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ORIGINAL RESEARCH

Boundary controllability for the Korteweg–de Vries–Burgers equation on a finite domain

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Abstract In this paper, we consider the controllability of the Korteweg–de Vries–Burgers equation on a finite domain. Because the third-order of the Korteweg–de Vries–Burgers equation, there needs three boundary conditions to secure the well-posedness of the system. We show that if we only use the right Neumann boundary condition, then the linear system is exactly contrallable if and only if the length of the spatial domain does not belong to a set of critical values. By contrast, if we also use the right Dirichlet boundary condition besides the right Neumann boundary condition, the linear system is exactly controllable without the limit of the length of the domain. Moreover, we also show that the nonlinear system is local exactly controllable via the method of contraction mapping principle.

Keywords Distributed parameter · Korteweg–de Vries–Burgers · Boundary controllability · Hilbert uniqueness method

1 Introduction

In this paper, we study a distributed parameter control system described by the Korteweg–de Vries–Burgers (KdV-B) equation posed on a finite domain [0,L] with nonhomogeneous boundary conditions:

$$\begin{cases} u_t + u_x - u_{xx} + u_{xxx} + uu_x = 0, \text{ for } (t,x) \in [0,T] \times [0,L], \\ u(0,x) = u_0, \quad u(t,0) = h_1(t), \quad u(t,L) = h_2(t), \quad u_x(t,L) = h_3(t), \end{cases} \tag{1.1}$$

The KdV-B equation describes one of the typical examples incorporating the effects of dissipation, dispersion and nonlinear problem. This model gives a description of the propagation of waves on an elastic tube filled with viscous fluid [1] and also the phenomenon of containing turbulance [2], gas bubbles [3]. The author gave the well-posedness of (1.1) in [4] using the nonlinear interpolation. Liu and Krstic considered a boundary feedback stabilization problem for the KdV-B equation on a finite spatial interval [5]. Chaohua. Jia and Bingyu. Zhang also gave a result of the well-posedness of the KdV-B equation on a finite domain with another kind of boundary conditions and the controllability using the method of feedback [6]. Smaoui. Nejib and Al-Jamal, Rasha H studied the Kdv-B equation on a periodic boundary conditions via the Karhunen-Loeve Galerkin method [7].

As we all known there is a phenomenon that if we only use the right Neumann boundary condition to control the KdV equation on a finite domain, a critical set of the length of the domain is needed. Similarly, we also need a critical set of the length of the domain to ensure the controllability of the KdV-B equation if we only use the

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right Neumann condition. The first result of this paper is the following theorem associated to the linear problem of (1.1).

Theorem 1 *Let* $\mathcal{N} = \{\pi\sqrt{2(l^2 - k^2)}; l, k \in \mathbb{N}^* \quad and \quad l > k\}$ *and* $T > 0, \quad L \in (0, \infty) \setminus \mathcal{N}$ *be given, then for any* $u_0, u_T \in L^2(0, L)$*, there exists* $h_3 \in L^2(0, T)$ *such that the solution* u *of the following problem:*

$$\begin{cases} u_t + u_x - u_{xx} + u_{xxx} = 0, \quad for \quad (t, x) \in [0, T] \times [0, L], \\ u(x, 0) = u_0, \quad u(t, 0) = 0, \quad u(t, L) = 0, \quad u_x(t, L) = h_3(t), \end{cases} \tag{1.2}$$

satisfies $u(0, \cdot) = u_0$ *and* $u(T, \cdot) = u_T$*, with*

$$u \in C([0, T], L^2(0, L)) \cap L^2(0, T; H^1(0, L)).$$

With some estimates of the linear systems associated to (1.2), we have the local exactly controllability of the following nonlinear problem which is the main result of this paper:

$$\begin{cases} u_t + u_x - u_{xx} + u_{xxx} + uu_x = 0, \quad for \quad (t, x) \in [0, T] \times [0, L], \\ u(x, 0) = u_0, \quad u(t, 0) = 0, \quad u(t, L) = 0, \quad u_x(t, L) = h_3(t), \end{cases} \tag{1.3}$$

Theorem 2 *Let* $T > 0, \quad L \in (0, \infty) \setminus \mathcal{N}$ *be given, then there exists an positive number* δ *such that for any* $u_0, u_T \in L^2(0, L)$ *with* $\|u_0\|_{L^2(0,L)} \leq \delta$ *and* $\|u_T\|_{L^2(0,L)} \leq \delta$*, there exists* $h_3(t) \in L^2(0, T)$ *such that the solution* u *of* (1.3) *satisfies* $u(0, \cdot) = u_0$ *and* $u(T, \cdot) = u_T$*, with* $u \in C([0, T], L^2(0, L)) \cap L^2(0, T; H^1(0, L))$.

Remark 1 When $L \in \mathcal{N}$, we can also get the controllability of the (1.3) if we both use the right Neumann boundary condition and the right Dirichlet boundary condition. The detail of this method will be shown at the end of the proof of Theorem 2.

The paper is organized as follows. In Section 1, we first give some linear estimates and the critical set $\mathcal{N}$ of the length of the domain, and then, we use the Hilbert Uniqueness Method [8] and the observability inequality of the adjoint system to prove the controllability of the linear system (1.2). In Section 2, we use contraction mapping principle to get the local exactly controllability of (1.3).

2 Controllability of linear system

We first discuss the estimate of the solution u of the following linear system

$$\begin{cases} u_t + u_x - u_{xx} + u_{xxx} = 0, \quad for \quad (t, x) \in [0, T] \times [0, L], \\ u(0, x) = u_0, \quad u(t, 0) = 0, \quad u(t, L) = 0, \quad u_x(t, L) = 0. \end{cases} \tag{2.1}$$

Let A be the linear operator of (2.1) in the space $L^2(0, L)$ defined by

$$Af = -f''' + f'' - f',$$

and define its definition domain as

$$\mathbb{D}(A) = \{f \in H^3(0, L), f(0) = f(L) = f'(L) = 0\}.$$

We can describe (2.1) in an abstract form in the space $L^2(0, L)$

$$\frac{du}{dt} = Au, \qquad u(0) = u_0.$$

The adjoint operator A^* of operator A in the space $L^2(0, L)$ and its definition domain can be described as following

$$A^*g = g''' + g'' + g,$$
$$\mathbb{D}(A^*) = \{g \in H^3(0, L), g(0) = g(L) = g'(0) = 0\}.$$

Using semigroup theory [9], we know that both the linear operator A and its adjoint operator A^* are dissipative operators by the following results

$$\langle Af, f\rangle_{L^2(0,L)} \le 0, \qquad \langle A^*g, g\rangle_{L^2(0,L)} \le 0,$$

for any $f \in \mathbb{D}(A)$ and $g \in \mathbb{D}(A^*)$. So, the semigroup $W_0(t)$ which is generated by the infinitesimal generator A is a C_0-semigroup of contractions in the space $L^2(0, L)$ and we can write the solution u of (2.1) in the following form

$$u(t) = W_0(t)u_0$$

If $u_0 \in \mathbb{D}(A)$, then $u(t)$ belongs to the space $C(0, \infty; H^3(0, L)) \cap C^1(0, \infty; L^2(0, L))$ and also $u(t) \in \mathbb{D}(A)$ for all $t \ge 0$. From now on, we denote

$$Y = C([0, T], L^2(0, L)) \cap L^2(0, T; H^1(0, L)),$$

with the norm as following

$$\|u\|_Y = \sup_{t\in[0,T]} \|u(t, \cdot)\|_{L^2(0,L)} + \|u\|_{L^2(0,T;H^1(0,L))},$$

for any $u \in Y$.

Proposition 1 *The map $\Psi_1 : u_0 \in L^2(0, L) \mapsto u = W_0(t)u_0 \in Y$ is continuous, and we also have:*

(i) $\|u_x(\cdot, 0)\|_{L^2(0,T)} \le \|u_0\|_{L^2(0,L)}$;

(ii) $\|u_0\|^2_{L^2(0,L)} \le \frac{1}{T}\|u\|^2_{L^2(0,T;L^2(0,L))} + 2\|u_x\|_{L^2(0,T;L^2(0,L))} + \|u_x(\cdot, 0)\|^2_{L^2(0,T)}$,

for any $u_0 \in L^2(0, L)$.

Proof Because the semigroup $W_0(t)$ is a contractive C_0-semigroup in $L^2(0, L)$, it's obviously that

$$\|u\|_{C([0,T],L^2(0,L))} \le \|u_0\|_{L^2(0,L)}, \quad for \quad any \quad u_0 \in L^2(0, L).$$

We may assume u is a strong solution of (2.1) and also assume $u_0 \in \mathbb{D}(A)$. Multiply both sides of the equation (2.1) by $2xu$ and integrate over $(0, L) \times (0, T)$ with respect to x and t. Integration by parts leads to

$$\int_0^T \int_0^L u_x^2 dxdt \le \frac{1}{6}\int_0^T \int_0^L u^2 dxdt + \frac{1}{3}\int_0^L xu_0^2 dx,$$

$$\begin{aligned}\|u\|^2_{L^2(0,T;H^1(0,L))} &\le \frac{7}{6}\int_0^T \int_0^L u^2 dxdt + \frac{1}{3}\int_0^L xu_0^2 dx \\ &\le \frac{7T}{6}\|u_0\|^2_{L^2(0,L)} + \frac{L}{3}\int_0^L u_0^2 dx,\end{aligned}$$

i.e.

$$\|u\|_{L^2(0,T;H^1(0,L))} \le \left(\frac{7T}{6} + \frac{L}{3}\right)^{\frac{1}{2}} \|u_0\|_{L^2(0,L)}. \tag{2.2}$$

Because $\mathbb{D}(A)$ is dense in $L^2(0, L)$, the result can be extended to any $u_0 \in L^2(0, L)$. So we have proved that the map Ψ_1 is continuous, with

$$\|u\|_Y \le \left[\left(\frac{7T}{6} + \frac{L}{3}\right)^{\frac{1}{2}} + 1\right]\|u_0\|_{L^2(0,L)}. \tag{2.3}$$

Multiply both sides of the equation in (2.1) by u and integrate over $(0, L) \times (0, T)$ with respect to x and t, we have the following result

$$\|u(t, \cdot)\|^2_{L^2(0,L)} + \int_0^t u_x^2(\tau, 0)d\tau + 2\int_0^t \int_0^L u_x^2 dxd\tau = \|u_0\|^2_{L^2(0,L)},$$



and we get

$$\|u_x(\cdot,0)\|_{L^2(0,T)} \le \|u_0\|_{L^2(0,L)}. \tag{2.4}$$

Using the linear extension theory, the map $u_0 \in \mathbb{D}(A) \mapsto u_x(\cdot,0) \in L^2(0,L)$ can be uniquely extended to the space $L^2(0,L)$. With the same method, Multiply both sides of the equation in Eq. (2.2) by $(T-t)u$ and integrate over $(0,T)\times(0,L)$ with respect to t and x, we have the following result

$$\begin{aligned}\int_0^L u_0^2 dx &= \frac{1}{T}\int_0^T\int_0^L u^2 dxdt + \frac{2}{T}\int_0^T\int_0^L (T-t)u_x^2 dxdt \\ &\quad + \frac{1}{T}\int_0^T (T-t)u_x^2(t,0)dt \\ &\le \frac{1}{T}\|u\|^2_{L^2(0,T;L^2(0,L))} + 2\|u_x\|^2_{L^2(0,T;L^2(0,L))} + \|u_x(\cdot,0)\|^2_{L^2(0,T)}.\end{aligned}$$

So we have finished the proof of Proposition 1. □

The adjoint system of (1.2) is described as following

$$\begin{cases} y_t + y_x - y_{xx} + y_{xxx} = 0, \quad for \quad (t,x) \in [0,T]\times[0,L], \\ y(T,x) = y_T, \quad y(t,0) = y(t,L) = y_x(t,0) = 0. \end{cases} \tag{2.5}$$

According to the duality theorem, the system (1.2) is exactly controllable if and only if (2.5) is observable [10,11]. With the method of variable substitution, let $\tau = T-t$, $\xi = L-x$, then (2.5) can be changed into the following form

$$\begin{cases} y_\tau + y_\xi - y_{\xi\xi} + y_{\xi\xi\xi} = 0, \text{ for } (\tau,\xi) \in [0,T]\times[0,L], \\ y(\tau,0) = y(\tau,L) = y_\xi(\tau,L) = 0, \\ y(0,\xi) = y_T, \end{cases}$$

which is just (2.1). So, the observability of (2.5) is just the observability of (2.1).

In order to get the observability inequality of (2.1), we need the following Lemmas.

Lemma 1 *There exists* $\lambda \in \mathbb{C}$ *and* $u_0 \in H^3(0,L)\backslash\{0\}$ *satisfying*

$$\begin{aligned} &\{\lambda u_0 + u_0' - u_0'' + u_0''' = 0, \quad for \quad x \in [0,L], \\ &\quad u_0(0) = u_0(L) = u_0'(0) = u_0'(L) = 0, \end{aligned} \tag{2.6}$$

if and only if $L \in \mathcal{N} = \{\pi\sqrt{2(l^2-k^2)}; l,k \in \mathbb{N}^* \quad and \quad l > k\}$.

Proof Let's assume $u_0 \in H^3(0,L)\backslash\{0\}$ and satisfy (2.6). Denote $w \in H^2(\mathbb{R})$ be the prolongation of u_0 by 0 and so w satisfies the following equation

$$\lambda w + w' - w'' + w''' = u_0''(0)\delta_0 - u_0''(L)\delta_L,$$

where δ_0 and δ_L denote the Dirac measures at 0 and L. The proposition in 1 is equivalent to the following problem:

Do there exist complex numbers λ, α, β with $(\alpha,\beta) = (u_0''(0), u_0''(L)) \ne (0,0)$, and a function $w \in H^2(\mathbb{R})$ with compact support in $[-L,L]$ satisfying

$$\lambda w + w' - w'' + w''' = \alpha\delta_0 - \beta\delta_L, \text{ in } \mathcal{D}'(\mathbb{R}) \quad ?$$

Taking Fourier transform yields

$$(\lambda + (i\xi) + (i\xi)^2 + (i\xi)^3)\hat{w}(\xi) = \alpha - \beta e^{-iL\xi}.$$

we have

$$\hat{w}(\xi) = i\frac{\alpha - \beta e^{-iL\xi}}{\xi^3 - i\xi^2 - \xi - \lambda/i}. \tag{2.7}$$

Since both $\alpha-\beta e^{-iL\xi}$ and $\xi^3-i\xi^2-\xi-\lambda/i$ are analytic functions, the function $\hat{w}(\xi)$ is an entire function if and only if the roots of $\xi^3-i\xi^2-\xi-\lambda/i$ are contained in the roots of $\alpha-\beta e^{-iL\xi}$. Using the Paley-Wiener Theorem [12] and characterization analysis by means of Fourier transform, we know that the proposition in Lemma 1 is equivalent to the existence of complex numbers $\lambda, \alpha, \beta \in \mathbb{C}$ with $(\alpha, \beta) \neq (0, 0)$ such that the function (2.7) satisfies:

(i) $\hat{w}(\xi)$ is an entire function in $\mathbb{C}$;

(ii) $\displaystyle\int_{\mathbb{R}} |\hat{w}(\xi)|^2(1+|\xi|^2)^2 d\xi < \infty$;

(iii) There exist positive constants C, N such that
$$|\hat{w}(\xi)| \leq C(1+|\xi|)^N e^{L|Im\xi|} \text{ for any } \xi \in \mathbb{C}.$$

It's obvious that if (i) is true, then both (ii) and (iii) are satisfied. Since all the roots of $\alpha-\beta e^{-iL\xi}$ are simple, (i) is true provided that the three roots of $\xi^3-i\xi^2-\xi-\lambda/i$ are simple and contained in the roots of $\alpha-\beta e^{-iL\xi}$. So, the proposition of Lemma 1 is true if there exist complex numbers λ, ξ_1 and positive integers k, l such that

$$\xi_2 = \xi_1 + k\frac{2\pi}{L}, \quad \xi_3 = \xi_1 + (k+l)\frac{2\pi}{L},$$

and

$$\xi^3 - i\xi^2 - \xi - \lambda/i = (\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3).$$

So we have the following equations:

$$\begin{cases} \xi_1+\xi_2+\xi_3 = i, \\ \xi_1\xi_2+\xi_1\xi_3+\xi_2\xi_3 = -1, \\ \xi_1\xi_2\xi_3 = \lambda/i. \end{cases} \tag{2.8}$$

With easy calculations, we have the solutions of (2.8) as following

$$\begin{cases} L = \pi\sqrt{2(l^2-k^2)}, \quad l,k \in \mathbb{N}^*, \quad and \quad l > k, \\ \xi_1 = \frac{1}{3}\big[i - \frac{(l+2k)\sqrt{2(l^2-k^2)}}{l^2-k^2}\big], \\ \lambda = i\xi_1(\xi_1 + k\frac{2\pi}{L})(\xi_1 + (k+l)\frac{2\pi}{L}) \\ \quad = \frac{i}{27}\big[i - \frac{(l+2k)\sqrt{2(l^2-k^2)}}{l^2-k^2}\big]\big[i - \frac{\sqrt{2(l^2-k^2)}}{l+k}\big]\big[i + \frac{(2l+k)\sqrt{2(l^2-k^2)}}{l^2-k^2}\big]. \end{cases}$$

Taking $\lambda = \lambda(k, l, L)$ into (2.6) and solving it, we have

$$\begin{cases} u_0''(0) = \alpha = -\sum_{i=1}^3 C_i\xi_i^2, \\ u_0''(L) = \beta = \big(-\sum_{i=1}^3 C_i\xi_i^2\big)e^{i\xi_1 L}, \end{cases}$$

where C_1, C_2 and C_3 satisfy the following equation

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ C_1\xi_1 + C_2\xi_2 + C_3\xi_3 = 0. \end{cases}$$

We have proved that the proposition in Lemma 1 is true if and only if

$$L \in \mathcal{N} = \{\pi\sqrt{2(l^2-k^2)}; \quad l,k \in \mathbb{N}^* \quad and \quad l > k\}.$$

□

Lemma 2 *For any $T > 0$, we denote*

$$K_T = \{u_0 \in L^2(0, L); \|u_x(\cdot, 0)\|_{L^2(0,T)} = 0\},$$

where u is the solution of (2.1). *Then $K_T = \{0\}$, if $L \in (0, +\infty)\backslash\mathcal{N}$.*

Proof We denote B being the unit ball in K_T as

$$B = \{w \in K_T; \|w\|_{L^2(0,L)} \leq 1\},$$

and $\{u_0^n\}$ being any sequence in B. Let $u^n = W_0(t)u_0^n$. We use the similar method presented in [8]. With the estimate (2.2) and $u_t^n = -u_x^n + u_{xx}^n - u_{xxx}^n$, we know that $\{u^n\}$ and $\{u_t^n\}$ are bounded in $L^2(0, T; H^1(0, L))$ and $L^2(0, T; H^{-2}(0, L))$, respectively. By the following embedding

$$H^1(0, L) \hookrightarrow L^2(0, L) \hookrightarrow H^{-2}(0, L),$$

we know that the sequence $\{u^n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(0, T; L^2(0, L))$ [13]. According to Proposition 1, we have

$$\begin{aligned}\|u_0^n\|^2_{L^2(0,L)} \leq & \frac{1}{T}\|u^n\|^2_{L^2(0,T;L^2(0,L))} + 2\|u_x^n\|^2_{L^2(0,T;L^2(0,L))} \\ & + \|u_x^n(\cdot, 0)\|^2_{L^2(0,T)}.\end{aligned} \tag{2.9}$$

The third term $u_x^n(\cdot, 0)$ on the right in (1.9) equals zero in $L^2(0, T)$. Furthermore, the second term $\{u_x^n\}$ is bounded in $L^2(0, T; L^2(0, L))$ and so there exists a convergent subsequence in the sequence $\{u_x^n\}$ in $L^2(0, T; L^2(0, L))$. Therefore, there exists a Cauchy subsequence $\{u_0^{n_i}\}$ in $\{u_0^n\}$ in $L^2(0, L)$ from the result in [13]. So the unit ball B is compact by Riesz theorem and the dimension of K_T is finite. Since $K_{T'} \subseteq K_T$ if $T < T'$, we know that we can find some T and δ with $T + \delta < T'$ such that $dim K_T = dim K_{T+\delta}$, i.e $K_T = K_{T+\delta}$. Since $\|[W_0(T)W_0(\epsilon)u_0]_x(\cdot, 0)\|_{L^2(0,T)} = 0$ and $\|[W_0(T)u_0]_x(\cdot, 0)\|_{L^2(0,T)} = 0$ for any $0 < \epsilon \leq \delta$, $u_0 \in K_T$, we have

$$\frac{W_0(\epsilon)u_0 - u_0}{\epsilon} \in K_T. \tag{2.10}$$

Since we have showed that $dim(K_T) < \infty$, we have the dimension of the space $\Omega_T = \{u = W_0(s)u_0; 0 \leq s \leq T, u_0 \in K_T\} \subset C([0, T]; L^2(0, L))$ is finite. So Ω_T is a closed subspace in $C([0, T]; L^2(0, L))$. For (2.10), we have

$$\frac{u(\epsilon + \cdot) - u}{\epsilon} \in \Omega_T, \quad for \quad any \quad \epsilon \in (0, \delta],$$

and also

$$\lim_{\epsilon\to 0^+} \frac{u(\epsilon + \cdot) - u}{\epsilon} = u_t \in \Omega_T \subset C([0, T]; L^2(0, L)).$$

So we have

$$u_t(0) = \lim_{\epsilon\to 0^+} \frac{W_0(\epsilon)u_0 - u_0}{\epsilon} = Au_0 \in L^2(0, L).$$

Since $dim(K_T) < \infty$ and with (2.10), we have $Au_0 \in K_T$. With the discussion above, there is a linear map:

$$\begin{aligned}\Pi : K_T &\to K_T \\ u_0 &\mapsto Au_0.\end{aligned}$$

If $K_T \neq \{0\}$, then there exists a complex number λ and $u_0 \in H^3(0, L)\backslash\{0\}$ such that

$$\begin{cases} \lambda u_0 = -u_{0x} + u_{0xx} - u_{0xxx}, \quad for \quad x \in [0, L] \\ u_0(0) = u_0(L) = u_{0x}(0) = u_{0x}(L) = 0. \end{cases} \tag{2.11}$$

According to Lemma 1, there exists a complex number λ and $u_0 \in H^3(0, L)\backslash\{0\}$ satisfying (2.11) if and only if $L \in \mathcal{N}$. So we have proved that if $L \in (0, +\infty)\backslash\mathcal{N}$, then $K_T = \{0\}$. The proof is completed. □

With the help of Lemmas 1 and 2, we can prove the following observability inequality of (2.1).

Proposition 2 *Let $L \in (0, +\infty)\backslash\mathcal{N}, \quad T > 0$ be given, then there exists a positive constant C such that the solution u of* (2.1) *satisfying*

$$\|u_0\|_{L^2(0,L)} \leq C\|u_x(\cdot, 0)\|_{L^2(0,T)}, \tag{2.12}$$

where the positive constant C depends onlu on L and T.

Proof If (2.12) is false, then there exists a sequence $\{u_0^n\}_{n\in\mathbb{N}} \in L^2(0, L)$ with $\|u_0^n\|_{L^2(0,L)} = 1$ for any $n \in \mathbb{N}$ which satisfy

$$\|u_0^n\|_{L^2(0,L)} > n\|u_x(\cdot, 0)\|_{L^2(0,T)}. \tag{2.13}$$

What we want to do is to get the contradiction, so we should prove that our assumption above does not hold. For (2.13), we know $\|u_x(\cdot, 0)\|_{L^2(0,T)} \to 0$ as $n \to \infty$. By the same argument showed in Lemma 2, we know that there exists a Cauchy subsequence $\{u_0^{n_i}\}$ in $\{u_0^n\}$ in $L^2(0, L)$. We denote $u_0 = \lim\limits_{n_i\to+\infty} u_0^{n_i}$ and u be the corresponding solution of (2.1) with initial value u_0. Since $\|u_0^n\|_{L^2(0,L)} = 1$ for any n and $\lim\limits_{n_i\to+\infty} u_x^{n_i}(\cdot, 0) = u_x(t, 0) = 0$ in the space $L^2(0, T)$. By Lemma 2, we know that $u \equiv 0$, therefore $u_0 \equiv 0$ which contradicts the assumption above that $u_0 = 1$. So our assumption at the beginning of the proof does not hold, and we have finished the proof. □

The following proposition reveals that we can get a certain weak sense solution of the Eq. (2.1).

Proposition 3 *Let $\widetilde{C}_0^2[0, T] = \{g \in C^2([0, T], \mathbb{R}\}; g(0) = 0\}$. There exists a unique solution u of* (2.1)*, for any $u_0 \in \mathbb{D}(A)$ and $h_3 \in \widetilde{C}_0^2[0, T]$. Moreover, the following solution map:*

$$\begin{aligned}\widetilde{\Phi} : \mathbb{D}(A) \times \widetilde{C}_0^2[0, T] &\to Y\\ (u_0, h_3) &\mapsto u,\end{aligned}$$

can be uniquely extended to the following linear continuous map:

$$\begin{aligned}\Phi : L^2(0, L) \times L^2(0, T) &\to Y\\ (u_0, h_3) &\mapsto u.\end{aligned}$$

Proof We first assume $u_0 \in \mathbb{D}(A)$ and $h_3 \in \widetilde{C}_0^2[0, T]$. Choose $\varphi \in C^\infty([0, L])$ with $\varphi(0) = \varphi(L) = 0$, $\varphi'(L) = -1$, then with the change of $w(t, x) = u(t, x) - W_0(t)u_0(x) + h_3(t)\varphi(x)$, we can transform (2.1) into the following form:

$$\begin{cases} w_t + w_x - w_{xx} + w_{xxx} \\ \qquad = h_3'(t)\varphi(x) + h_3(t)(\varphi'(x) - \varphi''(x) + \varphi'''(x)), \quad for \quad (t, x) \in [0, T] \times [0, L], \\ w(t, 0) = w(t, L) = w_x(t, L) = 0, \\ w(0, x) = 0, \text{ for } (t, x) \in [0, T] \times [0, L]. \end{cases} \tag{2.14}$$

Since the right part $h_3'(t)\varphi(x) + h_3(t)(\varphi'(x) - \varphi''(x) + \varphi'''(x))$ of (2.14) belongs to $C^1([0, T]; L^2(0, L))$ and by semigroup theory (Paz83), there exists a unique solution w of (2.14) with

$$w \in C([0, T]; \mathbb{D}(A)) \cap C^1([0, T]; L^2(0, L)).$$

For the definition of w and the condition $u_0 \in \mathbb{D}(A)$ and $h_3 \in \widetilde{C}_0^2[0, T]$, we know that there exists a unique classical solution u of (2.1) with

$$u \in C([0, T]; H^3(0, L)) \cap C^1([0, T]; L^2(0, L)).$$

Since we have already proved the existence of the solution of (2.1), we can discuss the estimate of the solution u of (2.1). We still assume $u_0 \in \mathbb{D}(A)$ and $h_3 \in \widetilde{C}_0^2[0, T]$. Multiply both sides of the equation in (2.1) by u and integrate over $(0, S) \times (0, L)$ with respect to t and x, where $S \in [0, T]$. With simple calculation, we have the following estimate



$$\int_0^L u^2(S,x)dx + \int_0^S u_x^2 dt + 2\int_0^S \int_0^L u_x^2 dxdt$$
$$= \int_0^L u^2(0,x)dx + \int_0^S h_3^2 dt \le \|u_0\|^2_{L^2(0,L)} + \|h_3\|^2_{L^2(0,T)} = \|(u_0,h_3)\|^2_G,$$

where $\|(u_0,h_3)\|_G = \left(\|u_0\|^2_{L^2(0,L)} + \|h_3\|^2_{L^2(0,T)}\right)^{\frac{1}{2}}$. So we have

$$\|u\|_{C([0,T];L^2(0,L))} \le \|(u_0,h_3)\|_G, \tag{2.15}$$

$$\|u\|_{L^2(0,T;L^2(0,L))} \le \sqrt{T}\|(u_0,h_3)\|_G,$$

and

$$\|u_x(t,0)\|_{L^2(0,T))} \le \|(u_0,h_3)\|_G.$$

Now we multiply both sides of the equation in (2.1) by xu and integrate over $(0,T)\times(0,L)$ with respect to t and x. With simple calculation, we have

$$\int_0^T \int_0^L (x+\frac{3}{2})u_x^2 dxdt = \frac{1}{2}\Big(\int_0^T \int_0^L u^2 dxdt + \int_0^T h_3^2 dt + \int_0^L xu^2(0,x)dx$$
$$- \int_0^L xu(T,x)dx\Big).$$

So we have the following estimate

$$\int_0^T \int_0^L u_x^2 dxdt \le \frac{1}{3}\Big(\int_0^T \int_0^L u^2 dxdt + L\int_0^T h_3^2 dt + L\int_0^L u_0^2 dx\Big).$$

With (2.15), we also have

$$\|u\|_{L^2(0,T;H^1(0,L))} \le \Big(\frac{4T+L}{3}\Big)^{\frac{1}{2}}\|(u_0,h_3)\|_G,$$

$$\|u\|_Y \le \Big(\frac{4T+L}{3}+1\Big)^{\frac{1}{2}}\|(u_0,h_3)\|_G. \tag{2.16}$$

With the help of (2.16) and the fact that $\mathbb{D}(A)$, $\widetilde{C}_0^2[0,T]$ are dense in $L^2(0,L)$, $L^2(0,T)$, respectively, the linear map

$$\widetilde{\Phi} : \mathbb{D}(A)\times\widetilde{C}_0^2[0,T] \to Y$$
$$(u_0,h_3) \mapsto u,$$

can be uniquely extended to the following linear continuous map:

$$\Phi : L^2(0,L)\times L^2(0,T) \to Y$$
$$(u_0,h_3) \mapsto u.$$

We have finished the proof of Proposition 3. □

We will use Hilbert Uniqueness Method to prove the controllability of system (1.2), so we should get the observability inequality of the adjoint system of (2.1) first. The adjoint system of (2.1) is given as following

$$\begin{cases} y_t + y_x + y_{xx} + y_{xxx} = 0, \text{ for } (t,x) \in [0,T] \times [0,L], \\ y(t,0) = y(t,L) = y_x(t,0) = 0, \\ y(T,x) = y_T. \end{cases} \tag{2.17}$$

If we use the method of variable substitution with $\tau = T - t, \quad \xi = L - x$, then (2.17) can be changed into the following form

$$\begin{cases} y_\tau + y_\xi - y_{\xi\xi} + y_{\xi\xi\xi} = 0, \text{ for } (\tau,\xi) \in [0,T] \times [0,L], \\ y(\tau,0) = y(\tau,L) = y_\xi(\tau,L) = 0, \\ y(0,\xi) = y_T, \end{cases}$$

which is just (2.1). So, the observability of (2.17) is just the observability of (2.1). With Proposition 1 and Proposition 2, we can have the observability inequality of (2.17) directly which is present in the following proposition.

Proposition 4 *Let $L > 0$, $T > 0$ be given, then for any $y_T \in L^2(0,L)$, the solution y of* (2.17) *belongs to the space Y with $y_x(t,L) \in L^2(0,T)$. Moreover, if $L \notin \mathcal{N}$, we have the following estimate*

$$\|y_x(t,L)\|_{L^2(0,T)} \le \|y_T\|_{L^2(0,L)} \le C\|y_x(t,L)\|_{L^2(0,T)} \tag{2.18}$$

for any $y_T \in L^2(0,L)$, where C is a positive constant which depends only on L and T.

With the help of the observability inequality (2.18), we can give the exact controllability of the linear system (1.2) which is one of the main results of this paper.

Theorem 3 *Let $L \in (0,\infty) \setminus \mathcal{N}, \quad T > 0$ be given, then for any $u_0, u_T \in L^2(0,L)$, there exists $h_3 \in L^2(0,T)$ such that there exists an unique solution u of* (1.2) *which satisfies $u(0,\cdot) = u_0$ and $u(T,\cdot) = u_T$, with $u \in Y$.*

Proof Since the map Φ in Proposition 3 is bilinear, we can write the solution u of (1.2) as $u(t,x) = \Phi(u_0,h_3) = W_0(t)u_0 + \Phi(0,h_3)$. Without loss of generality, we may assume $u_0 = 0$. Let $y_T \in L^2(0,L), \quad h_3 \in L^2(0,T)$ and y, u be solutions of (2.15), (1.2), respectively. Multiply both sides of (2.1) by y and integrate over $[0,T] \times [0,L]$ with respect to t and x, then we have the following equality

$$\int_0^L y_T u_T dx = \int_0^T y_x(t,L) h_3 dt. \tag{2.19}$$

Let Υ denote the linear and bounded map as following

$$\begin{aligned} \Upsilon : L^2(0,L) &\to L^2(0,L) \\ y_T &\mapsto u_T, \end{aligned}$$

where u is the solution of (1.2) with $h_3 = y_x(t,L) \in L^2(0,T)$ and y being the solution of (2.15). With (2.18) and (2.19), we have

$$< \Upsilon(y_T), y_T >= \|y_x(t,L)\|^2_{L^2(0,T)} \ge C^{-2}\|y_T\|^2_{L^2(0,L)}.$$

Using Lax-Milgram theorem, we know that Υ is invertible. Then, for any given $u_T \in L^2(0,L)$, we solve (2.15) with $y_T = \Upsilon^{-1}u_T$ and its solution y belongs to Y. Solving (1.2) with $h_3 = y_x(t,L)$, we get the solution u of it which satisfies

$$u(0,x) = u_0, \quad u(T,x) = u_T, \quad with \quad u \in Y.$$

The proof is completed. □

Remark 2 If we both use the right Dirichlet boundary condition $u(L,t) = h_2(t)$ and the right Neumann boundary condition $u_x(L,t) = h_3(t)$ as control inputs, then the restriction on L with $L \notin \mathcal{N}$ can be removed. Since the set $\mathcal{N} = \{\pi\sqrt{2(l^2 - k^2)}; l,k \in \mathbb{N}^* \text{ and } l > k\}$ are discrete, if $L \in \mathcal{N}$, we let $\widetilde{L} = L + \tau$ with $\widetilde{L} \notin \mathcal{N}$, where τ is some very little positive number. Using Proposition 2, we consider the (1.2) with $\widetilde{u}_0, \widetilde{u}_T \in L^2(0,\widetilde{L})$ and let $\widetilde{u}$ be its solution, where $\widetilde{u}_0, \widetilde{u}_T$ are the prolongations by zero of u_0, u_T, respectively. Then we just need to restrict the solution $\widetilde{u}$ to the domain $(0,T) \times (0,L)$ to get the control inputs $h_2(t) = \widetilde{u}(L,t)$ and $h_3(t) = \widetilde{u}_x(L,t)$.

3 Controllability of nonlinear system

In this section, we consider the boundary controllability of the following nonlinear system

$$\begin{cases} u_t + u_x - u_{xx} + u_{xxx} + uu_x = 0, \text{ for } (t,x) \in [0,T] \times [0,L], \\ u(0,x) = u_0, \quad u(t,0) = 0, \quad u(t,L) = 0, \quad u_x(t,L) = h_3(t), \end{cases} \tag{3.1}$$

with only one control input $h_3(t) \in L^2(0,T)$. We first consider the following nonhomogeneous system

$$\begin{cases} u_t + u_x - u_{xx} + u_{xxx} = f, \text{ for } (t,x) \in [0,T] \times [0,L], \\ u(0,x) = u_0, \quad u(t,0) = 0, \quad u(t,L) = 0, \quad u_x(t,L) = h_3(t). \end{cases} \tag{3.2}$$

Since the solution u of (3.2) can be written in the form $u = u_1 + u_2 + u_3$, where u_1, u_2, u_3 are the solutions of the following linear systems (3.3), (3.4), (3.5), respectively.

$$\begin{cases} u_{1t} + u_{1x} - u_{1xx} + u_{1xxx} = 0, \text{ for } (t,x) \in [0,T] \times [0,L], \\ u_1(0,x) = u_0, \quad u_1(t,0) = 0, \quad u_1(t,L) = 0, \quad u_{1x}(t,L) = 0, \end{cases} \tag{3.3}$$

$$\begin{cases} u_{2t} + u_{2x} - u_{2xx} + u_{2xxx} = 0, \text{ for } (t,x) \in [0,T] \times [0,L], \\ u_2(0,x) = 0, \quad u_2(t,0) = 0, \quad u_2(t,L) = 0, \quad u_{2x}(t,L) = h_3, \end{cases} \tag{3.4}$$

$$\begin{cases} u_{3t} + u_{3x} - u_{3xx} + u_{3xxx} = f, \text{ for } (t,x) \in [0,T] \times [0,L], \\ u_3(0,x) = 0, \quad u_3(t,0) = 0, \quad u_3(t,L) = 0, \quad u_{3x}(t,L) = 0. \end{cases} \tag{3.5}$$

According to Propositions 1 and 3, we have already known that the map $\Phi : (u_0, h_3) \in L^2(0,L) \times L^2(0,T) \mapsto u_1 + u_2 \in Y$ is a bounded bilinear continuous operator. So we know that the solution map of (3.4) $\Psi_2 : h_3 \in L^2(0,T) \mapsto u_2 \in Y$ is also a bounded linear continuous operator. Before we discuss the properties of the solution map of (3.5), we first give the following lemma.

Lemma 3 *If $f, g \in L^2(0,T; H^1(0,L))$, then $fg_x \in L^1(0,T; L^2(0,L))$, and the ap $\Gamma : f \in L^2(0,T; H^1(0,L)) \mapsto ff_x \in L^1(0,T; L^2(0,L))$ is continuou.*

Proof Using Gagliardo-Nirenberg interpolation inequality, we have

$$\begin{aligned} \|f(t,\cdot)g_x(t,\cdot)\|_{L^2(0,L)} &\le \|f(t,\cdot)|\|_{L^\infty(0,L)} \|g_x(t,\cdot)\|_{L^2(0,L)} \\ &\le C\left(\|f(t,\cdot)\|_{L^2(0,L)} + \|f_x(t,\cdot)\|_{L^2(0,L)}\right) \|g_x(t,\cdot)\|_{L^2(0,L)}, \end{aligned}$$

where C is a positive constant depending only on L. Integrating the two terms in the right part of the above inequality on the interval $(0,T)$ with respect to t and using *Hölder* inequality, we have

$$\begin{aligned} &\int_0^T \|f(t,\cdot)\|_{L^2(0,L)} \|g_x(t,\cdot)\|_{L^2(0,L)} dt \\ &\qquad \le \|f(t,x)\|_{L^2(0,T;L^2(0,L))} \|g_x(t,x)\|_{L^2(0,T;L^2(0,L))} \\ &\qquad \le \|f\|_{L^2(0,T;H^1(0,L))} \|g\|_{L^2(0,T;H^1(0,L))}, \end{aligned}$$

and

$$\begin{aligned} &\int_0^T \|f_x(t,\cdot)\|_{L^2(0,L)} \|g_x(t,\cdot)\|_{L^2(0,L)} dt \\ &\qquad \le \|f_x(t,x)\|_{L^2(0,T;L^2(0,L))} \|g_x(t,x)\|_{L^2(0,T;L^2(0,L))} \\ &\qquad \le \|f\|_{L^2(0,T;H^1(0,L))} \|g\|_{L^2(0,T;H^1(0,L))}. \end{aligned}$$

Then we have

$$\|fg_x\|_{L^1(0,T;L^2(0,L))} \le C_1 \|f\|_{L^2(0,T;H^1(0,L))} \|g\|_{L^2(0,T;H^1(0,L))},$$

where C_1 is a positive constant depending only on L. The proof is completed. □

Proposition 5 *For any $f \in L^1(0,T;L^2(0,L))$, the mild solution u_3 of (3.5) belongs to the space Y and the solution map*

$$\Psi_3 : f \in L^1(0,T;L^2(0,L)) \mapsto u_3 \in Y$$

is continuous.

Proof We may assume u_3 is a strong solution of (3.5). Multiply both sides of the equation in (3.5) by u_3 and integrate over $(0, L)$ with respect to x. Integration by pars leads to

$$\frac{d}{dt}\int_0^L u_3^2 dx + u_{3x}^2(t,0) + 2\int_0^L u_{3x}^2 dx \le 2\|f(t,\cdot)\|_{L^2(0,L)}\|u_3(t,\cdot)\|_{L^2(0,L)}, \tag{3.6}$$

where $t \in [0,T]$. Integrate both sides of (3.6) with respect to t, we have

$$\begin{aligned}\|u_3(t,\cdot)\|^2_{L^2(0,L)} + \|u_{3x}(\cdot,0)\|^2{}_{L^2(0,t)} + 2\int_0^t\int_0^L u_{3x}^2(\tau,x)dxd\tau \\ \le 2\|f\|_{L^1(0,t;L^2(0,L))}\|u_3\|_{L^\infty(0,t;L^2(0,L))},\end{aligned}$$

for any $t \in [0,T]$. Since the solution u_3 of (3.5) can be written as $u_3(t) = \int_0^t W_0(t-\tau)f(\tau)d\tau$ where $W_0(t)$ is a semigroup of contractions, we have the following estimate

$$\begin{aligned}\|u_3\|_{L^\infty(0,t;L^2(0,L))} &\le \|u_3(t',\cdot)\|_{L^2(0,L)} \\ &\le \int_0^{t'} \|f(\tau,\cdot)\|_{L^2(0,L)}d\tau \\ &\le \|f\|_{L^1(0,t;L^2(0,L))},\end{aligned}$$

for any $t \in [0,T]$ and some $t' \in [0,t]$. So we have

$$\|u_3(t,\cdot)\|^2_{L^2(0,L)} + \|u_{3x}(\cdot,0)\|^2_{L^2(0,t)} + 2\|u_{3x}\|^2_{L^2(0,t;L^2(0,L))} \le 2\|f\|^2_{L^1(0,t;L^2(0,L))}, \tag{3.7}$$

for any $t \in [0,T]$. According to (3.7), we have

$$\begin{aligned}&\|u_3\|_{C([0,T];L^2(0,L))} \le \sqrt{2}\|f\|_{L^1(0,T;L^2(0,L))}, \\ &\|u_3\|_{L^2(0,T;L^2(0,L))} \le \sqrt{2T}\|f\|_{L^1(0,T;L^2(0,L))}, \\ &\|u_{3x}\|_{L^2(0,T;L^2(0,L))} \le \|f\|_{L^1(0,T;L^2(0,L))}.\end{aligned}$$

So we have the following inequalities

$$\|u_3\|_{L^2(0,T;H^1(0,L))} \le \sqrt{1+2T}\|f\|_{L^1(0,T;L^2(0,L))}$$

and

$$\|u_3\|_Y \le (\sqrt{2}+\sqrt{1+2T})\|f\|_{L^1(0,T;L^2(0,L))}. \tag{3.8}$$

With (3.8), it's obvious that the mild solution u_3 belongs to the space Y and the solution map Ψ_3 is continuous. The proof is completed. □

Now we can give the other main result of this paper.

Theorem 4 *Let $T > 0$, $L \in (0,\infty) \setminus \mathcal{N}$ be given, then there exists a positive number δ such that for any $u_0, u_T \in L^2(0,L)$ with $\|u_0\|_{L^2(0,L)} \le \delta$ and $\|u_T\|_{L^2(0,L)} \le \delta$, there exists $h_3(t) \in L^2(0,T)$ such that the solution u of (3.1) satisfies $u(0,\cdot) = u_0$ and $u(T,\cdot) = u_T$, with $u \in Y$.*



Proof We first consider the case $L \notin \mathcal{N}$. We can write the solution u of Eq. (3.1) in an integral form as following

$$u(t) = W_0(t)u_0 + \Psi_2 h_3 - \int_0^t W_0(t-\tau)(uu_x)(\tau)d\tau. \tag{3.9}$$

Let's denote

$$\omega(T, v) = \int_0^T W_0(T-\tau)(vv_x)d\tau, \text{ for any } v \in Y.$$

According to Theorem 3, there exists a linear bounded map of (2.1)

$$\begin{aligned} \Lambda : L^2(0, L) \times L^2(0, L) &\to L^2(0, T) \\ (u_0, u_T) &\mapsto h_3. \end{aligned}$$

Let $h_3 = \Lambda(u_0, u_T + \omega(T, v))$, then we have

$$v(t) = W_0(t)u_0 + \Psi_2\Lambda(u_0, u_T + \omega(T, v)) - \int_0^t W_0(t-\tau)(vv_x)(\tau)d\tau,$$

and v satisfies

$$v|_{t=0} = u_0, \qquad v|_{t=T} = u_T.$$

We can define a map Γ as following

$$\Gamma(v) = W_0(t)u_0 + \Psi_2\Lambda(u_0, u_T + \omega(T, v)) - \int_0^t W_0(t-\tau)(vv_x)(\tau)d\tau.$$

In order to finish the proof, it's sufficient that if we can show the map Γ is a map of contractions in a certain metric space and then its fixed point u is right the solution of (3.1) with boundary control $h_3 = \Lambda(u_0, u_T + \omega(T, v))$ where v satisfies

$$v|_{t=0} = u_0 \text{ and } v|_{t=T} = u_T.$$

We define a ball as following

$$B_r = \{v \in Y; \|v\|_Y \le r\},$$

where r will be appropriately chosen later and we will show that the map Γ is indeed a contraction map in B_r. According to (2.3), (2.16) and (3.8), we have the following estimate

$$\|\Gamma(v)\|_Y \le C_1 \left(\|u_0\|_{L^2(0,L)} + \|\Lambda(u_0, u_T + \omega(T, v))\|_{L^2(0,T)} + \int_0^T \|vv_x\|_{L^2(0,L)}(t)dt \right),$$

where C_1 is a positive constant depending only on T, L. Since Λ is a bounded and linear operator, we have

$$\|\Lambda(u_0, u_T + \omega(T, v))\|_{L^2(0,T)} \le C_2 \left(\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} + \|\omega(T, v)\|_{L^2(0,L)} \right),$$

where C_2 is a positive constant depending only on T, L. For the definition of $\omega(T, v)$, we have

$$\|\omega(T, v)\|_{L^2(0,L)} \le \int_0^T \|vv_x\|_{L^2(0,L)}(t)dt.$$

According to Lemma 3, we have

$$\int_0^T \|vv_x\|_{L^2(0,L)}(t)dt \le C_3\|v\|^2_{L^2(0,T;H^1(0,L))} \le C_3\|v\|^2_Y,$$

where C_3 is a positive constant depending only on L. So there are two positive constants C_4, C_5 depending only on T, L,such that

$$\|\Gamma(v)\|_Y \le C_4\left(\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)}\right) + C_5\|v\|^2_Y, \tag{3.10}$$

for any $v \in Y$. If we choose some positive numbers δ and r satisfying the following conditions

$$r = 3C_4\delta, \quad 9C_4C_5\delta \le 1 \quad and \quad 3C_1(C_2+T)C_3C_4\delta < 1,$$

and let $\|u_0\|_{L^2(0,L)} < \delta$, $\|u_T\|_{L^2(0,L)} < \delta$, $v \in B_r$, then we have

$$\|\Gamma(v)\|_Y \le 2C_4\delta + 9C_4C_5\delta C_4\delta \le 3C_4\delta = r, \quad \Gamma(v) \in B_r.$$

For any $v_1, v_2 \in B_r$, we have

$$\Gamma(v_1) - \Gamma(v_2) = \Psi_2\Lambda(0, \omega(T, v_1) - \omega(T, v_2)) + \frac{1}{2}\int_0^t W_0(t-\tau)[(v_1+v_2)(v_1-v_2)]_x(\tau)d\tau.$$

Using generalized Minkowski inequality, we have the following estimate

$$\begin{aligned}
&\|\Gamma(v_1) - \Gamma(v_2)\|_Y \\
&\le C_1\big(\|\Lambda(0, \omega(T, v_1) - \omega(T, v_2))\|_{L^2(0,T)} \\
&\quad + \frac{1}{2}\Big\| \int_0^t W_0(t-\tau)[(v_1+v_2)(v_1-v_2)]_x(\tau)d\tau \Big\|_{L^1(0,T;L^2(0,L))}\big) \\
&\le C_1C_2\|\omega(T, v_1) - \omega(T, v_2)\|_{L^2(0,L)} \\
&\quad + \frac{1}{2}C_1\int_0^T \Big(\int_0^L \Big|\int_0^t W_0(t-\tau)[(v_1+v_2)(v_1+v_2)]_x(\tau)d\tau\Big|^2 dx\Big)^{\frac{1}{2}} dt \\
&= \frac{1}{2}C_1C_2\Big(\int_0^L \Big|\int_0^T W_0(T-\tau)[(v_1+v_2)(v_1-v_2)]_x(\tau)d\tau\Big|^2 dx\Big)^{\frac{1}{2}} \\
&\quad + \frac{1}{2}C_1\int_0^T \Big(\int_0^L \Big|\int_0^t W_0(t-\tau)[(v_1+v_2)(v_1+v_2)]_x(\tau)d\tau\Big|^2 dx\Big)^{\frac{1}{2}} dt \\
&\le \frac{1}{2}C_1C_2\int_0^T \Big(\int_0^L |[(v_1+v_2)(v_1-v_2)]_x|^2 dx\Big)^{\frac{1}{2}} dt \\
&\quad + \frac{1}{2}C_1\int_0^T \Big(\int_0^t \Big(\int_0^L |[(v_1+v_2)(v_1-v_2)]_x|^2 dx\Big)^{\frac{1}{2}} d\tau\Big) dt \\
&\le \frac{1}{2}C_1C_2C_3\|v_1+v_2\|_Y\|v_1-v_2\|_Y \\
&\quad + \frac{1}{2}C_1\int_0^T \Big(\int_0^T \Big(\int_0^L |[(v_1+v_2)(v_1-v_2)]_x|^2 dx\Big)^{\frac{1}{2}} d\tau\Big) dt \\
&\le \frac{1}{2}C_1C_2C_3\|v_1+v_2\|_Y\|v_1-v_2\|_Y + \frac{1}{2}C_1C_3T\|v_1+v_2\|_Y\|v_1-v_2\|_Y \\
&= \frac{1}{2}C_1(C_2+T)C_3\|v_1+v_2\|_Y\|v_1-v_2\|_Y \\
&\le \frac{1}{2}C_1(C_2+T)C_3(\|v_1\|_Y + \|v_2\|_Y)\|v_1-v_2\|_Y \\
&\le k\|v_1-v_2\|_Y.
\end{aligned}$$

with $k = 3C_1(C_2+T)C_3C_4\delta < 1$, where C_1,C_2 and C_3 are positive constants depending only on T and L. We have proved that the map Γ is a contraction map in the ball B_r if $\|u_0\|_{L^2(0,L)} \le \delta$ and $\|u_T\|_{L^2(0,L)} \le \delta$. The proof is complete. □

Remark 3 With the same method we have presented in Remark 2, we know that if $L \in \mathcal{N}$, the system (3.1) is local exactly controllable if we both use the right Dirichlet boundary $u(t, L) = h_2(t)$ and the right Neumann boundary condition $u_x(t, L) = h_3(t)$ as control inputs.

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