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ORIGINAL RESEARCH

Radon-Nikodým theorems for additive interval multimeasures in Banach spaces

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Abstract In this paper, we deal with Radon-Nikodým theorems for additive interval multimeasures defined on $\mathcal{I}$ and taking values in the hyperspace $cwk(X)$, where $\mathcal{I}$ is the family of all closed non-degenerate subintervals of a fixed closed non-degenerate subinterval W of $\mathbb{R}^m$ and $cwk(X)$ is the family of all convex weakly compact non-empty subsets of a non-separable Banach space X.

Keywords Radon-Nikodým theorem · Additive interval multimeasures · Convex weakly compact set · Non-separable Banach space

Mathematics Subject Classification Primary 28B05 · 28B20 · 46B28; Secondary 46B22 · 46G10

1 Introduction and preliminaries

The Radon-Nikodým theorems for multimeasures were motivated for their applications to mathematical economics, control theory and other mathematical fields. A striking and pretty useful result for existing of Radon-Nikodým derivatives of multimeasures in non-separable locally convex topological vector spaces was achieved by B. Cascales, V. Kadets and J. Rodriguez [7, Theorem 3.1 and Theorem 4.1]. Some recent results devoted to this topic can be found in the papers [2,12,20,26,27]. The first result concerning the existence of a Radon-Nikodým derivative of additive interval multimeasures in non-separable Banach spaces with respect to one dimensional Lebesgue measure have been obtained by L. Di Piazza and G. Porcello [9, Theorem 4.2 and Theorem 4.6]

In this paper, we present three Radon-Nikodým theorems for additive interval multimeasures defined on $\mathcal{I}$ and taking values in the hyperspace $cwk(X)$ of a non-separable Banach space X. The first one works for strongly absolutely continuous (sAC) additive interval multimeasures with "nonempty weakly density" (Theorem 2.2). The second one works for weakly dominated additive interval multimeasures (Theorem 2.3) and the last theorem works for sAC additive interval multimeasures taking convex weakly compact values in Banach spaces possessing the Radon-Nikodým property (Theorem 2.4). Theorem 2.3 improves [20, Theorem 2.7].

Throughout this paper, X is a non-separable Banach space with its dual X^*. The closed unit ball of X^* is denoted by B_{X^*}. We denote by 2^X the family of all non-empty subsets of X and by $cwk(X)$ ($bcc(X)$) is denoted the subfamily of 2^X of all convex weakly compact (bounded convex closed) subsets of X. We consider on $cwk(X)$ the Minkowski addition

$$A \oplus B = A + B, \quad (A + B = \{a + b : a \in A, b \in B\})$$

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and the standard multiplication by scalars. By $\mathcal{H}$ is denoted the subfamily of 2^X of all bounded closed subsets of X. The family $\mathcal{H}$ is a complete metric space with respect to the Hausdorff distance, given by

$$d_{\mathcal{H}}(A, B) = \max\{e(A, B), e(B, A)\},$$

where

$$e(A, B) = \sup_{a\in A} \operatorname{dist}(a, B), \quad \operatorname{dist}(a, B) = \inf\{||a - b|| : b \in B\}.$$

It is easy to see that $bcc(X)$ is a closed subspace of the complete metric space $(\mathcal{H}, d_{\mathcal{H}})$. For every $C \in 2^X$ the *support* of C is defined by equality

$$\sigma(x^*, C) = \sup\{\langle x^*, x\rangle : x \in C\}.$$

Let $\ell_\infty(B_{X^*})$ be the Banach space of all bounded real valued functions defined on B_{X^*} endowed with the supremum norm $||\cdot||_\infty$ and let

$$\kappa : cwk(X) \to \ell_\infty(B_{X^*}), \quad \kappa(C) = \sigma(\cdot, C). \tag{1.1}$$

Then the map κ satisfies the statements $(i) - (iv)$ in [5, Lemma 1.1] (see also [3, Theorem 2.4]).

Let $\alpha = (a_1, \ldots, a_m)$ and $\beta = (b_1, \ldots, b_m)$ with $-\infty < a_j < b_j < +\infty$, for $j = 1, \ldots, m$. The set $[\alpha, \beta] = \prod_{j=1}^m [a_j, b_j]$ is called a *closed non-degenerate interval* in $\mathbb{R}^m$. If $b_1 - a_1 = \ldots = b_m - a_m$, then $I = [\alpha, \beta]$ is called a *cube* and we set $l_I = b_1 - a_1$. We fix a closed non-degenerate subinterval $W \subset \mathbb{R}^m$ and denote by $\mathcal{I}$ the family of all closed non-degenerate subintervals of W. Two intervals $I, J \in \mathcal{I}$ are said to be *non-overlapping* if $I^o \cap J^o = \emptyset$. The Euclidean space $\mathbb{R}^m$ is equipped with the maximum norm. We shall also find it convenient to use the symbols $B_m(t, r)$ for the open ball in $\mathbb{R}^m$ with center t and radius $r > 0$, ∂B and B^o for *boundary* and *interior* of a subset $B \subset \mathbb{R}^m$, respectively. We denote by $\mathcal{L}$ the family of all Lebesgue measurable subsets of W. The Lebesgue measure of a set $E \in \mathcal{L}$ is denoted by $|E|$. Thus, if I is a cube, then $|I| = (l_I)^m$. The word "at almost all" always refers to the Lebesgue measure λ on W.

A mapping $\Gamma : W \to 2^X$ is called a *multifunction* and a function $f : W \to X$ is said to be a *selection* of Γ if $f(t) \in \Gamma(t)$ for all $t \in W$. By $\mathcal{S}_\Gamma$ is denoted the family of all selections of Γ. We say that a mapping $\Phi : \mathcal{I} \to cwk(X)$ is an *additive interval multimeasure (or additive interval multifunction)*, if for each two non-overlapping intervals $I, J \in \mathcal{I}$ with $I \cup J \in \mathcal{I}$, we have $\Phi(I \cup J) = \Phi(I) + \Phi(J)$. An additive interval function $\varphi : \mathcal{I} \to X$ is said to be a *selection* of the additive interval multimeasure $\Phi : \mathcal{I} \to cwk(X)$ if $\varphi(I) \in \Phi(I)$ for every $I \in \mathcal{I}$. We denote by $\mathcal{S}_\Phi$ the family of all selections of Φ.

Definition 1.1 We say that an additive interval multimeasure $\Phi : \mathcal{I} \to cwk(X)$ is *strongly absolutely continuous* (shortly sAC), if for every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that for every finite collection π of pairwise non-overlapping intervals in $\mathcal{I}$, we have

$$\sum_{I\in\pi} |I| < \eta_\varepsilon \Rightarrow \sum_{I\in\pi} d_{\mathcal{H}}\left(\Phi(I), \{\theta\}\right) < \varepsilon,$$

where θ is the zero vector in X. Replacing the last inequality with $d_{\mathcal{H}}\left(\sum_{I\in\pi} \Phi(I), \{\theta\}\right) < \varepsilon$ we obtain the notion AC (*absolutely continuous*) for Φ.

Given a sequence (B_n) of non-empty subsets of X, we write $\sum_n B_n$ to denote the set of all elements of X which can be written as the sum of an unconditionally convergent series $\sum_n x_n$, where $x_n \in B_n$ for every $n \in \mathbb{N}$.

Definition 1.2 A mapping $M : \mathcal{L} \to 2^X$ is said to be a *strong multimeasure* if:

(i) $M(\emptyset) = \{\theta\}$,
(ii) for each sequence (E_n) of pairwise disjoint members of $\mathcal{L}$, we have $M(\bigcup_n E_n) = \sum_n M(E_n)$.

A *selector* m of M is a vector-valued function $m : \mathcal{L} \to X$ such that $m(A) \in M(A)$ for every $A \in \mathcal{L}$. The mapping $M : \mathcal{L} \to 2^X$ is said to be a *weak multimeasure* or simple *multimeasure*, if for each $x^* \in X^*$, the function $\sigma(x^*, M(\cdot))$ is a signed measure on $\mathbb{R} \cup \{+\infty\}$. The multimeasure M is said be λ-*continuous*, if $M(Z) = \{\theta\}$ whenever $Z \subset W$ satisfies $|Z| = 0$. By [17, Proposition 4.7, p.851] if M is a strong multimeasure,



then M is a multimeasure. The two definitions are equivalent in the case of $cwk(X)$-valued multimeasures (see [15, Proposition 3]). If $M : \mathcal{L} \to 2^X$ is a strong multimeasure we define

$$|M|(E) = \sup \sum_i ||M(E_i)||,$$

where the supremum is taken over all finite partitions (E_i) of E in $\mathcal{L}$ and

$$||A|| = \sup\{||x|| : x \in A\} \quad \left(A \in 2^X\right).$$

We say that M has *bounded variation* if $|M|(W) < +\infty$.

Definition 1.3 A multifunction $\Gamma : W \to bcc(X)$ is called *Pettis integrable* in $bcc(X)$ ($cwk(X)$) if

(i) $\sigma(x^*, \Gamma(\cdot))$ is Lebesgue integrable for every $x^* \in X^*$,
(ii) for each $E \in \mathcal{L}$, there is $C_E \in bcc(X)$ ($C_E \in cwk(X)$) such that

$$\sigma\left(x^*, C_E\right) = \int_E \sigma(x^*, \Gamma(t))d\lambda \quad \text{for every } x^* \in X^*.$$

We call C_E the *Pettis integral* of Γ over E and set $(P)\int_E \Gamma(t)d\lambda = C_E$.

The Pettis integral for multifunctions was first considered by Castaing and Valadier [4, Chapter V] and has been widely studied in papers [6,10,11,14,16,22,24,25]. For the definition and the properties of Pettis and Bochner integrable functions we refer to [1,8,23,30,31].

A pair (I, t) of an interval $I \in \mathcal{I}$ and a point $t \in W$ is called an $\mathcal{M}$-*tagged interval* in W. A finite collection $\{(I_i, t_i) : i = 1, \ldots, p\}$ of $\mathcal{M}$-tagged intervals in W is called an $\mathcal{M}$-*partition* in W, if $\{I_i \in \mathcal{I} : i = 1, \ldots, p\}$ is a collection of pairwise non-overlapping intervals. A positive function $\delta : E \subset W \to (0, +\infty)$ is called a *gauge* on E. We say that an $\mathcal{M}$-partition π in W is

1. a partition of W if $\bigcup_{(I,t)\in\pi} I = W$,
2. δ-fine if for each $(I, t) \in \pi$, we have $I \subset B_m(t, \delta(t))$.

We now recall the definitions of McShane integrability and variationally McShane integrability (or strongly McShane integrability) of vector valued functions, cf. [30, Definition 3.2.1 and Definition 3.6.2].

Definition 1.4 A function $f : W \to X$ is said to be *McShane integrable* if there exists a vector $I_f \in X$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on W such that for every δ-fine $\mathcal{M}$-partition π of W we have

$$\left\| I_f - \sum_{(I,t)\in\pi} f(t)|I| \right\| < \varepsilon.$$

We write then $I_f = (M)\int_W f(t)d\lambda$. The function f is said to be McShane integrable over $E \in \mathcal{L}$, if the function $f\mathbb{1}_E$ is McShane integrable, where $\mathbb{1}_E$ is the characteristic function of E. In this case, we write

$$(M)\int_E f(t)d\lambda = (M)\int_W f(t)\mathbb{1}_E(t)d\lambda.$$

Definition 1.5 A function $f : W \to X$ is said to be *variationally McShane integrable (or strongly McShane integrable)* if there is an additive interval function $\varphi : \mathcal{I} \to X$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on W such that for every δ-fine $\mathcal{M}$-partition π of W we have

$$\sum_{(I,t)\in\pi} ||f(t)|I| - \varphi(I)|| < \varepsilon.$$

In this case, the additive interval function φ is called the *variational McShane primitive* of f. It is clear that if $f : W \to X$ is variationally McShane integrable with the primitive φ, then for each $I \in \mathcal{I}$ the function f is McShane integrable over I and

$$\varphi(I) = (M)\int_I f(t)d\lambda.$$

A function $f : W \to X$ is variationally McShane integrable with the primitive φ, if and only if f is Bochner integrable (cf. [30, Theorem 5.1.4]). In this case,

$$\varphi(I) = (M)\int_I f(t)d\lambda = (B)\int_I f(t)d\lambda, \quad \text{for every } I \in \mathcal{I}.$$

Definition 1.6 Let $\varphi : \mathcal{I} \to X$ be an additive interval function and let $t \in W^o$. We set

$$\mathcal{I}(t) = \{I \in \mathcal{I} : t \in I, I \text{ is a cube}\}, \quad \mathcal{I}^o(t) = \{I \in \mathcal{I}(t) : t \in I^o\}.$$

If $I \in \mathcal{I}^o(t)$, then we write

$$\Delta\varphi(t, I) = \frac{\varphi(I)}{|I|}, \quad \mathcal{I}^o(t, I) = \{J \in \mathcal{I}^o(t) : J \subset I\}$$

and define a partial ordering $\preceq_t$ on $\mathcal{I}^o(t)$ by saying that $I' \preceq_t I''$ if and only if $I' \supset I''$. Then $(\mathcal{I}^o(t), \preceq_t)$ is a directed set. For the concepts of nets and subnets we refer to [21]. We now define

$$L_\varphi(t) = \bigcap_{I \in \mathcal{I}^o(t)} \overline{L_\varphi(t, I)}^{\sigma(X,X^*)}, \tag{1.2}$$

where $L_\varphi(t, I) = \{\Delta\varphi(t, J) : J \in \mathcal{I}^o(t, I)\}$ and $\overline{L_\varphi(t, I)}^{\sigma(X,X^*)}$ is the closure of $L_\varphi(t, I)$ with respect to the weak topology $\sigma(X, X^*)$. By [21, Theorem 7, p.72] follows that $L_\varphi(t)$ is the set of all weak limit points of the net $(\Delta\varphi(t, I))_{I \in \mathcal{I}^o(t)}$.

Definition 1.7 Let $\Phi : \mathcal{I} \to cwk(X)$ be an additive interval multimeasure and let $t \in W^o$. For each $I \in \mathcal{I}^o(t)$ we write $\Delta\Phi(t, I) = \frac{\Phi(I)}{|I|}$ and, then,

$$L_\Phi(t) = \bigcap_{I \in \mathcal{I}^o(t)} \overline{L_\Phi(t, I)}, \tag{1.3}$$

where $L_\Phi(t, I)$ is the convex hull of the set $A_\Phi(t, I) = \bigcup_{J \in \mathcal{I}^o(t,I)} \Delta\Phi(t, J)$, i.e.,

$$L_\Phi(t, I) = \operatorname{co}(A_\Phi(t, I)),$$

and $\overline{L_\Phi(t, I)}$ is the closure of the convex set $L_\Phi(t, I)$ in X. Since $L_\Phi(t, I)$ is a convex set it follows that

$$\overline{L_\Phi(t, I)} = \overline{L_\Phi(t, I)}^{\sigma(X,X^*)},$$

cf. [28, Proposition 8, p.34] or [29, Corollary 2, p.65].

2 The main results

The main results are Theorems 2.2, 2.3 and 2.4. The following lemma shows a schematic display of the major implications involved in proving of the first result.

Lemma 2.1 *Let X be a Banach space and let $\Phi : \mathcal{I} \to cwk(X)$ be an additive interval multimeasure. Then,*

(i) for each $x^* \in X^*$ and for each $I \in \mathcal{I}$, we have

$$\Phi x^*(I) \in cwk(X), \qquad \left(\Phi x^*(I) = \{x \in \Phi(I) : x^*(x) = \sigma(x^*, \Phi(I))\}\right),$$

(ii) for each $x^* \in X^*$, the mapping $\Phi x^*(\cdot)$ is an additive interval multimeasure,

(iii) if Φ is sAC and there exists $x^* \in X^*$ such that $L_{\Phi x^*}(t) \neq \emptyset$ at almost all $t \in W^o$, then

(a) $L_\Phi(t) \in bcc(X)$ at almost all $t \in W^o$,

(b) there exists a Lebesgue integrable function $g : W \to \mathbb{R}$ such that

$$\sigma(x^*, \Phi x^*(I)) = \sigma(x^*, \Phi(I)) = \int_I g(t)d\lambda \ \text{ for all } I \in \mathcal{I}. \tag{2.1}$$

Proof (i) Since each $x^* \in X^*$ is a continuous linear functional, we see that $\Phi x^*(I) \in cwk(X)$ whenever $I \in \mathcal{I}$. (ii) Fix an arbitrary $x^* \in X^*$ and let I and J be non-overlapping intervals in $\mathcal{I}$ with $I \cup J \in \mathcal{I}$. Then, $\Phi(I \cup J) = \Phi(I) + \Phi(J)$ and

$$\sigma(x^*, \Phi(I) + \Phi(J)) = \sigma(x^*, \Phi(I)) + \sigma(x^*, \Phi(J)) \quad (cf.[4, Theorem II - 17]).$$

It follows that $\Phi x^*(I) + \Phi x^*(J) \subset \Phi x^*(I \cup J)$. To prove the reverse inclusion, pick $z \in \Phi x^*(I \cup J) \subset \Phi(I) + \Phi(J)$ and write $z = x + y$, where $x \in \Phi(I)$ and $y \in \Phi(J)$. Suppose if possible that $x \notin \Phi x^*(I)$. Then $x^*(u) > x^*(x)$ for some $u \in \Phi(I)$. Hence,

$$x^*(z) = x^*(x) + x^*(y) < x^*(u) + x^*(y) = x^*(u + y) \leq \sigma(x^*, \Phi(I \cup J)),$$

which contradicts the choice of z. Similarly, $y \notin \Phi x^*(J)$ contradicts the choice of z. Therefore, $x \in \Phi x^*(I)$ and $y \in \Phi x^*(J)$. This means that $\Phi x^*(I \cup J) \subset \Phi x^*(I) + \Phi x^*(J)$. Thus,

$$\Phi x^*(I \cup J) = \Phi x^*(I) + \Phi x^*(J).$$

Since x^* was arbitrary, the last result holds for every $x^* \in X^*$. (iii) (a) We set $\Psi = \Phi x^*$. By hypothesis there exists $Z_\emptyset \subset W^o$ with $|Z_\emptyset| = 0$ such that $L_\Psi(t) \neq \emptyset$ for all $t \in W^o \setminus Z_\emptyset$, and since $L_\Psi(t) \subset L_\Phi(t)$ follows that

$$L_\Phi(t) \neq \emptyset \text{ for every } t \in W^o \setminus Z_\emptyset. \tag{2.2}$$

Since Φ is sAC, the additive interval function $\Phi^\infty = \kappa \circ \Phi$ is also sAC, where κ is defined by (1.1). Therefore, by [18, Lemma 2.3] there exists a unique λ-continuous countably additive vector measure $M^\infty : \mathcal{L} \to \kappa(cwk(X))$ of bounded variation $|M^\infty|(W) < +\infty$ and such that $M^\infty(I) = \Phi^\infty(I)$ for all $I \in \mathcal{I}$. From the fact that the variation $|M^\infty|$ of the vector measure M^∞ is a λ-continuous countable additive real-valued measure, it follows that there exists a Lebesgue integrable function $h : W \to \mathbb{R}$ which is the Radon-Nikodým derivative of $|M^\infty|$ with respect to λ (cf. [13, Theorem III.10.2]), i.e.,

$$|M^\infty|(A) = \int_A h(t)d\lambda \text{ for all } A \in \mathcal{L}. \tag{2.3}$$

If we define an additive interval function as follows

$$|\Phi^\infty| : \mathcal{I} \to [0, +\infty), \quad |\Phi^\infty|(I) = |M^\infty|(I), \tag{2.4}$$

then

$$||\Phi(I)|| = d_{\mathcal{H}}(\Phi(I), \{\theta\}) = ||\Phi^\infty(I)||_\infty = ||M^\infty(I)||_\infty \leq |\Phi^\infty|(I) \text{ for every } I \in \mathcal{I},$$

and therefore

$$||\Delta\Phi(t, I)|| \leq \Delta|\Phi^\infty|(t, I), \qquad (t \in W^o,\ I \in \mathcal{I}^o(t)).$$

Hence,

$$||L_\Phi(t, I)|| = ||A_\Phi(t, I)|| \leq \sup_{J \in \mathcal{I}^o(t,I)} \Delta|\Phi^\infty|(t, J) = \sup L_{|\Phi^\infty|}(t, I), \qquad (t \in W^o,\ I \in \mathcal{I}^o(t)),$$

and since $||L_\Phi(t)|| \leq ||L_\Phi(t, I)||$ for all $I \in \mathcal{I}^o(t)$, it follows that

$$||L_\Phi(t)|| \leq \inf_{I \in \mathcal{I}^o(t)} \left(\sup_{J \in \mathcal{I}^o(t,I)} \Delta|\Phi^\infty|(t, J) \right) = \limsup_{I \in \mathcal{I}^o(t)} \Delta|\Phi^\infty|(t, I) \quad (t \in W^o). \tag{2.5}$$

By (2.3) and (2.4) we obtain

$$|\Phi^\infty|(I) = |M^\infty|(I) = \int_I h(t)d\lambda \text{ for all } I \in \mathcal{I}.$$

Therefore, by [18, Lemma 2.4] and [19, Lemma 2.2] there exists $Z_b \subset W^o$ with $|Z_b| = 0$ such that the net $(\Delta|\Phi^\infty|(t, I))_{I \in \mathcal{I}^o(t)}$ converges to $h(t)$ for every $t \in W^o \setminus Z_b$. Thus,

$$\lim_{I \in \mathcal{I}^o(t)} \Delta|\Phi^\infty|(t, I) = \limsup_{I \in \mathcal{I}^o(t)} \Delta|\Phi^\infty|(t, I) = h(t) < +\infty \text{ for every } t \in W^o \setminus Z_b$$

and by (2.5) follows

$$||L_\Phi(t)|| \le \limsup_{I \in \mathcal{I}^o(t)} \Delta|\Phi^\infty|(t, I) = h(t) < +\infty \text{ for all } t \in W^o \setminus Z_b. \tag{2.6}$$

The last result together with (2.2) yields

$$L_\Phi(t) \in bcc(X) \text{ for every } t \in W^o \setminus Z \qquad (Z = Z_\emptyset \cup Z_b).$$

(*b*) For each $A \in \mathcal{L}$ there exists the unique weakly compact convex set $M(A) \in cwk(X)$ such that $\kappa(M(A)) = M^\infty(A)$. Thanks to [5, Lemma 2.3] the mapping

$$M : \mathcal{L} \to cwk(X), \quad \kappa(M(A)) = M^\infty(A),$$

is a λ-continuous strong multimeasure of bounded variation. Hence the mapping

$$\nu : \mathcal{L} \to \mathbb{R}, \quad \nu(A) = \sigma(x^*, M(A))$$

is a λ-continuous countable additive real-valued measure. Then, there exists a Lebesgue integrable function $g : W \to \mathbb{R}$ which is the Radon-Nikodým derivative of ν with respect to λ, i.e.,

$$\nu(A) = \int_A g(t)d\lambda \text{ for every } A \in \mathcal{L}.$$

Since $\Phi(I) = M(I)$ for all $I \in \mathcal{L}$, the last result yields (2.1) and the proof is complete.

We say that an additive interval multimeasure $\Phi : \mathcal{I} \to cwk(X)$ has *nonempty weakly density* if for each $x^* \in X^*$ we have

$$L_{\Phi x^*}(t) \neq \emptyset \text{ at almost all } t \in W^o,$$

where the exceptional set may vary with x^*. We are now ready to prove the first result.

Theorem 2.2 *Let $\Phi : \mathcal{I} \to cwk(X)$ be an additive interval multimeasure such that*

- *Φ is sAC,*
- *Φ has nonempty weakly density.*

Then, there exists a multifunction $\Gamma : W \to bcc(X)$ Pettis integrable such that

$$\Phi(I) = (P)\int_I \Gamma(t)d\lambda \text{ for every } I \in \mathcal{I}.$$

Proof By Lemma 2.1 there exists $Z \subset W^o$ with $|Z| = 0$ such that

$$L_\Phi(t) \in bcc(X) \text{ for every } t \in W^o \setminus Z.$$

Then we can define

$$\Gamma : \Omega \to bcc(X), \quad \Gamma(t) = \begin{cases} L_\Phi(t) & t \in W^o \setminus Z \\ \{\theta\} & t \in Z \cup \partial W \end{cases}. \tag{2.7}$$

We are going to prove that Γ is the required multifunction.

Fix an arbitrary $x^* \in X^*$ and write $\Psi = \Phi x^*$,

$$\psi : \mathcal{I} \to \mathbb{R}, \quad \psi(I) = \sigma(x^*, \Psi(I)).$$

Then there exists $N^{x^*} \subset W^o$ with $|N^{z^*}| = 0$ such that

$$L_\Psi(t) \neq \emptyset \text{ for all } t \in W^o \setminus N^{x^*} \tag{2.8}$$

By Lemma 2.1 there exists a Lebesgue integrable function $g : W \to \mathbb{R}$ such that

$$\psi(I) = \sigma(x^*, \Psi(I)) = \sigma(x^*, \Phi(I)) = \int_I g(t)d\lambda, \quad \text{for all } I \in \mathcal{I}. \tag{2.9}$$



Thanks to [18, Lemma 2.4] and [19, Lemma 2.2] there exists $Z^{x^*} \subset W^o$ with $|Z^{x^*}| = 0$ such that the net $(\Delta\psi(t, I))_{I\in\mathcal{I}^o(t)}$ converges to $g(t)$ for all $t \in W^o \setminus Z^{x^*}$, i.e.,

$$\lim_{I\in\mathcal{I}^o(t)} \sigma(x^*, \Delta\Psi(t, I)) = \lim_{I\in\mathcal{I}^o(t)} \sigma(x^*, \Delta\Phi(t, I)) = \lim_{I\in\mathcal{I}^o(t)} \Delta\psi(t, I) = g(t) \quad (t \in W^o \setminus Z^{x^*}). \tag{2.10}$$

Since for each $t \in W^o$ and $I \in \mathcal{I}^o(t)$,

$$x^*(\Delta\Psi(t, I)) = \{\sigma(x^*, \Delta\Phi(t, I))\} = \{\Delta\psi(t, I)\},$$

we obtain

$$x^*(A_\Psi(t, I)) = \bigcup_{J\in\mathcal{I}^o(t,I)} x^*(\Delta\Psi(t, J)) = \{\Delta\psi(t, J) : J \in \mathcal{I}^o(t, I)\} = L_\psi(t, I), \tag{2.11}$$

and since

$$x^*(L_\Psi(t, I)) = \operatorname{co}\left(x^*(A_\Psi(t, I))\right) = \operatorname{co}\left(L_\psi(t, I)\right), \quad (t \in W^o, I \in \mathcal{I}^o(t))$$

it follows that

$$x^*(L_\Psi(t)) \subset \bigcap_{I\in\mathcal{I}^o(t)} \overline{x^*(L_\Psi(t, I))} \subset \bigcap_{I\in\mathcal{I}^o(t)} \overline{\operatorname{co}\left(L_\psi(t, I)\right)} \text{ for every } t \in W^o.$$

The last result together with (2.8), (2.10) and the equality

$$\left[\liminf_{I\in\mathcal{I}^o(t)} \Delta\psi(t, I), \ \limsup_{I\in\mathcal{I}^o(t)} \Delta\psi(t, I)\right] = \bigcap_{I\in\mathcal{I}^o(t)} \overline{\operatorname{co}\left(L_\psi(t, I)\right)}$$

yields

$$x^*(L_\Psi(t)) = \{g(t)\} \text{ for all } t \in W^o \setminus (N^{x^*} \cup Z^{x^*}). \tag{2.12}$$

We have also

$$\begin{aligned}\sigma(x^*, L_\Phi(t)) \le& \sigma(x^*, \overline{L_\Phi(t, I)}) = \sigma(x^*, L_\Phi(t, I))\\ =& \sigma(x^*, A_\Phi(t, I)) = \sigma(x^*, A_\Psi(t, I)) \quad (t \in W^o, I \in \mathcal{I}^o(t)),\end{aligned}$$

and since

$$\sigma(x^*, A_\Psi(t, I)) = \sup_{J\in\mathcal{I}^o(t,I)} \Delta\psi(t, J) \quad (t \in W^o, I \in \mathcal{I}^o(t)) \quad \text{(see (2.11))},$$

it follows that

$$\sigma(x^*, L_\Phi(t)) \le \inf_{I\in\mathcal{I}^o(t)} \left(\sup_{J\in\mathcal{I}^o(t,I)} \Delta\psi(t, J)\right) = \limsup_{I\in\mathcal{I}^o(t)} \Delta\psi(t, I) \quad (t \in W^o).$$

Hence, we obtain by (2.7) and (2.10) that

$$\sigma(x^*, \Gamma(t)) = \sigma(x^*, L_\Phi(t)) \le \limsup_{I\in\mathcal{I}^o(t)} \Delta\psi(t, I) = g(t) \ (t \in W^o \setminus (N^{x^*} \cup Z^{x^*} \cup Z)).$$

Thus, for each $t \in W^o \setminus (N^{x^*} \cup Z^{x^*} \cup Z)$, we have

$$g(t) = \sigma(x^*, L_\Psi(t)) \le \sigma(x^*, L_\Phi(t)) \le g(t) \qquad \text{(see (2.12))}.$$

This means that $\sigma(x^*, \Gamma(t)) = g(t)$ at almost all $t \in W$. Consequently the function $\sigma(z^*, \Gamma(\cdot))$ is Lebesgue integrable and by (2.9) follows that

$$\sigma(x^*, \Phi(I)) = \int_I \sigma(z^*, \Gamma(t))d\lambda, \quad \text{for all } I \in \mathcal{I}.$$

Since x^* was arbitrary, the last result together with [18, Lemma 2.4] implies that Γ is a scalar derivative of the additive interval multimeasure Φ. Therefore, by [19, Theorem 2.6] we obtain that Γ is Pettis integrable with $\Phi(I) = (P)\int_I \Gamma(t)d\mu$ for all $I \in \mathcal{I}$, and this completes the proof.

The next Radon-Nikodým theorem works for weakly dominated additive interval multimeasures and improves [20, Theorem 2.7].

Theorem 2.3 *Let $\Phi : \mathcal{I} \to cwk(X)$ be an additive interval multimeasure and let $Q \in cwk(X)$ be a weakly compact set such that $\Phi(I) \subset |I|Q$ for every $I \in \mathcal{I}$. Then, there exists a multifunction $\Gamma : W \to cwk(X)$ Pettis integrable such that*

(i) $\Phi(I) = (P)\int_I \Gamma(t)d\lambda$ for every $I \in \mathcal{I}$,
(ii) for each $\varphi \in \mathcal{S}_\Phi$ there exists a variationally McShane integrable function $f \in \mathcal{S}_\Gamma$ with $(M)\int_I f(t)d\lambda = \varphi(I)$ for all $I \in \mathcal{I}$.

Proof Since for any finite collection $\pi \subset \mathcal{I}$ we have

$$\sum_{I\in\pi} ||\Phi(I)|| \leq \left(\sum_{I\in\pi} |I|\right) ||Q||,$$

it follows that Φ is sAC, and since $L_\Phi(t, I) \subset Q$ for all $t \in W^o$ and $I \in \mathcal{I}^o(t)$, we obtain by [21, Theorem 1, p.136] that $L_\Phi(t) \neq \emptyset$ for all $t \in W^o$. Hence, $L_\Phi(t) \in cwk(X)$ for all $t \in W^o$ and we can define

$$\Gamma : \Omega \to cwk(X), \quad \Gamma(t) = \begin{cases} L_\Phi(t) & t \in W^o \\ \{\theta\} & t \in \partial W \end{cases}. \tag{2.13}$$

and if $\varphi \in \mathcal{S}_\Phi$, then

$$L_\varphi(t) \neq \emptyset \text{ and } L_\varphi(t) \subset L_\Phi(t) \text{ for all } t \in W^o.$$

If we define a function $f : W \to X$ such that $f(t) \in L_\varphi(t)$ for all $t \in W^o$ and $f(t) = \theta$ for every $t \in \partial W$, then $f \in \mathcal{S}_\Gamma$ and f is variationally McShane integrable with $\varphi(I) = (M)\int_I f(t)d\lambda$ for all $I \in \mathcal{I}$ (see [19, Lemma 2.3]).

It remains to prove that Γ is Pettis integrable with $\Phi(I) = (P)\int_I \Gamma(t)d\lambda$ for every $I \in \mathcal{I}$. Thanks to Theorem 2.2, it is enough to show that Φ has nonempty weakly density. To do this, fix an arbitrary $x^* \in X^*$ and write $\Psi = \Phi x^*$. Then, $L_\Psi(t, I) \subset Q$ for all $t \in W^o$ and $I \in \mathcal{I}^o(t)$ and, consequently, we obtain by [21, Theorem 1, p.136] that $L_\Psi(t) \neq \emptyset$ for all $t \in W^o$. Since x^* was arbitrary, the last result implies that Φ has nonempty weakly density, and this completes the proof.

The last Radon-Nikodým theorem works for sAC additive interval multimeasures taking convex weakly compact values in Banach spaces possessing the Radon-Nikodým property.

Theorem 2.4 *Let $\Phi : \mathcal{I} \to cwk(X)$ be an additive interval multimeasure. Assume that*

- *X has the Radon-Nikodým property,*
- *Φ is sAC.*

Then, there exists a multifunction $\Gamma : W \to bcc(X)$ Pettis integrable such that

(i) $\Phi(I) = (P)\int_I \Gamma(t)d\lambda$ for every $I \in \mathcal{I}$,
(ii) for each $\varphi \in \mathcal{S}_\Phi$ there exists a variationally McShane integrable function $f \in \mathcal{S}_\Gamma$ with $(M)\int_I f(t)d\lambda = \varphi(I)$ for all $I \in \mathcal{I}$.

Proof (i) Fix an arbitrary $x^* \in X^*$ and write $\Psi = \Phi x^*$. By [19, Lemma 1.1] there exists $\psi \in \mathcal{S}_\Psi$. Since $||\Psi(I)|| \leq ||\Phi(I)||$ for every $I \in \mathcal{I}$, it follows that Ψ is sAC. Hence, the additive interval function ψ is also sAC. Therefore, by [18, Lemma 2.3] there exists a λ-continuous countably additive vector measure $m : \mathcal{L} \to X$ of bounded variation such that $m(I) = \psi(I)$ for all $I \in \mathcal{I}$. From the fact that X has the Radon-Nikodým property, it follows that there exists a Bochner (or variationally McShane) integrable function $f : W \to X$ which is the Radon-Nikodým derivative of m with respect to λ, i.e.,

$$\psi(I) = m(I) = (B)\int_I f(t)d\lambda = (M)\int_I f(t)d\lambda \text{ for every } I \in \mathcal{I}.$$

By [18, Theorem 2.7] and [19, Lemma 2.2] the net $(\Delta\psi(t, I))_{I\in\mathcal{I}^o(t)}$ converges to $f(t)$ at almost all $t \in W^o$. Thus, $f(t) \in L_\psi(t)$ at almost all $t \in W^o$, and since $L_\psi(t) \subset L_\Psi(t)$ for all $t \in W^o$, it follows that $f(t) \in L_\Psi(t)$ at almost all $t \in W^o$. Thus, $L_\Psi(t) \neq \emptyset$ at almost all $t \in W^o$. Since x^* was arbitrary, the last result yields that

Φ has nonempty weakly density. Consequently, by Theorem 2.2 there exists a multifunction $\Gamma : W \to bcc(X)$ Pettis integrable such that $\Gamma(t) = L_\Phi(t)$ at almost all $t \in W$ and $\Phi(I) = (P)\int_I \Gamma(t)d\lambda$ for every $I \in \mathcal{I}$. (ii) Assume that $\varphi \in \mathcal{S}_\Phi$. Then, there exists a variationally McShane integrable function $g : W \to X$ such that

$$\varphi(I) = (M)\int_I g(t)d\lambda \text{ for every } I \in \mathcal{I}.$$

Hence, there exists a subset $Z \subset W$ with $|Z| = 0$ such that

$$g(t) \in L_\varphi(t) \subset L_\Phi(t) = \Gamma(t) \text{ for all } t \in W^o \setminus Z.$$

Define a function $h : W \to X$ as follows

$$h(t) = \begin{cases} g(t) & t \in W^o \setminus Z \\ x_t & t \in Z \cup \partial W \end{cases},$$

where $x_t \in \Gamma(t)$ for all $t \in Z \cup \partial W$. It follows that $h \in \mathcal{S}_\Gamma$ and h is variationally McShane integrable with $\varphi(I) = (M)\int_I h(t)d\lambda$ for every $I \in \mathcal{I}$, and this completes the proof.

Declarations

Conflict of interest: The author declare that there is no conflict of interests regarding the publication of this paper.

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