Indian J Pure Appl Math (2026) 57:2096–2104
https://doi.org/10.1007/s13226-025-00785-7

ORIGINAL RESEARCH

Some generalizations of ideals in near-rings

Ahmed Y. Abdelwanis · Shakir Ali · Vaishali Varshney · Olfat A. Elmenofy

Received: 4 February 2024 / Accepted: 4 April 2025 / Published online: 18 April 2025
© The Indian National Science Academy 2025

Abstract Let N be a near-ring with identity 1 and $M_n(N)$ be the near-ring of $n \times n$ matrices over N for an arbitrary natural number n. The main aims of this paper are to introduce and study the notions of quasi-ideal and equiprime quasi-ideal of a near-ring N. It is shown that there is a one-to-one correspondence between the set of quasi-ideals of N and the set of full quasi-ideals of matrix near-ring $M_n(N)$. Furthermore, there is a one-to-one correspondence between the set of equiprime quasi-ideals of N and the set of equiprime quasi-ideals of matrix near-ring $M_n(N)$.

Keywords Equiprime quasi-ideal · Ideal · Near-ring · Matrix near-ring · Quasi-ideal

Mathematics Subject Classification 16Y30

1 Introduction

The theory of right near-rings is presented in [10]. We briefly recall the related definitions. We say that $(N, +, ".")$ is a right near-ring with identity 1 if (1) $(N, +)$ is a group (not necessarily abelian); (2) $(N, .)$ is a semi-group; (3) $(a + b)c = ac + bc$ for all $a, b, c \in N$ and; (4) $a.1 = 1.a = a$ for all $a \in N$. A near ring N is said to be a zero symmetric if $a.0 = 0$ for all $a \in N$. A non-empty subset U of N is said to be an ideal of N if (1) $(U, +)$ is a normal subgroup of $(N, +)$; (2) $UN \subseteq U$; (3) $r(a + b) - rb \in U$ for each $r, b \in N$, $a \in U$. Moreover, if U satisfies the conditions (1) and (2) then U is said to be a right ideal of N and if U satisfies (1) and (3) then U is

Communicated by Bakshi Gurmeet Kaur, Ph.D.

A. Y. Abdelwanis (✉)
Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt
E-mail: ayunis@sci.cu.edu.eg

S. Ali
Department of Mathematics, Faculty of Science, Aligarh Muslim University, Aligarh 202002, India
E-mail: shakir.ali.mm@amu.ac.in

S. Ali
Faculty of Mathematics and Natural Sciences, University Gadjah Mada, Yogyakarta, Indonesia

V. Varshney
Institute of Applied Sciences and Humanities, GLA University, Bharthia, India
E-mail: vaishali.varshney@gla.ac.in

O. A. Elmenofy
Department of Mathematics, Faculty of Women for Arts, Science and Education, Ain Shams University, Cairo, Egypt
E-mail: olfatelmenofy@women.asu.edu.eg

said to be a left ideal of N. Thus U is called an ideal of N if it is a right and left ideal of N. In 1986, Meldrum and Van der Walt [7] introduced the concept of matrix near-ring $M_n(N)$ (where n is a natural number) as follow. Let N^n be the direct sum of n copies of $(N, +)$. The elements of N^n are considered as column vectors and written in transposed form $(r_1, \ldots, r_n)$ with $r_i \in N$, $1 \leq i \leq n$. The symbols l_j and π_j will denote the j^{th} coordinate injection and projection functions respectively. The elementary $n \times n$ matrices over N are the functions $f_{ij}^r : N^n \to N^n$, where $f_{ij}^r = l_i f^r \pi_j$ with $r \in N$ and $f^r : N \to N$ is the left multiplication $t \to rt$, for each $t \in N$. The subnear-ring of $M(N^n)$ generated by f_{ij}^r is the near-ring of $n \times n$ matrices over N, denoted by $M_n(N)$, and the elements of $M_n(N)$ are called matrices. The identity matrix $f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1$ will be denoted by I. A matrix of the form $\sum_{i=1}^{n} f_{ii}^{r_i}$ where $r_i \in N$ for each $1 \leq i \leq n$ is called the diagonal matrix. A matrix is, of course, a map from N^n to N^n, but we shall regularly require representations of matrices. For this reason, we utilize the set $E_n(N)$ of matrix expressions, i.e., the subset of the free semi-group on the alphabet of symbols $\{f_{ij}^r : r \in N, 1 \leq i, j \leq n\} \cup \{(,), +\}$ recursively defined by the following rules:

(1) $f_{ij}^r \in E_n(N)$ for $1 \leq i, j \leq n$ and all $r \in N$.
(2) If $A, B \in E_n(N)$, then $A + B \in E_n(N)$.
(3) If $A, B \in E_n(N)$, then $(A)(B) \in E_n(N)$.
(4) Nothing else is in $E_n(N)$.

The length $l(E)$ of a matrix expression $E \in E_n(N)$ where n is an arbitrary natural number, is the number of f_{ij}^r in E. The weight $\omega(Y)$ of a matrix Y is the length of a minimum length expression representing Y. The matrix represented by $A \in E_n(N)$ is denoted by $\mu(A)$. Each matrix is represented by at least one expression; however the same matrix can be represented by many and different expressions. Despite this, we generally do not distinguish between expressions and matrices, except where this distinction is necessary to avoid any ambiguity. Moreover, we shall omit parentheses if the meaning is clear. For graphic reasons to type, in some cases we could type f_{ij}^r as $[r, i, j]$. Those functions f_{ij}^r, in the ring case, correspond to matrices with r in position (i, j) and 0 elsewhere, so every $n \times n$ matrix over a ring R is a sum of elements of the generating set $X_n(N) = \{f_{ij}^r : r \in N, 1 \leq i, j \leq n\}$. For more details on the matrix near rings and its generalizations (see [1,3,6,11,12,14] and [15] where further references can be found). Several researchers, for example, Meldrum , Van der walt, Booth and Groenewald have mentioned the relationship between ideals of the near-rings and ideals of the matrix near-rings (see [2,4,5,7–9] and [13] for details).

In this paper, we are following the similar path as these researchers, but with different concepts, which are quasi ideals of near-rings, that is, a generalization of the concept of ideals in near-rings. In this paper, we generalize the notion of an ideal of a near-ring N and call it a quasi-ideal of N. Moreover, we give some examples of quasi-ideals which are not ideals of N. We also prove that there is a one-to-one correspondence between the set of quasi-ideals of N and the set of full quasi-ideals of $M_n(N)$ as in Sect. 2. In addition, we introduce the concept of equiprime quasi-ideals of a near-ring N and prove that there is a one-to-one correspondence between the set of equiprime quasi-ideals of N and the set of equiprime quasi-ideals of $M_n(N)$ as in Sect. 3. All the results of this paper are generalizations of Booth and Groenewald [2].

In the following, N is a zero symmetric right near-ring with identity 1.

2 Quasi-ideals of near-rings

In this section we study the set of quasi-ideals of a near-ring N and we investigate the relationship between the set of quasi-ideals of N and those of $M_n(N)$. We begin our discussions with the following two lemmas which are useful in the proof of the main results.

Lemma 2.1 [2, Lemma 2.1] *Let $B \in M_n(N)$, $x \in N$, $\underline{r} \in N^n$ and $\underline{e} = (1, \ldots, 1) \in N^n$. Then*

$$(B\underline{r}).(x\underline{e}) = B(\underline{r}.x\underline{e}).$$

Lemma 2.2 [2, Lemma 2.3] *Let $B \in M_n(N)$, $x \in N$, $i, j \in \{1, \ldots, n\}$. Then there exist $b_1 \ldots, b_n \in N$ such that*

$$Bf_{ij}^x = [b_1, 1, j] + \cdots + [b_n, n, j].$$

Definition 2.3 A non-empty set I of a near-ring N is said to be a quasi-ideal of N if the following hold

(1) $(I, +)$ is a subgroup of $(N, +)$ and $K + I = I + K$ for any subgroup K of $(N, +)$;
(2) $IN \subseteq I$;
(3) $r(x + y) - ry \in I$,$\forall\, r, y \in N, \forall\ x \in I$.

Definition 2.4 (i) If I, in the pervious definition, satisfied the conditions (1) and (2)(resp. (1) and (3)) then I is said to be a right (resp. a left) quasi-ideal of N.
(ii) I is called a quasi-ideal of N if I is both a right and left quasi-ideal of N.

Remark 2.5 It is clearly that every ideal of N is a quasi-ideal of N. But the reverse is not true. The following examples justify this fact.

Example 2.6 Let $N = A \times C$ be the direct product of two groups A and C where $A =< a, b \mid\ 9a = 3b = 0, b + a - b = 4a >$ and $C =< c >, |c| = 9$. Define multiplication on N by $g.h = 0$ for all $g, h \in N$. Then $(N, +, .)$ is a near ring which is called the zero near ring on the group $(N, +)$. Let $U =< b > \times C$. It is clear that U is a subgroup of N which permutes with each subgroup of N. However, $U =< b > \times C$ is not normal in N. Hence, U is a quasi-ideal which is not an ideal of N.

Example 2.7 If we define a multiplication on N ,which is defined in the pervious example, by $g.h = g$ for all $g, h \in N$. Then $(N, +, .)$ is a near ring which is called the trivial constant near-ring on the group $(N, +)$. Then $U =< b > \times C$ is a quasi-ideal of N which is not an ideal.

Example 2.8 If we define a multiplication on N ,which is defined also in Example 2.6, by $g.h = \{^{0 \text{ if } h=0}_{g \text{ if } h\neq 0}$ for all $g \in N$. Then $(N, +, .)$ is a near ring which is called the trivial zero-symmetric near-ring on $(N, +)$. Then $U =< b > \times C$ is a quasi-ideal which is not an ideal of N.

Proposition 2.9 *Let L be a quasi-ideal of N. Then*

$$L^* := \{A \in M_n(N) : A\alpha \in L^n, \textit{ for any}\alpha \in N^n\}$$

is a quasi-ideal of $M_n(N)$.

Proof (1) We first have $(L^n, +)$ is a subgroup of $(N^n, +)$ since $(L, +)$ is a subgroup of $(N, +)$. Thus we have $(L^*, +)$ is a subgroup of $(M_n(N), +)$. Now we prove that $L^* + H = H + L^*$ for all subgroup H of $(M_n(N), +)$. Let H be a subgroup of $(M_n(N), +)$ and $u \in N^n$. Then $Hu = \{Au, A \in H\}$ is a subgroup of $(N^n, +)$. But L commutes with all subgroups of $(N, +)$ and hence $(L^n, +)$ commutes with all subgroups of $(N^n, +)$. Let us now show that

$$L^*u + Hu = Hu + L^*u \quad \forall\, u \in N^n.$$

Let us take $u \in N^n$, $Au \in Hu$ and $Du \in L^*u$. Thus, we have

$$Au + Du \in Hu + L^n = L^n + Hu.$$

So there is $Bu \in Hu$, $Z \in L^n$ such that

$$Au + Du = Z + Bu$$

and then

$$Au + Du - Bu = Z \in L^n.$$

Which implies that

$$(A + D - B)u = Z \in L^n \quad \forall\, u \in N^n.$$

This forces to

$$A + D - B = C \in L^*.$$

Therefore, we get

$$A + D = C + B$$

i.e.,

$$L^*u + Hu = Hu + L^*u, \quad \forall\, u \in N^n. \tag{1}$$

Now we use Eq. (1) to prove the required conclusion. For any $u \in N^n$, we have

$$\begin{aligned}(L^* + H)u &= L^*u + Hu,\\ &= Hu + L^*u,\\ &= (H + L^*)u, \quad \forall\, u \in N^n.\end{aligned}$$

Henceforward, we conclude that

$$L^* + H = H + L^*$$

for all subgroup H of $(M_n(N), +)$.

(2) Note that if $D \in L^*$, $B \in M_n(N)$, then we have

$$(DB)u = D(Bu) \in L^n$$

for all $u \in N^n$. This implies that $L^*M_n(N) \subseteq L^*$.

(3) Finally we show that if for all $A, C \in M_n(N)$ and $D \in L^*$ then $A(C + D) - AC \in L^*$, i.e., $(A(C + D) - AC)w = A(Cw + Dw) - (AC)w \in L^n$, for all $w \in N^n$. Further, we show that for all $A \in M_n(N)$, $\alpha \in N^n$ and $\beta \in L^n$, we have $A(\alpha + \beta) - A\alpha \in L^n$. That is, there is $\gamma \in L^n$ such that $A(\alpha + \beta) - A\alpha = \gamma$ for all $\alpha \in N^n$ and $\beta \in L^n$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in N^n$ *and* $\beta = (b_1, \ldots, b_n) \in L^n$. We use induction on $w(A)$. If $w(A) = 1$, i.e., $A = f_{ij}^r$ where $r \in N$ and $1 \leq i, j \leq n$. Then, we have

$$\begin{aligned}f_{ij}^r(\alpha + \beta) - f_{ij}^r(\alpha) &= f_{ij}^r(\alpha_1 + b_1, \ldots, \alpha_n + b_n) - f_{ij}^r(\alpha_1, \ldots, \alpha_n),\\ &= (0, 0, ..., r(\alpha_j + b_j), ..., 0) - (0, 0, ..., r\alpha_j, ..., 0),\\ &= (0, 0, ..., r(\alpha_j + b_j) - r\alpha_j, ..., 0).\end{aligned}$$

But L is a left quasi-ideal of N, then $r(\alpha_j + b_j) - r\alpha_j \in L$ and hence

$$f_{ij}^r(\alpha + \beta) - f_{ij}^r(\alpha) = (0, 0, ..., r(\alpha_j + b_j) - r\alpha_j, ..., 0) \in L^n.$$

Next, suppose that the result holds for any matrix A, $\omega(A) < m;\ m \geq 2$. Let $\omega(A) = m$. Then $A = L + M$ or $A = LM$, where $\omega(L),\ \omega(M) < \omega(A)$. Thus, we have

$$\begin{aligned}A(\alpha + \beta) &= (L + M)(\alpha + \beta),\\ &= L(\alpha + \beta) + M(\alpha + \beta),\\ &= \delta + L\alpha + \eta + M\alpha, \quad \delta, \eta \in L^n,\\ &= \delta + \theta + L\alpha + M\alpha \quad \theta \in L^n,\\ &= \delta + \theta + (L + M)(\alpha),\\ &= \delta + \theta + A(\alpha).\end{aligned}$$

Also, we have

$$\begin{aligned}A(\alpha + \beta) &= (LM)(\alpha + \beta),\\ &= L(M(\alpha + \beta)),\\ &= L(\delta + M\alpha) \quad \delta \in L^n,\\ &= \eta + L(M\alpha) \quad \eta \in L^n,\\ &= \eta + (LM)\alpha,\\ &= \eta + A\alpha.\end{aligned}$$

Hence we conclude that L^* is a quasi-ideal of $M_n(N)$. This completes the proof.

□

Let $QI(N)$ and $QI(M_n(N))$ be the sets of all quasi-ideals in N and $M_n(N)$, respectively.

Lemma 2.10 *The map*

$$\theta : QI(N) \to QI(M_n(N))$$

defined by $\theta(J) = J^*$ *is an injection.*

Proof Suppose that $L_1, L_2 \in QI(N)$ such that $L_1 \neq L_2$ and let $a \in L_2 \setminus L_1$. Then, $f_{11}^a \in L_2^*$ because L_2 is a quasi-ideal of N. But $f_{11}^a \notin L_1^*$ since $f_{11}^a(e) = (a, 0, ..., 0) \notin L_1^n$. □

Lemma 2.11 *Let* Q *be a quasi-ideal of* $M_n(N)$ *and*

$$Q_* = \{x \in N : x \in Im(\pi_j A) \text{ for some } A \in Q \text{ and } j, \text{ where } 1 \leq j \leq n\}.$$

Then $a \in Q_*$ *if and only if* $f_{11}^a \in Q$.

Proof ($\Rightarrow$) Suppose that Q is a quasi-ideal of $M_n(N)$, and $a \in Q_*$. Then there exist $A \in Q$, $\alpha \in N^n$ and $j \in \{1, 2, ..., n\}$ such that $\pi_j(A\alpha) = a$. But Q is right invariant so there exists $B \in Q$ such that $A\alpha = B\epsilon = B(1, 0, ..., 0)$. Thus, we have

$$\begin{aligned} B\epsilon &= B(1, 0, ..., 0) \\ &= B(f_{11}^1 + f_{21}^0 + \cdots + f_{n1}^0)\epsilon. \end{aligned}$$

Therefore, there are $a_1, ..., a_n$ such that

$$B(f_{11}^1 + f_{21}^0 + \cdots + f_{n1}^0) = [a_1; 1, 1] + \cdots + [a_n; n, 1].$$

It follows that $a_j = a$. But we have

$$E_{1j}B(f_{11}^1 + f_{21}^0 + \cdots + f_{n1}^0) = f_{11}^a \in Q.$$

($\Leftarrow$) Assume that $f_{11}^a \in Q$, then $f_{11}^a\epsilon = f_{11}^a(1, 0, ..., 0) = (a, ..., 0)$. So $a = \pi_1(f_{11}^a\epsilon)$ and then $a \in Q_*$. □

Lemma 2.12 *Let* Q *be a quasi-ideal of* $M_n(N)$. *Then* Q_* *is a quasi-ideal of* N.

Proof (1) We first prove that $(Q_*, +)$ is a subgroup of $(N, +)$. Let $x, y \in Q_*$, thus by Lemma 2.11, we have $f_{11}^x, f_{11}^y \in Q$. This implies $f_{11}^{x-y} = f_{11}^x - f_{11}^y \in Q$. Again by Lemma 2.11 we obtain $x - y \in Q_*$ and hence $(Q_*, +)$ is a subgroup of $(N, +)$. We prove that $Q_* + H = H + Q_*$ for each subgroup H of $(N, +)$. Let H be a subgroup of $(N, +)$, $z \in H$ and $d \in Q_*$ therefore $f_{11}^d \in Q$. Then $S = \{f_{11}^x : x \in H\}$ is a subgroup of $(M_n(N), +)$. Then $Q + S = S + Q$ since Q is a quasi ideal of $M_n(N)$. There is an element $h \in H$ and $q \in Q$ such that

$$f_{11}^z + f_{11}^d = q + f_{11}^h.$$

This implies that

$$f_{11}^{z+d-h} = f_{11}^z + f_{11}^d - f_{11}^h = q \in Q.$$

Application of Lemma 2.11 gives

$$z + d - h = l \in Q_*.$$

This leads to

$$z + d = l + h$$

i.e.,

$$Q_* + H = H + Q_* \text{ for all subgroup } H \text{ of } N.$$

(2) Let $r \in N$ and $x \in Q_*$. Then, we have

$$f_{11}^{xr} = f_{11}^x f_{11}^r \in Q\text{Mat}_n(N) \subseteq Q$$

Therefore, we have $xr \in Q_*$ and hence $Q_*N \subseteq Q_*$.

(3) Finally, let $r, s \in N$ and $x \in Q_*$. Then, we have

$$f_{11}^{r(s+x)-rs} = f_{11}^r(f_{11}^s + f_{11}^x) - f_{11}^r f_{11}^s \in Q.$$

By Lemma 2.11, we get

$$r(s+x) - rs \in Q_*.$$

Hence in view of (1),(2) and (3), we conclude that Q_* is a quasi-ideal of N. This proves the lemma. □

Lemma 2.13 *For quasi-ideals Q in $M_n(N)$ and L in N, we have*

(1) $(Q_*)^* \supseteq Q$;
(2) $(L^*)_* = L$;
(3) $((Q_*)^*)_* = Q_*$.

Proof (1) Suppose that $A \notin (Q_*)^*$. Then there is $\alpha \in N^n$ such that $A\alpha \notin Q_*^n$, and so there exist j; $1 \leq j \leq n$ with $\pi_j(A\alpha) \notin Q_*$. This implies that $A \notin Q$.
(2) From Lemma 2.11 we have $a \in (L^*)_*$ if and only if $f_{11}^a \in L^*$ if and only if $a \in L$.
(3) In (2) if we put $L = Q_*$, then we have $((Q_*)^*)_* = Q_*$. □

Definition 2.14 A quasi-ideal of $M_n(N)$ is called full if it is of the form L^* for some quasi-ideal L of N.

Recall that $\{O\}$ and $M_n(N)$ are full quasi-ideals of $M_n(N)$.

Theorem 2.15 *There is a bijection between the set of quasi-ideals of N and the set of full quasi-ideals of $M_n(N)$ given by*

$$L \to L^* \text{ and its inverse by } Q \to Q_*,$$

such that $(L^*)_* = L$ and $(Q_*)^* = Q$ for a quasi-ideal L of N and a full quasi-ideal Q of $M_n(N)$.

Proof Define a map

$$\Psi : \mathrm{QI}(N) \to \mathrm{QI}(\mathrm{M}_n(N))$$
$$\text{such that} \Psi(L) = L^*.$$

From Lemma 2.10, it follows that Ψ is injection. Further, by Lemma 2.12, if $Q \in QI(M_n(N))$, then $Q_* \in QI(N)$. Also if Q is full quasi-ideal of $M_n(N)$, then $Q = (Q_*)^*$. Finally, $Q = (Q_*)^* = \Psi(Q_*)$. Hence, Ψ is onto. This completes the proof of the theorem. □

3 Equiprime quasi-ideals of near-rings

This section devoted with the study of equiprime quasi-ideals of near-rings. First, we define the equiprime quasi-ideals of N and then we prove that there is a one-to-one correspondence between the set of equiprime quasi-ideals of N and the set of equiprime quasi-ideals of a matrix near-ring $M_n(N)$. We begin with the definition of equiprime quasi-ideals of near-rings.

Definition 3.1 Let P be a quasi-ideal of N. Then P is called an equiprime quasi-ideal of N if and only if for all $a \in N \setminus P$ and $x, y \in N$, $arx - ary \in P$ for all $r \in R$, implies $x - y \in P$. Also a near ring N is called an equiprime near ring if the zero quasi-ideal of N is an equiprime.

Proposition 3.2 *Let P be an equiprime quasi-ideal of N. Then, P^* is an equiprime quasi-ideal of the matrix near-ring $M_n(N)$.*

Proof By Proposition 2.9 we have P^* is a quasi-ideal of $M_n(N)$, so it remains to prove that it is an equiprime. Suppose that $A, B, C \in M_n(N)$ and that $A \notin P^*$, $B - C \notin P^*$. Then there exists $\alpha, \beta \in N^n$ such that $A\alpha \notin P^n$, $B\beta - C\beta \notin P^n$. Thus, there exists $i, j \in \{1, 2, ..., n\}$ such that $[A\alpha]_i \notin P$ and $[B\beta]_j - [C\beta]_j \notin P$. But P is an equiprime quasi-ideal of N, so there exists $x \in N$ such that $[A\alpha]_i x[B\beta]_j - [A\alpha]_i x[C\beta]_j \notin P$. It implies that

$$(A\alpha).(x[B\beta]_j e) - (A\alpha).(x[C\beta]_j e) \notin P^n.$$

Hence by Lemma 2.1, we get

$$A(\alpha.(x[B\beta]_j e)) - A(\alpha.(x[C\beta]_j e)) \notin P^n. \tag{2}$$

Suppose $\alpha = (\alpha_1, ..., \alpha_n)$. A simple calculation shows that

$$\begin{aligned}\alpha.(x[B\beta]_j e) &= [\alpha_1, 1, 1] + \cdots + [\alpha_n, , n, n])(f_{ij}^x + \cdots + f_{nj}^x)B\beta \\ &= \Phi B\beta \text{ (say)}.\end{aligned}$$

Similarly, we have

$$\alpha.(x[C\beta]_j e) = \Phi C\beta.$$

Then Eq. (2) becomes $A\Phi B\beta - A\Phi C\beta \notin P^n$, whence $A\Phi B - A\Phi C \notin P^*$. Thus, P^* is an equiprime quasi ideal of $M_n(N)$, as required. □

Proposition 3.3 *Let Q be an equiprime quasi-ideal of $M_n(N)$. Then Q_* is an equiprime quasi-ideal of N.*

Proof By Lemma 2.12 we have Q_* is a quasi-ideal of N so it remains to prove that it is an equiprime. Suppose that $a, b, c \in N$, and that $a \notin Q_*, b - c \notin Q_*$. By Lemma 2.11 we have $f_{11}^a \notin Q$ and $f_{11}^b - f_{11}^c \notin Q$. Since Q is an equiprime quasi-ideal of $M_n(N)$, then there exists $A \in M_n(N)$ such that

$$f_{11}^a A\, f_{11}^b - f_{11}^a A f_{11}^c \notin Q, \text{ i.e., } f_{11}^a A f_{11}^1\, f_{11}^b - f_{11}^a A f_{11}^1 f_{11}^c \notin Q. \tag{3}$$

By Lemma 2.2 there exist $a_1, ..., a_n \in R$ such that

$$Af_{11}^1 = [a_1, 1, 1] + \cdots + [a_n, n, 1].$$

We can check by direct calculation, that is,

$$f_{11}^a A f_{11}^1 = f_{11}^a([a_1, 1, 1] + \cdots + [a_n, n, 1]) = f_{11}^a[a_1, 1, 1].$$

Hence, Eq. (3) becomes

$$f_{11}^a[a_1, 1, 1]f_{11}^b - f_{11}^a[a_1, 1, 1]f_{11}^c \notin Q.$$

This gives

$$[aa_1b - aa_1c, 1, 1] \in Q.$$

Thus, we obtain $aa_1b - aa_1c \notin Q_*$. Consequently, Q_* is an equiprime quasi-ideal of N. □

Lemma 3.4 *Let U be a quasi-ideal of $M_n(N)$. Then , $A \in (U_*)^*$ if and only if $[[A\alpha]_j;\ i,\ k] \in U$ for all $\alpha \in N^n$ and $1 \le i,\ j,\ k \le n$.*

Proof Let U be a quasi-ideal of $M_n(N)$, then we have

$$\begin{aligned}A \ &\in \ (U_*)^* \\ &\Leftrightarrow A\alpha \in (U_*)^n \qquad \text{for all } \alpha \in N^n \\ &\Leftrightarrow [A\alpha]_j \in U_* \qquad \text{for all } \alpha \in N^n,\ j \in \{1, ..., n\} \\ &\Leftrightarrow [[A\alpha]_j, 1, 1] \qquad \text{for all } \alpha \in N^n,\ j \in \{1, ..., n\} \\ &\qquad\qquad \text{(by Lemma 2.11).}\end{aligned}$$

But $[[A\alpha]_j; i, k] = f_{i1}^1[[A\alpha]_j, 1, 1]f_{1k}^1$, so we have the result. □

Proposition 3.5 *Let P be an equiprime quasi-ideal of $M_n(N)$. Then $P = (P_*)^*$.*

Proof Suppose that $B \notin (P_*)^*$. Then there exists an element $\alpha \in N^n$ such that $(B\alpha) \notin (P_*)^n$, and hence there is $j \in \{1, 2, ..., n\}$ with $[B\alpha]_j \notin P_*$. This implies that $B \notin P$. Thus , $P \subseteq (P_*)^*$. Let $A \in M_n(R)$, $A \notin P$. But

$$A = (f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1)A,$$

so there exists $k \in \{1, 2, ..., n\}$ such that

$$f_{kk}^1 A \notin P. \tag{4}$$

Since P is an equiprime quasi-ideal of $M_n(N)$, then there exists $B \in M_n(R)$ such that

$$ABf_{kk}^1 A \notin P.$$

By Lemma 2.2, then there is $a_1, ..., a_n \in N$ such that

$$Bf_{kk}^1 = f_{1k}^{a_1} + f_{2k}^{a_2} + \cdots + f_{nk}^{a_n}.$$

Consequently,

$$A(f_{1k}^{a_1} + f_{2k}^{a_2} + \cdots + f_{nk}^{a_n})A \notin P.$$

Which means that

$$A(f_{1k}^{a_1} + f_{2k}^{a_2} + \cdots + f_{nk}^{a_n}) \notin P.$$

By reasoning similar to that used to deduce Eq. (4), there exists $i \in \{1, 2, ..., n\}$ such that

$$f_{ii}^1 A(f_{1k}^{a_1} + f_{2k}^{a_2} + \cdots + f_{nk}^{a_n}) \notin P.$$

If we let $\delta = (a_1, ..., a_n)$. Then, application of Lemma 2.2 yields that, $[[A\delta]_i; i, k] \notin P$. It follows from Lemma 3.4, that $A \notin (P_*)^*$. Hence the proof is completed. □

Let $E(N)$ and $E(M_n(N))$ are the sets of equiprime quasi-ideals of N and $M_n(N)$, respectively. Our next result gives one to one correspondence between $E(N)$ and $E(M_n(N))$, respectively.

Theorem 3.6 *The mapping*

$$\varphi \ : \ E(N) \to E(M_n(N))$$

$$\textit{defined by } \ \varphi(P) = P^* \ \textit{ for all } P \in E(N).$$

is a one-one correspondence between the sets of equiprime quasi-ideals of N and equiprime quasi-ideals of $M_n(N)$.

Proof Consider the function

$$\varphi \ : \ \mathrm{E}(N) \to \mathrm{E}(\mathrm{M}_n(N))$$

$$\text{defined by } \ \varphi(P) = P^* \ \text{, for all } P \in \mathrm{E}(N).$$

It is easy to show that φ is an injection. Suppose that $P \in E(M_n(N))$, then by Proposition 3.3 we have $P_* \in E(N)$. It follows from Proposition 3.5 that $P = (P_*)^* = \varphi(P_*)$. Therefore, the function φ is onto. Hence, we conclude the result. □

Corollary 3.7 *N is an equiprime near-ring if and only if so is* $M_n(N)$.

Proof ($\Rightarrow$) We have $\{O\} = \{0\}^*$, where $\{O\}$ is the zero quasi-ideal of $M_n(N)$ and $\{0\}$ is the zero quasi-ideal of N. But N is an equiprime near-ring, so $\{0\}$ is an equiprime quasi-ideal of N. In view of Proposition 3.2, we conclude that $\{O\}$ is an equiprime quasi-ideal of $M_n(N)$. This implies $M_n(N)$ is an equiprime near-ring. ($\Leftarrow$) Since $\{0\} = \{O\}_*$ and $\{O\}$ is an equiprime quasi-ideal of $M_n(N)$ so from Proposition 3.3 we have $\{0\}$ is an equiprime quasi-ideal of N, i.e., N is an equiprime near-ring. □

Acknowledgements The authors thank to the anonymous reviewers for their valuable suggestions and comments which significantly improved both the quality and the presentation of this paper.

Declarations

Conflict of interest The authors declare no competing interests.

References

1. Abbasi, S. J., Meldrum J. D. P. (1991). On matrix near-rings. *Math. Pannon.* 2(2):107-113.
2. Booth, G. L., Groenewald, N. J. (1991). On primeness in matrix near-ring. *Arch. Math.* 56:539-546.
3. Chen, X. W. Singularity Categories, Schur Functors and Triangular Matrix Rings. *Algebr. Represent. Theory* 12:181-191.
4. Groenewald, N. J. (1988), Strongly prime near-rings. *Proc. Edinb. Math. Soc.* 31:337-343.
5. Groenewald, N. J. (1989). Strongly prime near-rings 2. *Comm. Algebra* 17(3):735-749.
6. Ke, W. F., Meyer, J. H. (2012). Matrix near-rings and θ-primitivity. *Monatsh. Math.* 165:353-363.
7. Meldrum, J. D. P., Van der Walt, A. P. J. (1986). Matrix near-ring. *Arch. Math.* 47:312-319.
8. Meyer, J. H. (1994). Chains of intermediate ideals in matrix near-rings. *Arch. Math.* 63:311-315.
9. Meldrum, J. D. P., Meyer, J. H. (1996). Intermediate ideals in matrix near-rings. *Comm. Algebra* 24(5):1601-1619.
10. Pilz, G. (1983). *Near-rings*. Revised edition, Amsterdam.
11. Smith, K. C. (1996). Generalized matrix near-rings. *Comm. Algebra* 24(6):2065-2077.
12. Van der Walt, A. P. J. (1986). Primitivity in matrix near-rings. *Quaest. Math.* 9:459-469.
13. Van der Walt, A. P. J. (1987). On two-sided ideals in matrix near-rings. Near-rings and Near-fields, C. Bersch (editor) Elsevier Science Publishen B.V. (North-Holland).
14. Van der Walt, A. P. J. (1992). Matrix near-rings contained in 2-primitive near-rings with minimal subgroups. *J. Algebra* 148:296-304.
15. Zayed, M. (2001). On matrix near-rings. *Arch. Math.* 77:163-169.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.