition 3.1 *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition. For each $i \geq 1$, let S_i be any set of positive integers with $|S_i| = \lambda_i$. Then there exists a unique filling $\sigma \in \mathcal{F}(\lambda)$ with $\mathrm{inv}(\sigma) = n(\lambda')$ such that the (multi)set $R_i(\sigma)$ of entries in the i^{th} row of σ is S_i for all i. Furthermore, σ satisfies $\mathrm{maj}(\sigma) = \mathrm{ch}(w(\sigma))$.*

Proof Let σ be a filling with $\mathrm{inv}(\sigma) = n(\lambda')$. Then from the definition of inv in (4), and the maximality of $\mathrm{inv}(\sigma)$ one can observe the following facts about σ:

1. For each i, the entries in the i^{th} row of σ must be distinct
2. σ is strictly decreasing in row 1, and
3. for every cell u not in row 1, if v is the cell directly above u, and S is the set consisting of u and the cells to its right in the same row, then
   (a) If $\sigma(u) \leq \sigma(v)$, then $\sigma(u) = \max\{x \in \sigma(S) : x \leq \sigma(v)\}$,
   (b) If $\sigma(u) > \sigma(v)$ then $x > \sigma(v)$ for all $x \in \sigma(S)$, and $\sigma(u) = \max \sigma(S)$.

From the above facts we can conclude that for a given collection $\{S_i : S_i \subset \mathbb{N}, |S_i| = \lambda_i \ \forall\ i\}$, the filling σ with the required properties is unique if it exists. To establish existence, we construct σ as follows:

1. Take the entries of σ in row 1 (from left to right) to be the elements of S_1 in strictly decreasing order.
2. Inductively, once we have constructed rows 1 through $i - 1$, the entries $\sigma(u)$ in row i are determined one-by-one, moving from left to right, as follows: (a) let v be the cell directly above u and define $\sigma(u)$ to be the largest unused element x in S_i such that $x \leq \sigma(v)$, provided such an x exists (b) if no such x exists, define $\sigma(u)$ to be the largest unused element in S_i.

It is straightforward to verify that σ satisfies the conditions in the first part of the proposition.

To prove $\mathrm{maj}(\sigma) = \mathrm{ch}(w(\sigma))$, let w_1 be the first standard subword of $w(\sigma)$ in the classical definition of charge (Section 2.1.1), and for all i, let $t_{i,1}$ denote the letter i in w_1. From the definition, $t_{1,1}$ is the rightmost 1 in $w(\sigma)$, and from the construction of $w(\sigma)$ (Section 2.2), $t_{1,1}$ corresponds to the largest entry in the first row of σ, that is $t_{1,1}$ corresponds to $\sigma(1, 1)$. For $i \geq 1$, $t_{i+1,1}$ will be the rightmost $(i + 1)$ to the left of $t_{i,1}$ in $w(\sigma)$ if

$\sigma(i,1) \geq \sigma(i+1,1)$, and $t_{(i+1),1}$ will be the rightmost $i+1$ in $w(\sigma)$ otherwise. Observe that $t_{(i+1),1}$ lies to the right of $t_{i,1}$ in w_1 if and only if the pair $\sigma(i+1,1) > \sigma(i,1)$ is a descent. Therefore

$$\mathrm{ch}(w_1) = \sum_{u \in \mathrm{Des}(C_1(\sigma))} (\mathrm{leg}(u)+1)\,,$$

where $\mathrm{Des}(C_1(\sigma))$ denotes the collection of cells u where the pair $\sigma(u) > \sigma(v)$ is a descent in the first column of σ.

If we remove the first column of σ, the remaining will be a filling σ' of $\mathrm{dg}(\mu)$, where μ is the shape of σ'. It is also easy to see that σ' is an inv-maximal filling of $\mathrm{dg}(\mu)$, and the word $w(\sigma')$ corresponding to σ' is nothing but the subword of $w(\sigma)$ obtained by removing w_1.

Therefore, we can conclude that

$$\mathrm{ch}(w(\sigma)) = \mathrm{ch}(w_1) + \mathrm{ch}(w(\sigma'))\,,$$

and

$$\mathrm{maj}(\sigma) = \mathrm{maj}(\sigma') + \sum_{u \in \mathrm{Des}(C_1(\sigma))} (\mathrm{leg}(u)+1)\,.$$

It follows by induction that $\mathrm{ch}(w(\sigma)) = \mathrm{maj}(\sigma)$. □

Remark 3.2 The unique inv-maximal filling corresponding to a given collection of sets is exactly the transpose of the unique *coinversion* zero non-attacking filling corresponding to that same collection of sets (see [11, Proposition 17]).

Example 3.3 Let $\lambda = (7,5,4,2)$, and let $S_1 = \{1,2,3,4,5,6,7\}$, $S_2 = \{2,5,7,9,10\}$, $S_3 = \{6,8,9,10\}$, and $S_4 = \{7,8\}$, then the corresponding inv-maximal filling is

$$\sigma = \begin{array}{|c|c|c|c|c|c|c|}\hline 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \hline 7 & 5 & 2 & 10 & 9 \\ \cline{1-5} 6 & 10 & 9 & 8 \\ \cline{1-4} 8 & 7 \\ \cline{1-2}\end{array}$$

Then $w(\sigma) = 1211121314214332 32$, and $\mathrm{ch}(w(\sigma)) = \mathrm{maj}(\sigma) = 7$.

3.2 quinv maximisers

The following is the analogue of Proposition 3.1 for the quinv statistic.

Proposition 3.4 *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition. For each $i \geq 1$, fix a set $S_i \subset \mathbb{N}$ of size λ_i. There exists a unique filling $\tau \in \mathcal{F}(\lambda)$ with*
(a) $S_i = R_i(\tau)$ for all i, (b) $\mathrm{quinv}(\tau) = n(\lambda')$*, and (c)* $\mathrm{maj}(\tau) = \mathrm{ch}(w'(\tau))$.

Proof Consider a filling τ with $\mathrm{quinv}(\tau) = n(\lambda')$. As before, from the definition of quinv in (5) and the maximality, we can derive the following observations about τ:

1. For each i, the entries in the i^{th} row of τ must be distinct.
2. For a cell u, if there is no cell directly below it, let S be the set consisting of u and the cells to its right in $\mathrm{dg}(\lambda)$. Then $\tau(u) = \min \tau(S)$. In particular, τ is strictly increasing from left to right in the bottom row.
3. For a cell u, if v is the cell directly below u, and S be the set consisting of u and the cells to its right in the same row, then:
   (a) if $\tau(u) \geq \tau(v)$, then $\tau(u)$ equals $\min\{x \in \tau(S) : x \geq \tau(v)\}$,
   (b) while $\tau(u) < \tau(v)$ implies that $x < \tau(v)$ for all $x \in \tau(S)$, and $\tau(u) = \min \tau(S)$.

The above facts about quinv-maximum fillings implies that given a collection of sets S_i, if a filling τ with the desired properties exists, then it must be unique. For the existence of τ, we proceed as follows. Let n denote the number of nonzero parts of λ.

1. The entries of τ in the n^{th} row must be the elements of S_n in strictly increasing order.
2. Once the rows n through $i+1$ have been constructed, the entries $\tau(u)$ in the i^{th} row are determined one by one, moving from left to right:

(a) If there is a cell v directly below u, then $\tau(u)$ is the smallest unused element x in S_i such that $x \geq \tau(v)$. If there is no such x exists, then $\tau(u)$ is the smallest unused element in S_i.
(b) If there is no cell directly below u, then $\tau(u)$ is the smallest unused element in S_i.

This satisfies the properties of the first part of the proposition.

It remains to prove that $\mathrm{maj}(\tau) = \mathrm{ch}(w'(\tau))$. Let v_1 be the first standard subword of $w(\tau)$ in the Killpatrick description of charge (Section 2.1.2). Denote the occurrence of the letter i in v_1 by $s_{i,1}$. From the definition, $s_{n,1}$ is the leftmost n in the charge word $w'(\tau)$. From the construction (Section 2.2) of $w(\tau)$, $s_{n,1}$ corresponds to the smallest entry in the n^{th} row of τ, i.e., $\tau(n, 1)$. For $i < n$, (i) $s_{i,1}$ is the leftmost i to the right of $s_{i+1,1}$ in $w(\tau)$ if $\tau(i, 1) \geq \tau(i + 1, 1)$ and (ii) $s_{i,1}$ is the leftmost i otherwise.

Observe that $s_{i+1,1}$ lies to the right of $s_{i,1}$ if and only if the pair $\tau(i + 1, 1) > \tau(i, 1)$ is a descent. Therefore

$$\mathrm{ch}(v_1) = \sum_{u \in \mathrm{Des}(C_1(\tau))} (\mathrm{leg}(u) + 1)\,,$$

where $\mathrm{Des}(C_1(\tau))$ denotes the collection of cells u where the pair u, v is a descent in the first column of τ.

As before, if we remove the first column of τ, then the remaining is a filling τ' of $\mathrm{dg}(\mu)$, where μ is the shape of τ'. It follows from definitions that τ' is a quinv-maximal filling of $\mathrm{dg}(\mu)$. The charge word $w'(\tau')$ corresponding to τ' is same as the subword obtained by removing v_1 from $w'(\tau)$. Thus

$$\mathrm{ch}(w'(\tau)) = \mathrm{ch}(v_1) + \mathrm{ch}(w'(\tau'))\,,$$

and

$$\mathrm{maj}(\tau) = \mathrm{maj}(\tau') + \sum_{u \in \mathrm{Des}(C_1(\tau))} (\mathrm{leg}(u) + 1)\,.$$

We have by induction that

$$\mathrm{ch}(w'(\tau)) = \mathrm{maj}(\tau)\,.$$

□

Example 3.5 For the same collection of sets from Example 3.3, the corresponding quinv-maximal filling is

$$\tau = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 7 & 3 & 5 & 4 & 6 \\ \hline 9 & 10 & 7 & 2 & 5 \\ \cline{1-5} 8 & 9 & 6 & 10 \\ \cline{1-4} 7 & 8 \\ \cline{1-2} \end{array}$$

Then $w'(\tau) = 12111213142143323 2$, $\mathrm{ch}(w'(\tau)) = 7$, and $\mathrm{maj}(\tau) = 7$.

3.3 The first main theorem

Given a partition λ, let $\mathrm{IMF}(\lambda) \subset \mathcal{F}(\lambda)$ denote the set of all inv-maximal fillings of $\mathrm{dg}(\lambda)$. Likewise, let $\mathrm{QMF}(\lambda)$ denote the set of all quinv-maximal fillings. Propositions 3.1 and 3.4, allow us to establish the desired bijection between $\mathrm{IMF}(\lambda)$ and $\mathrm{QMF}(\lambda)$.

Theorem 3.6 *There exists a bijection* $\Phi : \mathrm{IMF}(\lambda) \to \mathrm{QMF}(\lambda)$ *such that for all* $\sigma \in \mathrm{IMF}(\lambda)$*:*

1. $\sigma \sim \Phi(\sigma)$,
2. $\mathrm{maj}(\sigma) = \mathrm{maj}(\Phi(\sigma))$.

Proof For $\sigma \in \mathrm{IMF}(\lambda)$, let $R_i(\sigma)$ be the set of entries in the i^{th} row of σ. From Proposition 3.4, there exists a unique $\tau \in \mathrm{QMF}(\lambda)$ that satisfies $R_i(\tau) = R_i(\sigma)$ for all i. Define $\Phi(\sigma) := \tau$; the uniqueness of τ ensures that Φ is well defined. The combination of the existence and uniqueness assertions of Proposition 3.1 establishes that Φ is a bijection.

It is evident from the definition that Φ satisfies (1). Since $R_i(\tau) = R_i(\sigma)$ for all i, the ordering (as in Section 2.2) of the cells of $\mathrm{dg}(\lambda)$ with respect to σ and $<$ coincides with the ordering of the cells with respect to τ and $<'$. Therefore, the charge words $w(\sigma)$ and $w'(\tau)$ are also the same, and Propositions 3.1 and 3.4, together imply that $\mathrm{maj}(\sigma) = \mathrm{ch}(w(\sigma)) = \mathrm{ch}(w'(\tau)) = \mathrm{maj}(\tau)$. □

4 The bijection in the modifed Hall-Littlewood case

Theorem 2.3, Theorem 2.4, and (1) together imply that the modified Hall-Littlewood polynomial can be expressed as

$$\widetilde{H}_{\lambda'}(x;t) = \sum_{\substack{\sigma\in\mathcal{F}(\lambda)\\ \mathrm{inv}(\sigma)=0}} t^{\mathrm{maj}(\sigma)}x^{\sigma}\,, \tag{10}$$

or

$$\widetilde{H}_{\lambda'}(x;t) = \sum_{\substack{\sigma\in\mathcal{F}(\lambda)\\ \mathrm{quinv}(\sigma)=0}} t^{\mathrm{maj}(\sigma)}x^{\sigma}\,. \tag{11}$$

The following two propositions characterise the class of inv zero and quinv zero fillings of $\mathrm{dg}(\lambda)$.

4.1 inv-zero fillings

Proposition 4.1 *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition. For each $i \geq 1$, let M_i be a multiset of positive integers, with size $|M_i| = \lambda_i$. Then there exists a unique filling $\sigma \in \mathcal{F}(\lambda)$ such that* $\mathrm{inv}(\sigma) = 0$ *and $R_i(\sigma) = M_i$ for all i. Further, σ satisfies* $\mathrm{maj}(\sigma) = \mathrm{cch}(cw(\sigma))$.

The proof of Proposition 4.1 can be seen in [5, Proposition 7.1].

Example 4.2 Let $\lambda = (7,5,4,2)$, and let $M_1 = \{1,2,2,3,4,4,4\}$, $M_2 = \{1,2,3,4,4\}$, $M_3 = \{2,2,4,5\}$, and $M_4 = \{3,5\}$, then the corresponding inv zero filling is

$$\sigma = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&2&3&4&4&4\\\hline 2&3&4&4&1\\\cline{1-5} 4&5&2&2\\\cline{1-4} 5&3\\\cline{1-2}\end{array}$$

Then $cw(\sigma) = 34111223124112331 2$, and $\mathrm{maj}(\sigma) = \mathrm{cch}(cw(\sigma)) = 15$.

4.2 quinv-zero fillings

The proposition below is the analogue of Proposition 4.1 for quinv zero fillings.

Proposition 4.3 *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition. For each $i \geq 1$, let M_i be a multiset of positive integers with size $|M_i| = \lambda_i$. Then there exists a unique filling $\tau \in \mathcal{F}(\lambda)$ such that* $\mathrm{quinv}(\tau) = 0$ *and $R_i(\tau) = M_i$ for all i. Further, τ satisfies* $\mathrm{maj}(\tau) = \mathrm{cch}(cw'(\tau))$.

Proof We observe the following facts about a quinv zero filling $\tau \in \mathcal{F}(\lambda)$:

1. Let u be a cell in $\mathrm{dg}(\lambda)$ which does not have a cell below, and let S be the set consisting of u and the cells to its right in the same row. Then $\tau(u) = \max\{\tau(w) : w \in S\}$, in particular τ is non-increasing in the bottom row.
2. For a cell u, if there exists a cell v directly below it, and let S' be the set consisting of u and the cells to its right in the same row, then
   (a) if $\tau(u) \geq \tau(v)$, then $\tau(v) \leq x$ for all $x \in \tau(S')$ and $\tau(u) = \max\{\tau(w) : w \in S'\}$, or
   (b) if $\tau(u) < \tau(v)$, then $\tau(u) = \max\{\tau(w)$ for all $w \in S' : \tau(w) < \tau(v)\}$.

From the above observations, it is clear that if there exists a quinv zero filling corresponding to a given collection $\mathrm{M} = \{M_i : M_i \text{ is a multiset }, |M_i| = \lambda_i, i \geq 1\}$, then it must be unique.

For the existence, we construct a filling τ, with $\mathrm{quinv}(\tau) = 0$, and the multiset of entries in the i^{th} row of τ, $M_{\tau_i} = M_i$ for each i. Let n denote the number of nonzero parts of λ. Since $\mathrm{quinv}(\tau) = 0$, τ must be non increasing in the n^{th} row, therefore its entries must be the elements of M_n in non-increasing order. After constructing rows n through $i+1$, the entries $\tau(u)$ in row i are determined sequentially from left to right as follows

1. If there is a cell v directly below u, and if M_i contains an unused element $x < \tau(v)$, then $\tau(u)$ is defined to be the largest such x; otherwise $\tau(u)$ is assigned the largest unused $x \in M_i$.
2. If there is no cell directly below u, then $\tau(u)$ is assigned the largest unused $x \in M_i$.

This completes the the first part of the proof.

To prove $\mathrm{maj}(\tau) = \mathrm{cch}(cw'(\tau))$, let cw'_1 be the first standard subword of $cw'(\tau)$ in the Killpatrick description of charge (or cocharge). Denote each alphabet i in cw'_1 as $r_{i,1}$. From the definition, $r_{n,1}$ is the leftmost n in $cw'(\tau)$, and from the construction of $cw'(\tau)$, $r_{n,1}$ corresponds to the largest entry in the last row of τ, that is $r_{n,1}$ corresponds $\tau(n, 1)$. For $i < n$, $r_{i,1}$ will be (i) the leftmost i to the right of $r_{i+1,1}$ in $cw'(\tau)$ if $\tau(i, 1) \geq \tau(i+1, 1)$, and (ii) the rightmost i otherwise. And observe that $r_{i+1,1}$ lies in the right of $r_{i,1}$ in cw'_1 if and only if the pair $\tau(i + 1, 1) > \tau(i, 1)$ is a descent. Therefore

$$\mathrm{cch}(cw'_1) = \sum_{u \in \mathrm{Des}(C_1(\tau))} (\mathrm{leg}(u) + 1)\,,$$

where $\mathrm{Des}(C_1(\tau))$ denotes the collection of cells u where the pair $\tau(u) > \tau(v)$ is a descent in the first column of τ.

If we remove the first column of τ, then the remaining will be a quinv-zero filling τ' of $\mathrm{dg}(\mu)$, where μ is the shape of τ'. The cocharge word $cw'(\tau')$ corresponding to τ' is the same as the word obtained by removing cw'_1 from $cw'(\tau)$. Thus

$$\mathrm{cch}(cw'(\tau)) = \mathrm{cch}(cw'_1) + \mathrm{cch}(cw'(\tau'))$$

and

$$\mathrm{maj}(\tau) = \sum_{u \in \mathrm{Des}(C_1(\tau))} (\mathrm{leg}(u) + 1) + \mathrm{maj}(\tau')\,.$$

By induction we have

$$\mathrm{cch}(cw'(\tau)) = \mathrm{maj}(\tau)\,.$$

□

Example 4.4 For the same collection of multisets from Example 4.2, the unique quinv zero filling is

$$\tau = \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & 4 & 3 & 2 & 1 & 4 & 4 \\ \hline 3 & 1 & 4 & 4 & 2 \\ \cline{1-5} 4 & 2 & 5 & 2 \\ \cline{1-4} 5 & 3 \\ \cline{1-2} \end{array}$$

Then $cw'(\tau) = 34111223124112331 2$, and $\mathrm{maj}(\tau) = \mathrm{cch}(cw'(\tau)) = 15$.

4.3 The second main theorem

Given a partition λ, let $\mathrm{IZF}(\lambda)$ (resp. $\mathrm{QZF}(\lambda)$) denote the set of inv-zero (resp. quinv-zero) fillings of $\mathrm{dg}(\lambda)$. Propositions 4.1 and 4.3 , now give us the desired bijection between $\mathrm{IZF}(\lambda)$ and $\mathrm{QZF}(\lambda)$.

Theorem 4.5 *Let λ be a partition. There exist a bijection φ from $\mathrm{IZF}(\lambda)$ to $\mathrm{QZF}(\lambda)$ such that for all $\sigma \in \mathrm{IZF}(\lambda)$:*

1. $\sigma \sim \varphi(\sigma)$,
2. $\mathrm{maj}(\sigma) = \mathrm{maj}(\varphi(\sigma))$.

Proof Consider $\sigma \in \mathrm{IZF}(\lambda)$, and let M_{σ_i} be the multiset of entries in the i^{th} row of σ. From Proposition 4.3 there exist a unique filling τ in $\mathrm{QZF}(\lambda)$, such that $M_{\sigma_i} = M_{\tau_i}$. Define $\varphi(\sigma) := \tau$, then φ is well defined, and Proposition 4.1 shows that it is in fact a bijection.

It is evident from the definition that φ satisfies (1). Since $M_{\sigma_i} = M_{\tau_i}$ for all i, the orderings of the cells of $\mathrm{dg}(\lambda)$ with respect to σ and $<$ and the ordering with respect to τ and $<'$ must be same, so the cocharge words $cw(\sigma)$ and $cw'(\tau)$ must coincide. Therefore, the Propositions 4.1 and 4.3 together imply that $\mathrm{maj}(\sigma) = \mathrm{cch}(cw(\sigma)) = \mathrm{cch}(cw'(\tau)) = \mathrm{maj}(\tau)$. □

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Declarations

Conflict of interest The author declares that there are no competing interests.

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