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ORIGINAL RESEARCH

Stratified bundles on the Hilbert Scheme of n points

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Abstract Let k be an algebraically closed field of characteristic $p > 3$ and S be a smooth projective surface over k with k-rational point x. For $n \geq 2$, let $S^{[n]}$ denote the Hilbert scheme of n points on S. In this note, we compute the fundamental group scheme $\pi^{\text{alg}}(S^{[n]}, \tilde{nx})$ defined by the Tannakian category of stratified bundles on $S^{[n]}$.

Keywords Stratified bundles · Tannakian groups · Hilbert scheme of points on a surface

1 Introduction

For a variety X over $\mathbb{C}$, one has the classical notion of the fundamental group $\pi_1(X^{\text{an}}, x)$ defined using the analytic topology on X. Over arbitrary base fields k, one has several analogues of the fundamental group defined in terms of algebro-geometric information.

In [15], Grothendieck introduced the notion of étale fundamental group $\pi^{\text{ét}}(X, x)$, where X is a connected scheme and x is a geometric point of X, in terms of the finite etale covers of X. In [12], Nori defined the Nori fundamental group scheme $\pi^N(X, x)$, where X is a connected, reduced and complete scheme over a perfect field k and x is a k-rational point, via Tannakian reconstruction using the category of essentially finite vector bundles on X. The definition of $\pi^N(X, x)$ was extended to the case of connected and reduced k-schemes in [13]. Another analogue, the S-fundamental group scheme $\pi^S(X, x)$ was introduced and studied by Langer in [10] and [11] for smooth projective varieties X over an algebraically closed field k. It is defined via Tannakian reconstruction using the category of numerically flat vector bundles on X. The S-fundamental group scheme for a smooth projective curve C over an algebraically closed field k was already introduced and studied in [2].

The variant of the fundamental group scheme which is of prime importance in this note is the algebraic fundamental group $\pi^{\text{alg}}(X, x)$. In [7], Gieseker defined $\pi^{\text{alg}}(X, x)$ as the fundamental group scheme corresponding to the Tannakian category of $\mathcal{D}_X$-modules, where $\mathcal{D}_X$ is the sheaf of differential operators on X. For X smooth over a field of positive characteristic, Gieseker introduced the notion of *stratified bundles* and showed that the category of $\mathcal{D}_X$-modules is tensor equivalent to the category of stratified bundles on X. Stratifed bundles were further studied in [4] and [1]. Precise definitions and statements will be given in the next section.

Let S be a smooth projective surface over k. For $n \geq 2$, let $S^{[n]}$ denote the Hilbert scheme of n points on S. It is well known that $S^{[n]}$ is a smooth projective variety of dimension $2n$. In [14], the authors show that for char $k > 3$ and $n \geq 2$, there is an isomorphism of affine group schemes over k

$$\pi^{\dagger}(S, x)_{\text{ab}} \to \pi^{\dagger}(S^{[n]}, \tilde{nx})$$

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where $\dagger = S, N$ or ét.

In this note, we extend their results to the case of π^{alg} and prove the following theorem.

Theorem *Let char $k > 3$ and $n \geq 2$. There is an isomorphism of affine group schemes over k*

$$f : \pi^{alg}(S, x)_{ab} \to \pi^{alg}(S^{[n]}, n\tilde{x})$$

In section 2, we recall the definition of stratified bundles and some of their basic properties. The formalism of Tannakian reconstruction is recalled in section 3 and used to define the algebraic fundamental group $\pi^{\text{alg}}(X, x)$.

The geometrical properties of the Hilbert scheme of n points on a smooth projective surface are in section 4. In section 5, we prove a result about descent of stratified bundles which allows us to define the homomorphism f by defining the associated functor of Tannakian categories. The concluding section 6 establishes the main theorem by showing that f is an isomorphism.

2 Stratified bundles

Let k be a field of characteristic p and X be a noetherian scheme over k. Stratified bundles on X are sequences of coherent sheaves on X satisfying infinite Frobenius descent. More precisely, the category of stratified bundles on X, denoted $\mathcal{S}(X)$, consists of

- **Objects** $(\mathcal{E}_i, \alpha_i)$ are sequences of coherent $\mathcal{O}_X$-modules $\mathcal{E}_i$, $i \in \mathbb{N}$ along with isomorphisms

$$\alpha_i : F^*\mathcal{E}_{i+1} \to \mathcal{E}_i$$

  for all $i \in \mathbb{N}$, where F is the absolute Frobenius on X.
- **Morphisms** $\phi : (\mathcal{E}_i, \alpha_i) \to (\mathcal{F}_i, \beta_i)$ consists of a sequence of $\mathcal{O}_X$-module morphisms $\phi_i : \mathcal{E}_i \to \mathcal{F}_i$ such that $\phi_i \circ \alpha_i = \beta_i \circ F * (\phi_{i+1})$

Let $f : Y \to X$ be a morphism and $(\mathcal{E}_i, \alpha_i)$ be a stratified bundle on X. Then we can define the pullback along f, denoted $f^*(\mathcal{E}_i, \alpha_i)$, as consisting of the sequence of O_Y coherent sheaves f^*E_i and isomorphisms are given by the composite maps

$$F^*f^*\mathcal{E}_{i+1} \xrightarrow{\gamma_{\mathcal{E}_{i+1}}} f^*F^*\mathcal{E}_{i+1} \xrightarrow{f^*(\alpha_i)} f^*\mathcal{E}_i$$

where $\gamma : F^*f^* \to f^*F^*$ is the natural isomorphism of functors. We note that $\mathcal{S}(X)$ is contravariant in X. One also has a tensor product on $\mathcal{S}(X)$ defined by taking term by term tensor product. Also $\mathcal{S}(X)$ is an abelian category [see [1], Proposition 4.4].

We recall some well known results about stratified bundles [cf [4,7]].

Proposition *If $(\mathcal{E}_i, \alpha_i)$ is a stratified bundle on X, then $\mathcal{E}_i$ is a locally free $\mathcal{O}_X$-module for all $i \in \mathbb{N}$.*

This allows us to define duals of stratified bundles, making $\mathcal{S}(X)$ into an *abelian rigid tensor category*.

The *rank* of a stratified bundle $(\mathcal{E}_i, \alpha_i)$ is defined to be the rank of $\mathcal{E}_0$. The *trivial stratified bundles* on X are of the form $\oplus(\mathcal{O}_X, ...; F^*, ...)$.

Let $\mathcal{D}_X$ be the sheaf of differential operators on X. The category of $\mathcal{D}_X$-modules consists of

- **Objects** coherent $\mathcal{O}_X$ modules $\mathcal{E}$ equipped with a $\mathcal{D}_X$-action i.e a morphism of $\mathcal{O}_X$-algebras

$$\mathcal{D}_X \to \mathcal{E}\text{nd}_k(\mathcal{E})$$

- **Morphisms** $\mathcal{O}_X$-linear maps $\mathcal{E} \to \mathcal{F}$ compatible with the $\mathcal{D}_X$ action

A theorem of Katz [[7], Theorem 1.3] shows that for X smooth over k, then the category of stratified bundles on X and the category of $\mathcal{D}_X$-modules are tensor equivalent to each other.

We close this section with the definition of G equivariant stratified bundles on a variety X admitting action of a group G on it.

Definition A stratified bundle $(\mathcal{E}_i, \alpha_i)$ is said to be a G-equivariant stratified bundle if $\mathcal{E}_i$ are G-equivariant vector bundles and α_i are G-equivariant $\mathcal{O}_X$ module morphisms.

3 Tannakian categories and fundamental group schemes

In this section we recall the definition and basic properties of Tannakian categories. We then recall Gieseker's definition of the fundamental group scheme π^{alg} using the Tannakian formalism.

3.1 Tannakian Categories and affine group schemes

Tannakian categories were defined and studied in [3] to formalize the properties of $\text{Rep}_k(G)$, the category of finite dimensional k-representations of G, an affine group scheme over k.

Definition (*Neutral Tannakian Categories*) A rigid abelian tensor category $\mathcal{C}$ with $\text{End}\ \mathbb{I} = k$ is a *neutral Tannakian category* if it admits an exact faithful k-linear tensor functor $\omega : \mathcal{C} \to \text{Vec}_k$. Any such functor is said to be a *fiber functor* for $\mathcal{C}$.

Given a neutral Tannakian category $(\mathcal{C}, \otimes, \omega, \mathbb{I})$, we define the functor $\text{Aut}^{\otimes}(\omega) : k - \text{algebra} \to \text{Sets}$ such that for k-algebra R, $\text{Aut}^{\otimes}(\omega)(R)$ consists of the families (λ_X) for $X \in \text{ob}(\mathcal{C})$, where λ_X is a R-linear automorphism of $X \otimes R$ such that $\lambda_{X_1 \otimes X_2} = \lambda_{X_1} \otimes \lambda_{X_2}$, $\lambda_{\mathbb{I}} = id_R$, and

$$\lambda_Y \circ (\alpha \otimes 1) = (\alpha \otimes 1) \circ \lambda_X : X \otimes R \to Y \otimes R$$

for all morphisms $\alpha : X \to Y$.

Theorem (Main theorem for neutral Tannakian categories, [3], Theorem 2.11) *Let $(\mathcal{C}, \otimes)$ be a rigid abelian tensor category such that $k = End(\mathbb{I})$ and let $\omega : \mathcal{C} \to Vec_k$ be an exact faithful tensor functor. Then*

- *The functor $Aut^{\otimes}(\omega)$ of k-algebras is represented by an affine group scheme G.*
- *The functor $\mathcal{C} \to Rep_k(G)$ is an equivalence of tensor categories.*

Theorem *Let $(\mathcal{C}, \otimes, \omega, \mathbb{I})$ and $(\mathcal{C}', \otimes, \omega', \mathbb{I}')$ be neutral Tannakian categories which correspond to the representation categories of the affine k group schemes G and G' respectively. Then any functor of Tannakian categories from $\mathcal{C} \to \mathcal{C}'$ is induced by a unique morphism of affine k group schemes $G' \to G$.*

This theorem allows us to define many variants of fundamental groups of a scheme X by considering different Tannakian categories naturally associated with X. The following result is very useful in establishing a given morphism between affine group schemes is an isomorphism.

Theorem ([3], Theorem 2.21) *Let $f : G \to G'$ be a homomorphism of group schemes over k and $Rep\ (f) : Rep\ (G') \to Rep\ (G)$ be the corresponding functor of Tannakian categories. Then*

- *f is faithfully flat if and only if $Rep\ (f)$ is fully faithful and has essential image closed under subobjects i.e for $V' \in Rep\ (G')$ and suboject $W \subset Rep\ (f)(V')$, there is a subobject $W' \subset V'$ in $Rep\ (G')$ such that $Rep\ (f)(W') \simeq W$ in $Rep\ (G)$*
- *f is closed immersion if and only if every object of $Rep\ (G)$ is a subquotient of some object in the essential image of $Rep\ (f)$.*

We finish by recalling a basic result on affine group schemes (we refer to section 4.1 in [14] for details).

Let G be a affine group scheme over k, G_{ab} be its abelianization (i.e the maximal abelian quotient of G) and $\alpha : G \to G_{\text{ab}}$ be the (faithfully flat) quotient morphism . We can then define the composite morphism

$$\phi : G^n \xrightarrow{\alpha^n} G^n_{\text{ab}} \xrightarrow{m} G_{\text{ab}}$$

where m is the multiplication homomorphism. As $\mathfrak{S}_n$ acts on the k-group scheme G^n, we can define the notion of a $\mathfrak{S}_n$-invariant group morphism $\psi : G^n \to H$ for any k-group scheme H.

Lemma 3.1 *Let G and H be two group schemes over k. For an integer $n \geq 2$, the set of $\mathfrak{S}_n$-invariant group morphisms $G^n \to H$ is in bijective correspondence with the set of group morphism $G_{ab} \to H$ i.e any morphism of k-group schemes $\eta : G^n \to H$ which is $\mathfrak{S}_n$-invariant factors uniquely through a morphism $\psi : G_{ab} \to H$ such that $\eta = \psi \circ \phi$*

3.2 The group scheme $\pi^{\text{alg}}(X, x)$

Classically, over $\mathbb{C}$, the Riemann-Hilbert correspondence identifies the category of vector bundles equipped with integrable connections on a smooth connected projective variety $X/\mathbb{C}$ with the category of representations of the topological fundamental group $\pi^{\text{top}}(X, x)$ for some chosen base point x. Via GAGA, this gives a purely algebraic description of the category of representations of the topological fundamental group $\pi(X, x)$. This category (equipped with the fiber functor $(E, \nabla) \to E_x$) is a neutral Tannakian category and can be identified, via the Tannakian formalism, with the representation category of the proalgebraic completion of the topological fundamental group, denoted as $\pi^{\text{top}}(X, x)^{\text{alg}}$.

Over a field k of characteristic 0, the category of flat connections on a smooth variety X is tensor equivalent to the category of $\mathcal{D}_X$-modules. However over a field of characteristic p, the category of flat connections on X is not as well behaved as the category of $\mathcal{D}_X$-modules and one defines a fundamental group scheme for X by Tannakian formalism using the category of $\mathcal{D}_X$-modules. By Katz's theorem mentioned before, the fundamental group coincides with the one defined using $\mathcal{S}(X)$ below.

Let $x \in X(k)$ be a k-rational point. Then the abelian rigid tensor category $\mathcal{S}(X)$ is neutralized by the fiber functor

$$T_x : \mathcal{S}(X) \to Vec_k$$
$$(\mathcal{E}_i) \mapsto \mathcal{E}_0|_x$$

The fundamental group scheme defined by the neutral Tannakian category $(\mathcal{S}(X), \otimes, T_x, (\mathcal{O}_X, F^*))$ is called the algebraic fundamental group of X based at x and is denoted by $\pi^{\text{alg}}(X, x)$.

The following basic properties of π^{alg} are well known.

- (Independence of basepoint) Let X be a geometrically connected, smooth projective k-scheme. Then for all $x_1, x_2 \in X(k)$, one has

$$\pi^{\text{alg}}(X, x_1) \simeq \pi^{\text{alg}}(X, x_2)$$

- (Product rule) For X_1, X_2 geometrically connected and smooth over k and $x_i \in X_i(k)$, there is an isomorphism

$$\pi^{\text{alg}}(X_1, x_1) \times \pi^{\text{alg}}(X_2, x_2) \to \pi^{\text{alg}}(X_1 \times X_2, (x_1, x_2))$$

- For X smooth and open immersion $U \xrightarrow{i} X$ such that the complement of U in X has codimension ≥ 2 and $x \in U(k)$, then the homomorphism

$$\pi^{\text{alg}}(U, x) \to \pi^{\text{alg}}(X, x)$$

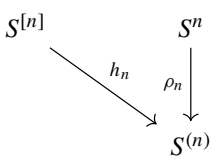
  associated to the restriction functor $i^* : \mathcal{S}(X) \to \mathcal{S}(U)$ is an isomorphism.

4 Geometry of Hilbert Scheme of points

Let S be a smooth projective surface over k. We fix notation as follows

- S^n denotes the n-fold cartesian product of S with itself.
- $S^{(n)}$ denotes the nth symmetric product of S defined as the quotient $S^n/\mathfrak{S}_n$, where $\mathfrak{S}_n$ denotes the symmetric group on n letters.
- $S^{[n]}$ denotes the Hilbert scheme of n points on S.

Let $\rho : S^n \to S^{(n)}$ be the quotient map and $h : S^{[n]} \to S^{(n)}$ be the Hilbert-Chow morphism. We write $S^{(n)}_\circ$ for the open subset of $S^{(n)}$ consisting of distinct points with $S^{[n]}_\circ := h^{-1}(S^{(n)}_\circ)$ and $S^n_\circ := \rho^{-1}(S^{(n)}_\circ)$. The map $h_{n,\circ} : S^{[n]}_\circ \to S^{(n)}_\circ$ is an isomorphism. We have the diagram:



In general, Hilbert schemes of points on a projective variety display a lot of pathological features. But in [5] the author shows that, in the case of smooth projective surface S, $S^{[n]}$ is a smooth projective variety. Thus, in this case, the Hilbert-Chow morphism $h : S^{[n]} \to S^{(n)}$ is a resolution of singularities.

One can consider $S^{(n)}$ as the set of effective 0-cycles of degree n on $S^{(n)}$. In this case it is easy to see that $S^{(n)}$ admits a stratification by *type*, where the type of a 0-cycle y of degree n is a tuple $(n_1, \ldots, n_r)$ where y can be written as

$$y = \Sigma_{j=1}^{r} n_j x_j$$

where x_j are distinct points of S with multiplicities $n_1 \geq n_2 \geq \cdots \geq n_r$, where n_j are positive integers.

Let $C(n_1, \ldots, n_r)$ denote the subset of $S^{(n)}$ consisting of points of the type $(n_1, \ldots, n_r)$. Let $S^{(n)}_* = C(1, \ldots, 1) \bigcup C(2, 1, \ldots, 1)$ denote the open subset of $S^{(n)}$ consisting of points of type $(1, \ldots, 1)$ and $(2, 1, \ldots, 1)$. Let $S^{[n]}_*$ and S^n_* denote the preimage of $S^{(n)}_*$ under h and ρ respectively.

We recall some basic properties below which we will need later (we refer to [5], [14] for details).

- The subsets $C(n_1, \ldots, n_r)$ are nonsingular of dimension $2r$.
- The closed subset $S^{(n)} \setminus S^{(n)}_*$ is of codimension ≥ 2 in $S^{(n)}$.
- The closed subset $S^{[n]} \setminus S^{[n]}_*$ is of codimension 2 in $S^{[n]}$.
- The closed subset $S^n \setminus S^n_*$ is of codimension ≥ 4 in S^n.
- The closed subset $S^{(n)}_* \setminus S^{(n)}_\circ$ is of codimension 2 in $S^{(n)}_*$.
- When characteristic of $k \neq 2$, for $y \in C(2, 1, \ldots, 1)$, the scheme theoretic fiber $h^{-1}(y)$ is isomorphic to $\mathbb{P}^1_k$. In fact, $S^{[n]}_*$ is the blowup of $S^{(n)}_*$ along $C(2, 1, \ldots, 1)$.

We end this section by recalling a result of Fogarty ([6], Proposition 3.6).

Proposition *If L is a $\mathfrak{S}_n$-invariant line bundle on S^n, there exists a line bundle L' on $S^{(n)}$ such that $h^*L' \simeq L$.*

It follows that L' in the proposition is isomorphic to $\sigma_*(L)^{\mathfrak{S}_n}$

5 The functor between Tannakian categories

Let S be a smooth projective surface over k and $(\mathcal{E}_i, \alpha_i)$ be a stratified bundle on $S^{[n]}$. Restricting to $S^{[n]}_*$ gives us a functor

$$i^* : \mathcal{S}(S^{[n]}) \to \mathcal{S}(S^{[n]}_*)$$

which is a equivalence of categories as $S^{[n]}_*$ is the complement of a codimension 2 closed subset of $S^{[n]}$.

Next we show that a stratified bundle on $S^{[n]}_*$ can be pushed forward under h to get a stratified bundle on $S^{(n)}_*$. First we begin by a result on descent of vector bundles along the morphism $h : S^{[n]}_* \to S^{(n)}_*$. Similar results have been established by authors in [9] and [14].

Proposition 1 *Assume char $k \neq 2$. Let $\mathcal{E}$ be a vector bundle on $S^{[n]}_*$ which restricts to trivial vector bundles on the fibers of h over $S^{(n)}_*$. Then $h_*\mathcal{E}$ is a locally free $\mathcal{O}_{S^{(n)}_*}$-module. Moreover the natural map*

$$h^*h_*(\mathcal{E}) \to \mathcal{E}$$

is an isomorphism.

Proof Let $x \in S^{(n)}_*$ be a point of type $(2, 1, \ldots, 1)$. Then by assumption, the fiber of h over x is isomorphic to $\mathbb{P}^1_k$. Let $\mathcal{J}$ be the ideal sheaf of the closed subscheme $h^{-1}(x)$ and $\mathcal{I}_x$ be the ideal sheaf of the closed point x. We have

$$\mathcal{J} = \mathcal{I}_x \mathcal{O}_{S^{[n]}_*}$$

For all $n \geq 1$, let Y_n denote the closed subscheme of $S^{[n]}_*$ corresponding to the ideal sheaf $\mathcal{J}^n$. Consider the following short exact sequence of sheaves on $S^{[n]}_*$

$$0 \to \mathcal{J} \otimes \mathcal{E} \to \mathcal{E} \to \mathcal{E}|_{Y_1} \to 0$$

Pushing forward by h, we get the following exact sequence of sheaves on $S_*^{(n)}$

$$h_*\mathcal{E} \to H^0(Y_1, \mathcal{E}|_{Y_1}) \to R^1h_*(\mathcal{J} \otimes \mathcal{E})$$

We claim that the completion of $R^1h_*(\mathcal{J} \otimes \mathcal{E})$ at the maximal ideal m_x in $\mathcal{O}_{S_*^{(n)},x}$ is 0. The proof uses the theorem of formal functions which says that

$$(R^1h_*(\mathcal{J} \otimes \mathcal{E}))^\wedge \simeq \varprojlim H^1(Y_n, \mathcal{J} \otimes \mathcal{E} \otimes \mathcal{O}_{S_*^{[n]}}/\mathcal{J}^n)$$

We prove by induction that $H^1(Y_n, \mathcal{J} \otimes \mathcal{E} \otimes \mathcal{O}_{S_*^{[n]}}/\mathcal{J}^n) = 0$. We note that the sheaves $\mathcal{J}^n/\mathcal{J}^{n+1} \simeq Sym^n(\mathcal{J}/\mathcal{J}^2)$ on $Y_1 \simeq \mathbb{P}^1_k$ are locally free. These sheaves are also globally generated over Y_1 as we have the surjection

$$m_x^n/m_x^{n+1} \otimes_{\mathcal{O}_{S_*^{(n)},x}} \mathcal{O}_{S_*^{[n]}} \simeq \mathcal{I}_x^n/\mathcal{I}_x^{n+1} \otimes_{\mathcal{O}_{S_*^{(n)}}} \mathcal{O}_{S_*^{[n]}} \twoheadrightarrow \mathcal{J}^n/\mathcal{J}^{n+1}$$

As $\mathcal{J}^n/\mathcal{J}^{n+1}$ is locally free on $Y_1 \simeq \mathbb{P}^1_k$ and globally generated, it is a direct sum of line bundles each of which has degree ≥ 0. Thus one gets the base case of induction from degree considerations, as

$$H^1(Y_1, \mathcal{J} \otimes \mathcal{E} \otimes \mathcal{O}_{S_*^{[n]}}/\mathcal{J} = H^1(Y_1, \mathcal{J}/\mathcal{J}^2 \otimes \mathcal{E}_{Y_1}) = 0$$

Assume that the claim is true for n. Then the proof for $n+1$ follows from the long exact sequence in cohomology attached to the short exact sequence of sheaves on Y_{n+1}

$$0 \to \mathcal{J}^{n+1}/\mathcal{J}^{n+2} \otimes \mathcal{E} \to \mathcal{J}/\mathcal{J}^{n+2} \otimes \mathcal{E} \to \mathcal{J}/\mathcal{J}^{n+1} \otimes \mathcal{E} \to 0$$

which gives us the exact sequence

$$H^1(Y_{n+1}, \mathcal{J}^{n+1}/\mathcal{J}^{n+2} \otimes \mathcal{E}) \to H^1(Y_{n+1}, \mathcal{J}/\mathcal{J}^{n+2} \otimes \mathcal{E}) \to H^1(Y_{n+1}, \mathcal{J}/\mathcal{J}^{n+1} \otimes \mathcal{E})$$

We know $H^1(Y_n, \mathcal{J}^{n+1}/\mathcal{J}^{n+2} \otimes \mathcal{E}) = H^1(Y_1, \mathcal{J}^{n+1}/\mathcal{J}^{n+2} \otimes \mathcal{E}) = 0$ (by degree consideration) and $H^1(Y_{n+1}, \mathcal{J}/\mathcal{J}^{n+1} \otimes \mathcal{E}) = H^1(Y_n, \mathcal{J}/\mathcal{J}^{n+1} \otimes \mathcal{E}) = 0$ (by induction hypothesis), thus we get

$$H^1(Y_{n+1}, \mathcal{J}/\mathcal{J}^{n+2} \otimes \mathcal{E}) = 0$$

Thus the stalk of $R^1h_*(\mathcal{J} \otimes \mathcal{E})$ at x is 0.

This shows that the natural map $h_*\mathcal{E} \to H^0(Y_1, \mathcal{E}|_{Y_1})$ is surjective in a neighbourhood of x. Let $f_1, ..., f_r$ be a basis of $H^0(Y_1, \mathcal{E}|_{Y_1})$. Let $\mathrm{Spec}(R)$ be an affine neighbourhood of x where the natural map is surjective and let $\tilde{f}_i \in \Gamma(\mathrm{Spec}(R), h_*\mathcal{E}) = \Gamma(h^{-1}(\mathrm{Spec}(R)), \mathcal{E})$ be lifts of f_i. Using $\tilde{f}_i$ one defines a homomorphism

$$\mathcal{O}_{S_*^{[n]}|_{h^{-1}(\mathrm{Spec}(R))}}^{\oplus r} \to \mathcal{E}$$

on $h^{-1}(\mathrm{Spec}(R)$ which is a surjection (and hence an isomorphism) on Y_1. As h is proper, there exists a smaller affine neighbourhood U of x over which there is an isomorphism

$$\mathcal{O}_V^{\oplus r} \simeq \mathcal{E}$$

where $V = h^{-1}(U)$. Applying h_*, we get

$$(h_*\mathcal{O}_V)^{\oplus r} \simeq h_*\mathcal{E}$$

As $S_*^{(n)}$ is normal and $h : S_*^{[n]} \to S_*^{(n)}$ is birational with connected fibers, by a form of Zariski's main theorem [cf [8], Corollary 11.3 and 11.4], we have that $h_*\mathcal{O}_V \simeq \mathcal{O}_U$ and thus $h_*\mathcal{E}$ is locally free. The natural morphism

$$h^*h_*(\mathcal{E}) \to \mathcal{E}$$

is clearly an isomorphism. □

Let $\mathbb{VB}_{S_*^{(n)}}$ be the category of locally free sheaves on $S_*^{(n)}$ and $\mathbb{VB}^h_{S_*^{[n]}}$ be the category of locally free sheaves on $S_*^{[n]}$ which restrict to trivial vector bundles on the fibers of h. Proposition 1 above gives us an equivalence of categories.

Proposition 2 *Assume char $k \neq 2$. The pushforward functor*

$$h_* : \mathbb{VB}^h_{S^{[n]}_*} \to \mathbb{VB}_{S^{(n)}_*}$$

is an equivalence of categories with the inverse given by

$$h^* : \mathbb{VB}_{S^{(n)}_*} \to \mathbb{VB}^h_{S^{[n]}_*}$$

Proof We observe that if $\mathcal{E}' \simeq h^*(\mathcal{E})$, then $\mathcal{E} \simeq h_*\mathcal{E}'$. This shows that h_* is essentially surjective. The natural map

$$\mathrm{Hom}_{S^{(n)}_*}(h_*\mathcal{E}, h_*\mathcal{F}) \to \mathrm{Hom}_{S^{[n]}_*}(\mathcal{E}, \mathcal{F})$$

is bijective. Thus h_* is an equivalence of categories. □

Corollary *For all $\mathcal{E} \in \mathbb{VB}^h_{S^{[n]}_*}$, the natural map*

$$F^*h_*(\mathcal{E}) \to h_*F^*(\mathcal{E})$$

is an isomorphism over $S^{(n)}_*$.

Proof As $F^*\mathcal{E}$ is also an object of $\mathbb{VB}^h_{S^{[n]}_*}$, thus both sheaves are locally free of the same rank. Thus it suffices to show that the natural map

$$F^*h_*(\mathcal{E}) \to h_*F^*(\mathcal{E})$$

is surjective. As F is faithfully flat on the smooth locus of $S^{(n)}_*$, the claim holds on the smooth locus. Let $x \in S^{(n)}_*$ be of type $(2, 1, \ldots, 1)$. Then the restriction of $F^*h_*(\mathcal{E})$ to x is naturally isomorphic to $H^0(Y_1, \mathcal{E}|_{Y_1})$ and the restriction of $h_*F^*(\mathcal{E})$ to x is $H^0(Y_1, F^*(\mathcal{E}|_{Y_1})$. The restriction of the natural map to x is the map

$$F^* : H^0(Y_1, \mathcal{E}_1) \to H^0(Y_1, F^*\mathcal{E}_1)$$

which is surjective. □

By Theorem 2.2 of [7], we have that every stratified bundle on $\mathbb{P}^1_k$ is trivial. Thus the above results give us

Proposition 3 *Assume char $k \neq 2$. Let $(\mathcal{E}_i, \alpha_i)$ be a stratified bundle on $S^{[n]}_*$. Then $h_*(\mathcal{E}_i)$ is locally free $\mathcal{O}_{S^{(n)}_*}$-module for all $i \in \mathbb{N}$. Moreover the natural map*

$$h^*h_*(\mathcal{E}_i) \to \mathcal{E}_i$$

is an isomorphism. Furthermore the natural map

$$F^*h_*(\mathcal{E}_i) \to h_*F^*(\mathcal{E}_i)$$

is an isomorphism over $S^{(n)}_*$.

This allows us to define the pushforward of a stratified bundle $(\mathcal{E}_i, \alpha_i)$ on $S^{[n]}_*$. The pushforward denoted $h_*(\mathcal{E}_i, \alpha_i)$ is given by the sequence of vector bundles $h_*\mathcal{E}_i$ for all $i \in \mathbb{N}$ and the isomorphisms are given by the composite

$$F^*h_*(\mathcal{E}_{i+1}) \xrightarrow{\eta_{\mathcal{E}_{i+1}}} h_*F^*(\mathcal{E}_{i+1}) \xrightarrow{h_*(\alpha_i)} h_*(\mathcal{E}_i)$$

where $\eta : F^*h_* \to h_*F^*$ is the natural transformation.

Thus we get a functor

$$h_* : \mathcal{S}(S^{[n]}_*) \to \mathcal{S}(S^{(n)}_*)$$

h_* is additive tensor functor as on the smooth locus $S^{(n)}_\circ$ we have the isomorphisms

$$h_*((\mathcal{E}_i, \alpha_i) \oplus (\mathcal{F}_i, \beta_i))|_{S^{(n)}_\circ} \simeq h_*(\mathcal{E}_i, \alpha_i)|_{S^{(n)}_\circ} \oplus h_*(\mathcal{F}_i, \beta_i))|_{S^{(n)}_\circ}$$
$$h_*((\mathcal{E}_i, \alpha_i) \otimes (\mathcal{F}_i, \beta_i))|_{S^{(n)}_\circ} \simeq h_*(\mathcal{E}_i, \alpha_i)|_{S^{(n)}_\circ} \otimes h_*(\mathcal{F}_i, \beta_i))|_{S^{(n)}_\circ}$$

which extend to $S^{(n)}_*$ due to codimension reasons.

The following commutative diagram shows that $h^*h_*(\mathcal{E}_i, \alpha_i)$ is isomorphic to $(\mathcal{E}_i, \alpha_i)$ as stratified bundles with the isomorphism given by the natural morphisms $h^*h_*\mathcal{E}_i \to \mathcal{E}_i$.

$$\begin{array}{ccccccc} F^*h^*h_*\mathcal{E}_{i+1} & \xrightarrow{\gamma_{h_*\mathcal{E}_{i+1}}} & h^*F^*h_*\mathcal{E}_{i+1} & \xrightarrow{h^*\eta_{\mathcal{E}_{i+1}}} & h^*h_*F^*\mathcal{E}_{i+1} & \xrightarrow{h_*h^*\alpha_i} & h^*h_*\mathcal{E}_i \\ \downarrow & & & & \downarrow & & \downarrow \\ F^*\mathcal{E}_{i+1} & & \xrightarrow{\quad=\quad} & & F^*\mathcal{E}_{i+1} & \xrightarrow{\alpha_i} & \mathcal{E}_i \end{array}$$

Consider the pullback functor

$$\rho^* : \mathcal{S}(S_*^{(n)}) \to \mathcal{S}(S_*^n)$$

which takes values in the category of $\mathfrak{S}_n$-equivariant stratified bundles on S_*^n. Also we have the extension functor

$$j_* : \mathcal{S}(S_*^n) \to \mathcal{S}(S^n)$$

which is an equivalence of categories. Composing these functors together, we get a functor

$$T : \mathcal{S}(S^{[n]}) \to \mathcal{S}(S^n)$$

given by

$$T = j_* \circ \rho^* \circ h_* \circ i^*$$

Clearly T is an additive tensor functor. Note that h_* is fully faithful, $\rho^* : \mathcal{S}(S_*^{(n)}) \to \mathcal{S}(S_*^n)$ is fully faithful (as $\rho : S_\circ^n \to S_\circ^{(n)}$ is finite étale) and $j_* : \mathcal{S}(S_*^n) \to \mathcal{S}(S^n)$ is an equivalence of categories (due to codimension reasons). Thus T is fully faithful.

5.1 The homomorphism

Fix n distinct k-valued points $x_1, \ldots, x_k \in S(k)$. Let $\tilde{x} \in S^{[n]}$ such that $h(\tilde{x}) = \sigma(x_1, \ldots, x_n) = z \in S_\circ^{(n)}$. Then the categories $\mathcal{S}(S^{[n]})$ and $\mathcal{S}(S^n)$ are neutralized by the respective fiber functors

$$\begin{aligned} &\tau_{\tilde{x}} : \mathcal{S}(S^{[n]}) \to \mathrm{Vec}_k \\ &(\mathcal{E}_i, \alpha_i) \mapsto (\mathcal{E}_0)_{\tilde{x}} \\ &\tau_{(x_1,\ldots,x_n)} : \mathcal{S}(S^n) \to \mathrm{Vec}_k \\ &(\mathcal{F}_i, \beta_i) \mapsto (\mathcal{F}_0)_{(x_1,\ldots,x_n)} \end{aligned}$$

If $T((\mathcal{E}_i, \alpha_i)) = (\mathcal{F}_i, \beta_i)$ that we have natural isomorphisms $(\mathcal{E}_0)_{\tilde{x}} \simeq h_*(\mathcal{E}_0)_z \simeq (\mathcal{F}_0)_{(x_1,\ldots,x_n)}$.
Thus we have a functor of Tannakian categories

$$T : (\mathcal{S}(S^{[n]}), \otimes, \tau_{\tilde{x}}, (\mathcal{O}_{S^{[n]}}, d)) \to (\mathcal{S}(S^n), \otimes, \tau_{(x_1,\ldots,x_n)}, (\mathcal{O}_{S^n}, d))$$

which by the independence of basepoint property of $\mathcal{S}$ induces a functor of Tannakian categories

$$T : (\mathcal{S}(S^{[n]}), \otimes, \tau_{\tilde{nx}}, (\mathcal{O}_{S^{[n]}}, d)) \to (\mathcal{S}(S^n), \otimes, \tau_{(x,\ldots,x)}, (\mathcal{O}_{S^n}, d))$$

and hence a morphisms of the associated fundamental group schemes

$$\tilde{f} : \pi^{\mathrm{alg}}(S^n, (x, \ldots, x)) \to \pi^{\mathrm{alg}}(S^{[n]}, \tilde{nx})$$

Note that by proposition 3.2 we have

$$\pi^{\mathrm{alg}}(S^n, (x, \ldots, x)) \simeq \pi^{\mathrm{alg}}(S, x)^n$$

.

As

$$T : (\mathcal{S}(S^{[n]}), \otimes, T_{\tilde{nx}}, (\mathcal{O}_{S^{[n]}}, d)) \to (\mathcal{S}(S^n), \otimes, T_{(x,\ldots,x)}, (\mathcal{O}_{S^n}, d))$$

takes stratified bundles on $S^{[n]}$ to $\mathfrak{S}_n$-equivariant stratified bundles on S^n and a $\mathfrak{S}_n$-equivariant stratified bundles on S^n corresponds to a $\mathfrak{S}_n$-invariant representation of $\pi^{\mathrm{alg}}(S, x)^n$, by 3.1, $\tilde{f}$ factors uniquely through

$$f : \pi^{\mathrm{alg}}(S, x)_{\mathrm{ab}} \to \pi^{\mathrm{alg}}(S^{[n]}, \tilde{nx})$$

6 Isomorphism of fundamental group schemes

In this section, we show that f is an isomorphism of affine group schemes. We begin by proving a result about $\mathfrak{S}_n$-equivariant stratified line bundles on S^n.

Proposition 4 *Let (L_i, α_i) be a $\mathfrak{S}_n$-equivariant stratified line bundles on S^n. Then there exists a stratified line bundle $(\mathcal{L}_i, \beta_i)$ such that $\rho^*(\mathcal{L}_i, \beta_i) \simeq (L_i, \alpha_i)$*

Proof By Fogarty's result mentioned above, for any $\mathfrak{S}_n$-equivariant line bundle L_i there exists line bundle $\mathcal{L}_i \simeq \rho_* L_i^{\mathfrak{S}_n}$ such that $\rho^* \mathcal{L}_i \simeq L_i$. Pushing forward α_i and taking $\mathfrak{S}_n$ invariants we get the isomorphism

$$\rho_*(F^* L_{i+1})^{\mathfrak{S}_n} \xrightarrow{\rho_*(\alpha_i)^{\mathfrak{S}_n}} \rho_*(L_i)^{\mathfrak{S}_n}$$

We show that the natural homomorphism

$$F^*(\rho_*(L_i)^{\mathfrak{S}_n}) \to (F^* \rho_*(L_i)^{\mathfrak{S}_n}) \to (\rho_* F^*(L_i)^{\mathfrak{S}_n})$$

is an isomorphism. Pulling back the morphism under ρ, we get the commutative diagram

$$\begin{array}{ccc} \rho^* F^*((\rho_* L_i)^{\mathfrak{S}_n}) & \longrightarrow & \rho^*((\rho_* F^* L_i)^{\mathfrak{S}_n}) \\ \downarrow & & \downarrow \\ F^* L_i & \xrightarrow{=} & F^* L_i \end{array}$$

where the vertical morphisms are the natural morphism which are isomorphisms by Fogarty's theorem. By pushing forward under ρ and taking $\mathfrak{S}_n$ invariants we get that

$$F^*(\rho_*(L_i)^{\mathfrak{S}_n}) \to (\rho_* F^*(L_i)^{\mathfrak{S}_n})$$

is an isomorphism. We define β_i to be the composite isomorphism

$$F^*(\rho_*(L_i)^{\mathfrak{S}_n}) \to (\rho_* F^*(L_i)^{\mathfrak{S}_n}) \xrightarrow{\rho_*(\alpha_i)^{\mathfrak{S}_n}} \rho_*(L_i)^{\mathfrak{S}_n}$$

The commutative diagram also gives us that $\rho^*(\mathcal{L}_i, \beta_i) \simeq (L_i, \alpha_i)$ □

6.1 Faithfully flat

Next we show that the morphism f is faithfully flat

Proposition 5 *The homomorphism*

$$f : \pi^{alg}(S, x)_{ab} \to \pi^{alg}(S^{[n]}, \tilde{nx})$$

is faithfully flat.

Proof By [[3] Theorem 2.21], this is equivalent to showing that the functor

$$T : \mathcal{S}(S^{[n]}) \to \mathcal{S}(S^n)$$

is fully faithful and the essential image of T is closed under taking subobjects. We already know that T is fully faithful. Let $\mathcal{E}_\bullet = (\mathcal{E}_i, \alpha_i)$ be a stratified bundle on $S^{[n]}$ and $\mathcal{F}_\bullet := T(\mathcal{E}_\bullet)$ be the corresponding $\mathfrak{S}_n$-equivariant stratified bundle on S^n. If $\mathcal{F}'_\bullet \subset \mathcal{F}_\bullet$ is a $\mathfrak{S}_n$-equivariant stratified subbundle, then we need to show there exists $\mathcal{E}'_\bullet \subset \mathcal{E}_\bullet$ such that $T(\mathcal{E}'_\bullet) = \mathcal{F}'_\bullet$.

The proof proceeds by induction on the rank of $\mathcal{E}_\bullet$. If rank $\mathcal{E}_\bullet = 1$, the proof is immediate. Let rank $\mathcal{E}_\bullet \geq 2$

Then the stratified bundles $\mathcal{F}_\bullet$ and $\mathcal{F}'_\bullet$ correspond to the representations

$$\pi^{\text{alg}}(S^n, (x, \ldots, x) \to \pi^{\text{alg}}(S, x)_{\text{ab}} \to GL(V)$$

and

$$\pi^{\text{alg}}(S^n, (x, \ldots, x) \to \pi^{\text{alg}}(S, x)_{\text{ab}} \to GL(V')$$

respectively.

As $\pi^{\text{alg}}(S, x)_{\text{ab}}$ is an abelian affine group scheme over k, all its irreducible representations are one dimensional. Thus one gets that the $\pi^{\text{alg}}(S, x)_{\text{ab}}$-module V/V' has a one dimensional quotient W. Thus there is a $\pi^{\text{alg}}(S, x)_{\text{ab}}$-module surjection $V \to W$ such that the kernel contains V'. Let $\mathcal{L}_\bullet$ be the $\mathfrak{S}_n$-equivariant stratified bundle corresponding to W. Thus we have a short exact sequence of $\mathfrak{S}_n$-equivariant stratified bundles

$$0 \to \mathcal{K}_\bullet \to \mathcal{F}_\bullet \to \mathcal{L}_\bullet \to 0$$

where $\mathcal{F}'_\bullet \subset \mathcal{K}_\bullet$.

By proposition 1 above, we know that $L_i := \rho_* \mathcal{L}_i^{\mathfrak{S}_n}$ is a line bundle on $S^{(n)}$ and $\rho^* L_i = \mathcal{L}_i$

We claim that the following complex of sheaves on $S^{(n)}_*$ is exact for all $i \in \mathbb{N}$

$$0 \to (\rho_* \mathcal{K}_i)^{\mathfrak{S}_n}|_{S^{(n)}_*} \to (\rho_* \mathcal{F}_i)^{\mathfrak{S}_n}|_{S^{(n)}_*} \to (\rho_* \mathcal{L}_i)^{\mathfrak{S}_n}|_{S^{(n)}_*} \to 0 \tag{1}$$

It is enough to show that $(\rho_* \mathcal{F}_i)^{\mathfrak{S}_n}|_{S^{(n)}_*} \to (\rho_* \mathcal{L}_i)^{\mathfrak{S}_n}|_{S^{(n)}_*}$ is surjective. We note that $(\rho_* \mathcal{F}_i)^{\mathfrak{S}_n}|_{S^{(n)}_*} = h_*(\mathcal{E}_i|_{S^{[n]}_*})$. Let C be the cokernel

$$h_*(\mathcal{E}_i|_{S^{[n]}_*}) \to (\rho_* \mathcal{L}_i)^{\mathfrak{S}_n}|_{S^{(n)}_*} \to C \to 0$$

Pulling back under ρ, we get the following commutative diagram on S^n_*

$$\begin{array}{ccccccc}
\rho^* h_*(\mathcal{E}_i|_{S^{[n]}_*}) & \longrightarrow & \rho^*((\rho_* \mathcal{L}_i)^{\mathfrak{S}_n}|_{S^{(n)}_*}) & \longrightarrow & \rho^* C & \longrightarrow & 0 \\
\downarrow = & & \downarrow = & & & & \\
\mathcal{F}_i & \longrightarrow & \mathcal{L}_i|_{S^n_*} & \longrightarrow & 0 & &
\end{array}$$

The rows are exact and hence $\rho^* C = 0$. As ρ is surjective, this implies $C = 0$. Thus $K_i := (\rho_* \mathcal{K}_i)^{\mathfrak{S}_n}|_{S^{(n)}_*}$ is locally free on $S^{(n)}_*$

Pulling back the exact sequence (1) under h, we get a short exact sequence of locally free sheaves on $S^{[n]}_*$

$$0 \to h^* K_i|_{S^{[n]}_*} \to \mathcal{E}_i|_{S^{[n]}_*} \to \tilde{L}_i|_{S^{[n]}_*} \to 0$$

where $\tilde{L}_i := h^* L_i$.

As the complement of $S^{[n]}_*$ in $S^{[n]}$ is of codimension ≥ 2 and $\mathcal{E}_i$, L are locally free, the surjective morphism

$$\mathcal{E}_i|_{S^{[n]}_*} \to \tilde{L}_i|_{S^{[n]}_*}$$

extends to a unique morphism $\tau_i : \mathcal{E}_i \to \tilde{L}_i$. This is surjective as L is of rank 1 and $\tau := (\tau_i)$ give a nonzero morphism of stratified bundles

$$\mathcal{E}_\bullet \to \tilde{L}_\bullet$$

where $\tilde{L}_\bullet := h^*(\rho_*(\mathcal{L}_\bullet)^{\mathfrak{S}_n})$. Let $\kappa_\bullet$ be the kernel of the morphism $\mathcal{E}_\bullet \to \tilde{L}_\bullet$. Then $T(\kappa_\bullet) = \mathcal{K}_\bullet$. Thus, by the induction hypothesis on rank, there exists a stratified subbundle $\mathcal{E}'_\bullet \subset \kappa_\bullet \subset \mathcal{E}_\bullet$ such that $T(\mathcal{E}'_\bullet) = \mathcal{F}'_\bullet$.

□

6.2 Closed immersion

We begin by recalling a result from [14].

Let $p \in S^{(n)}$ be a point of type $(n_1, n_2, \ldots, n_r)$. Let p'_i, for $i = 1, 2, \ldots m$ be the points in the fiber $h^{-1}(p)$. Let A be the local ring $\mathcal{O}_{S^{(n)},p}$ and B be the semilocal ring $\mathcal{O}_{S^n} \otimes_{\mathcal{O}_{S^{(n)}}} A$. Then B is a finite A module and $B^{\mathfrak{S}_n} = A$.

Lemma 6.1 *When char $k > n_1$, any $\mathfrak{S}_n$-equivariant surjective B-module homomorphism $f : M \to N$ of finitely generated B modules descends to surjective A-module homomorphism of the $\mathfrak{S}_n$-invariants $M^{\mathfrak{S}_n} \to N^{\mathfrak{S}_n}$*

This allows us to prove the following analogue of Proposition 5.3.6 in [14].

Proposition *Let $\mathcal{E}_\bullet = (\mathcal{E}_i, \alpha_i)$ be a $\mathfrak{S}_n$-equivariant stratified bundle on S^n*

1. *Let $p \in S^{(n)}$ be a point of type $(n_1, n_2, \ldots, n_r)$. If char $k > n_1$, then the sheaf $\rho_* \mathcal{E}_i^{\mathfrak{S}_n}$ is locally free in a neighbourhood of p for all i.*
2. *Let U denote the largest open subset where $\rho_* \mathcal{E}_i^{\mathfrak{S}_n}$ is locally free, then on $\rho^{-1}(U)$, the natural morphism*

$$\rho^* \rho_* \mathcal{E}_i^{\mathfrak{S}_n} \to \mathcal{E}_i$$

is an isomorphism for all $i \in \mathbb{N}$

Proof The first assertion is proved by induction on the rank of $\mathcal{E}_\bullet$. If $\mathcal{E}_\bullet$ is a $\mathfrak{S}_n$-equivariant stratified bundle of rank 1, then by proposition 1, $\rho_* \mathcal{E}_i^{\mathfrak{S}_n}$ is locally free on $S^{(n)}$ for all i. In general, as $\mathcal{E}_\bullet$ corresponds to a representation of the abelian group scheme $\pi^{\text{alg}}(S, x)_{\text{ab}}$, there exists a $\mathfrak{S}_n$-equivariant short exact sequence of locally free sheaves on S^n

$$0 \to \mathcal{K}_\bullet \to \mathcal{E}_\bullet \to \mathcal{L}_\bullet \to 0$$

Pushing forward by ρ and taking $\mathfrak{S}_n$-invariants we get the exact sequence for all i

$$0 \to \rho(\mathcal{K}_i)^{\mathfrak{S}_n} \to \rho(\mathcal{E}_i)^{\mathfrak{S}_n} \to \rho(\mathcal{L}_i)^{\mathfrak{S}_n}$$

We claim that the homomorphism on the right is surjective in the neighbourhood of a point p of type $(n_1, n_2, \ldots, n_r)$. Surjectivity can be checked after passing to a formal neighbourhood of p and thus reduces to lemma 6.1. By induction hypothesis on rank, both $\rho(\mathcal{K}_i)^{\mathfrak{S}_n}$ and $\rho(\mathcal{L}_i)^{\mathfrak{S}_n}$ are locally free on a neighbourhood of p and hence so is $\rho(\mathcal{E}_i)^{\mathfrak{S}_n}$.

The second assertion follows from the observation that the natural homomorphism

$$\rho^* \rho_* \mathcal{E}_i^{\mathfrak{S}_n} \to \mathcal{E}_i$$

is an isomorphism on $\rho^{-1}(S^{(n)}_\circ)$ as as $\rho : S^n_\circ \to S^{(n)}_\circ$ is finite étale. As the complement of $S^n_\circ$ in $\rho^{-1}(U)$ is of codimension ≥ 2 and both sheaves are locally free on $\rho^{-1}(U)$, thus the natural morphism is an isomorphism. □

Proposition 6 *Let char $k > 3$. The homomorphism*

$$f : \pi^{alg}(S, x)_{ab} \to \pi^{alg}(S^{[n]}, \tilde{nx})$$

is faithfully flat.

Proof By [[3], Theorem 2.21], it is enough to show that the functor

$$T : \mathcal{S}(S^{[n]}) \to \mathcal{S}(S^n)$$

is essentially surjective. Thus we want to show that for any $\mathfrak{S}_n$-equivariant stratified bundle $\mathcal{E}_\bullet$ on S^n, there exists a stratified bundle $\mathcal{F}_\bullet$ on $S^{[n]}$ such that $T(\mathcal{F}_\bullet) = \mathcal{E}_\bullet$.

Let U be the open subset of $S^{(n)}$ consisting of points of type $(1, 1, \ldots, 1)$, $(2, 1, \ldots, 1)$, $(3, 1, \ldots, 1)$ and $(2, 1, 1, \ldots, 1)$. By assumption on characteristic of k and the previous proposition, we get that $\rho_* \mathcal{E}_i^{\mathfrak{S}_n}$ is locally free on U. Also we have on $\rho^{-1}(U)$, the natural morphism

$$\rho^* \rho_* \mathcal{E}_i^{\mathfrak{S}_n} \to \mathcal{E}_i$$

is an isomorphism. Imitating proposition 1 above, this allows us to define a stratified bundle $(\rho_* \mathcal{E}_i^{\mathfrak{S}_n}, \beta_i)$ on U such that $\rho^*(\rho_* \mathcal{E}_i^{\mathfrak{S}_n}, \beta_i) \simeq \mathcal{E}_\bullet$. Pulling back under h to $h^{-1}(U)$ (whose complement in $S^{[n]}$ has codimension ≥ 3) and extending to $S^{[n]}$, we get a stratified bundle $\mathcal{F}_\bullet$ such that $T(\mathcal{F}_\bullet) = \mathcal{E}_\bullet$ □

As f is both faithfully flat and a closed immersion, we get the following theorem

Theorem 6.2 *Let char $k > 3$. The homomorphism*

$$f : \pi^{alg}(S, x)_{ab} \to \pi^{alg}(S^{[n]}, \tilde{nx})$$

is an isomorphism.

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