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ORIGINAL RESEARCH

Fermat or Mersenne numbers as products of two k-Pell numbers

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Abstract For an integer $k \geq 2$, let $P_n^{(k)}$ be the k-generalized Pell sequence which starts with $0, \ldots, 0, 1$ (k terms) and each term afterwards is given by the linear recurrence

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)}, \quad \text{for all } n \geq 2.$$

In this paper, our study focuses on Fermat and Mersenne numbers and we determine all of them, which are expressed as products of two k-Pell numbers.

Keywords k-Pell numbers · Fermat numbers · Mersenne numbers · Linear form in logarithms · Reduction method

Mathematics Subject Classification 11B39 · 11J86

1 Introduction

Numerous challenges in number theory can be simplified by identifying the common elements between two sequences of positive integers. Hence, the investigation into determining the intersection of two sequences has attracted interest from various number theorists. Mignotte [11] proved that the intersection of two sequences is finite unless the roots of their characteristic polynomials are multiplicatively dependent. But there is no effective result about the intersection of two sequences in general. For some special cases of intersection of two sequences, we can cite the works [3,4,6–8,12]. Here, we are interested in the problem of determining the intersection of the product of two k-Pell sequence and Fermat or Mersenne sequences.

A Fermat number is a number that can be expressed in the form $2^n + 1$, where n is a positive integer. It is named after Pierre de Fermat, a French mathematician who studied them in the 17-th century. The first few

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Table 1 First nonzero terms of k-Pell sequences for $k = 2, 3, 4$ and 5

k	Name	First nonzero terms
2	Pell	1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, ...
3	3-Pell	1, 2, 5, 13, 33, 84, 214, 545, 1388, 3535, 9003, 22929, 58396, ...
4	4-Pell	1, 2, 5, 13, 34, 88, 228, 591, 1532, 3971, 10293, 26680, 69156, ...
5	5-Pell	1, 2, 5, 13, 34, 89, 232, 605, 1578, 4116, 10736, 28003, 73041, ...

of them are 3, 5, 9, 17, 33, ... A Mersenne number is a number of the form $2^n - 1$, where n is a nonnegative integer. The first few Mersenne numbers are 1, 3, 7, 15, 31, 63, 127, 255, ... Mersenne numbers have many interesting properties and have been the subject of a deep research. One famous result is the discovery of the largest known prime number (as of 2021), which is a Mersenne prime with 82, 589, 933 digits. Some properties of these numbers have been well studied (see [6,8]).

The Pell numbers are a sequence of numbers that appear in mathematics and number theory. The Pell numbers are defined by the recurrence relation

$$P_0 = 0, \quad P_1 = 1 \quad \text{and} \quad P_n = 2P_{n-1} + P_{n-2} \quad \text{for all } n \geq 2.$$

The first few Pell numbers are 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, ... There are several generalizations of the Pell numbers, which are variations or extensions of the original sequence. Let $k \geq 2$ be an integer. We define a generalization of the Pell sequences $P_n^{(k)}$ recurrently as follows

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)}, \quad \text{for all } n \geq 2,$$

with the initial conditions $P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \cdots = P_0^{(k)} = 0$ and $P_1^{(k)} = 1$.
We note that this generalization is in fact a family of sequences where each new choice of k produces a distinct sequence. For example, the usual Pell sequence $(P_n)_{n\geq 0}$ is obtained for $k = 2$. Below we present the values of these numbers for the first few values of k and $n \geq 1$.

Some mathematicians have studied relations among Fermat/Mersenne numbers and other sequences [6–8,12]. As with Hernane [7], we determined all Fermat numbers and Mersenne numbers that are products of two k-Fibonacci numbers. Motived by this result, we will find all Fermat and Mersenne numbers that can be written as products of two k-Pell numbers. More precisely, in this paper, we will study the Diophantine equations presented in the following theorem.

Theorem 1.1 *The Diophantine equations*

$$P_n^{(k)} P_m^{(k)} = 2^a \pm 1 \tag{1.1}$$

have no solutions in positive integers a, k, n, m with $k, n, m \geq 2$ and $n \geq m$.

Remark 1.2 On can easily notice that if $m = 1$, then we have $(a, k, m, n) = (1, k, 1, 1)$ is a solution for the Mersenne sequence and $(a, k, m, n) = (2, k, 1, 3)$ is a solution for Fermat sequence.

To establish the aforementioned theorem, we use linear forms in logarithms of algebraic numbers along with a reduction algorithm originally introduced by Baker and Davenport [1]. The subsequent section provides an overview of these methods, accompanied by relevant results and definitions that will be applied throughout the remainder of this study.

2 The tools

2.1 Linear forms in logarithms

For a non-zero algebraic number η of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}\left(X - \eta^{(j)}\right)$, we denote by

$$h(\eta) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d}\log\left(\max\left\{1, |\eta^{(j)}|\right\}\right)\right)$$

the usual absolute logarithmic height of η. In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q \geq 1$, then $h(\eta) = \log \max\{|p|, q\}$. The following properties of the function logarithmic height $h(\cdot)$, which will be used in the next section without special reference, are well-known

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{2.1}$$
$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{2.2}$$
$$h(\eta^s) = |s|\, h(\eta) \quad (s \in \mathbb{Z}). \tag{2.3}$$

Now, we will recall a result proved by Matveev (see [10]) on a lower bound for a nonzero linear form in logarithms. Here, we consider the version due to Bugeaud, Mignotte and Siksek [4].

Theorem 2.1 *Let $\eta_1, \ldots, \eta_s$ be nonzero real algebraic numbers and let $b_1, \ldots, b_s$ be integer numbers. Let $d_{\mathbb{K}}$ be the degree of the number field $\mathbb{Q}(\eta_1, \ldots, \eta_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j \geq \max\{d_{\mathbb{K}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad \textit{for} \quad j = 1, \ldots, s.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_s|\}.$$

If $\eta_1^{b_1} \cdots \eta_s^{b_s} - 1 \neq 0$, then

$$|\eta_1^{b_1} \cdots \eta_s^{b_s} - 1| \geq \exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log B) A_1 \cdots A_s).$$

2.2 The reduction algorithm

Here, we present a variant of the reduction method of Baker-Davenport [1] due to Dujella-Pethő [5]. We use a slight variant of the version given by Dujella-Pethő [5].

Lemma 2.2 *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let*

$$\varepsilon = ||\mu q|| - M \cdot ||\gamma q||,$$

where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < u\gamma - v + \mu < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

The above lemma cannot be applied when μ is a linear combination of 1 and γ, since then $\varepsilon < 0$. In this case, we use the following nice property of continued fractions (see [9, pp. 30 and 37]).

Lemma 2.3 *Let p_i/q_i be the convergents of the continued fraction $[a_0, a_1, \ldots]$ of the irrational number γ. Let M be a positive integer and put $a_M := \max\{a_i | 0 \leq i \leq N + 1\}$ where $N \in \mathbb{N}$ is such that $q_N \leq M < q_{N+1}$. If $x, y \in \mathbb{Z}$ with $x > 0$, then*

$$|x\gamma - y| > \frac{1}{(a_M + 2)x} \quad \textit{for all } x < M.$$

2.3 Properties of the k-generalized Pell sequence

In this subsection, we recall some essential facts and properties of the k-Pell sequence, which will be pertinent in subsequent discussions. The characteristic polynomial associated with this sequence is given by

$$\Psi_k(x) = x^k - 2x^{k-1} - \cdots - x - 1.$$

In [2], Bravo et al. demonstrated that $\Psi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has just one root $\alpha(k)$ outside the unit circle. This root is real, positive, and satisfies $\alpha(k) > 1$. The remaining roots are strictly contained within the unit circle. Additionally, in the same paper, they demonstrated that

$$\varphi^2(1-\varphi^{-k}) < \alpha(k) < \varphi^2 \quad \text{for all } k \geq 2, \quad \text{with } \varphi := \frac{1+\sqrt{5}}{2}. \tag{2.4}$$

For simplicity, we generally omit the dependence on k for α. For $k \geq 2$, let

$$g_k(x) := \frac{x-1}{(k+1)x^2 - 3kx + k - 1} = \frac{x-1}{k(x^2-3x+1) + x^2 - 1}. \tag{2.5}$$

In [3], Bravo and Hererra proved that the inequalities

$$0.276 < g_k(\alpha) < 0.5 \quad \text{and} \quad \left|g_k(\alpha^{(i)})\right| < 1, \ 2 \leq i \leq k$$

hold, where $\alpha := \alpha^{(1)}, \ldots, \alpha^{(k)}$ are all the zeros of $\Psi_k(x)$. So, the number $g_k(\alpha)$ is not an algebraic integer. Indeed, if we suppose that $g_k(\alpha)$ is an algebraic integer, then we know that $g_k(\alpha) \neq 0$ and $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(g_k(\alpha)) \in \mathbb{Z}$, where $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(g_k(\alpha))$ is the norm of $g_k(\alpha)$ from $\mathbb{Q}(\alpha)$ to $\mathbb{Q}$. In addition, they proved that the logarithmic height of $g_k(\alpha)$ satisfies

$$h(g_k(\alpha)) < 4k\log\varphi + k\log(k+1), \quad \text{for all } k \geq 2. \tag{2.6}$$

With the above notation, Bravo et al. [2] showed that the inequalities

$$P_n^{(k)} = \sum_{i=1}^{k} g_k(\alpha^{(i)})\alpha^{(i)^n} \quad \text{and} \quad \left|P_n^{(k)} - g_k(\alpha)\alpha^n\right| < \frac{1}{2} \tag{2.7}$$

hold, for all $n \geq 1$ and $k \geq 2$. Furthermore, for $n \geq 1$ and $k \geq 2$, it was proved in [2] that

$$\alpha^{n-2} \leq P_n^{(k)} \leq \alpha^{n-1}. \tag{2.8}$$

Moreover, note that for $n \leq k+1$ we have

$$P_n^{(k)} = F_{2n-1}. \tag{2.9}$$

Lastly, we present the following crucial lemmas that will be used subsequently.

Lemma 2.4 ([3, Lemma 2]) *If $k \geq 30$ and $n > 1$ are integers satisfying $n < \varphi^{k/2}$, then*

$$g_k(\alpha)\alpha^n = \frac{\varphi^{2n}}{\varphi+2}(1+\zeta), \quad \textit{where} \quad |\zeta| < \frac{4}{\varphi^{k/2}}.$$

Lemma 2.5 ([13, Lemma 7]) *If $m \geq 1$, $T > (4m^2)^m$ and $T > y/(\log y)^m$. Then,*

$$y < 2^m T(\log T)^m.$$

Lemma 2.6 ([14, Lemma 2.2]) *If $|x| < d$ with $0 < d < 1$, then*

$$|\log(1+x)| < \frac{-\log(1-d)}{d}|x|. \tag{2.10}$$

3 The proof of Theorem 1.1

The proof of Theorem 1.1 will be done in four steps. We will first start by giving the following precision. If $m = 1$, then $P_m^{(k)} = 1$. In this case, the Diophantine equation (1.1) has been studied in [12]. For the cases $2 \leq m \leq n \leq k+1$, we use the identity (2.9) to rewrite the Diophantine equation (1.1) as

$$F_{2n-1}F_{2m-1} = 2^a \pm 1.$$

According to the main result in [7], we deduce that (1.1) has no solutions in this range. Thus, from now, we assume that $n \geq k+2$. Subsequently, we will establish the relationship between the sizes of a and n. By applying inequalities (2.8), we derive

$$2^{a-1} < 2^a \pm 1 = P_n^{(k)} P_m^{(k)} < \alpha^{n+m-2} < \alpha^{2n-2}$$

and so

$$a < (2n-2)\left(\frac{\log \alpha}{\log 2}\right) + 1 = n\left(\frac{2\log \alpha}{\log 2}\right) - \left(\frac{2\log \alpha}{\log 2}\right) + 1.$$

Furthermore, by (2.4) we obtain

$$a < 3n. \tag{3.1}$$

3.1 An inequality for n versus k

Here, we will establish an inequality for n in relation with k, by showing the following lemma.

Lemma 3.1 *If (a, k, m, n) is a solution in integers of equation (1.1) with $k \geq 2$ and $n \geq k+2$, then we have the following inequality*

$$n < 8.2 \cdot 10^{28} k^9 \log^5 k. \tag{3.2}$$

Proof Using estimate provided in (2.7), we obtain

$$P_n^{(k)} = g_k(\alpha)\alpha^n + e_k(n), \quad \text{where } |e_k(n)| < \frac{1}{2},$$

thus allowing us to express equation (1.1) as

$$(g_k(\alpha)\alpha^n + e_k(n))(g_k(\alpha)\alpha^m + e_k(m)) = 2^a \pm 1; \tag{3.3}$$

that is,

$$g_k^2(\alpha)\alpha^{n+m} - 2^a = -e_k(m)g_k(\alpha)\alpha^n - e_k(n)g_k(\alpha)\alpha^m - e_k(n)e_k(m) \pm 1.$$

By taking the absolute value and dividing both sides by $g_k^2(\alpha)\alpha^{n+m}$, we get

$$\left| g_k^{-2}(\alpha) \cdot \alpha^{-(n+m)} \cdot 2^a - 1 \right| \leq \frac{1.82}{\alpha^m} + \frac{1.82}{\alpha^n} + \frac{16.41}{\alpha^{n+m}} < \frac{20.1}{\alpha^m}, \tag{3.4}$$

where we have used the fact that $g_k(\alpha) > 0.276$. Define

$$\Gamma_1 := g_k^{-2}(\alpha) \cdot \alpha^{-(n+m)} \cdot 2^a - 1. \tag{3.5}$$

So, we have

$$|\Gamma_1| < \frac{20.1}{\alpha^m}. \tag{3.6}$$

Note that $\Gamma_1 \neq 0$, otherwise we would get

$$g_k^2(\alpha) = \alpha^{-(n+m)} \cdot 2^a,$$

which implies that $g_k^2(\alpha)$ is an algebraic integer, a contradiction.

Now, we will utilize Theorem 2.1 for Γ_1, considering the following parameters:

$$(\eta_1, b_1) := (g_k(\alpha)), -2), \quad (\eta_2, b_2) := (\alpha, -(n+m)), \quad (\eta_3, b_3) := (2, a).$$

As $\eta_1, \eta_2, \eta_3 \in \mathbb{K} := \mathbb{Q}(\alpha)$, we have $d_{\mathbb{K}} = k$. Now, we provide estimates for the usual absolute logarithmic heights of these numbers: $h(\eta_1) < h(g_k(\alpha)) < 4k \log \varphi + k \log(k+1) < 4.4k \log k$, $h(\eta_2) = (\log \alpha)/k < (2 \log \varphi)/k$, and $h(\eta_3) = \log 2$. Therefore, we have the flexibility to choose

$$A_1 := 4.4k^2 \log k > \max\{kh(\eta_1), |\log \eta_1|, 0.16\},$$

$$A_2 := 2 \log \varphi = \max\{kh(\eta_2), |\log \eta_2|, 0.16\},$$

and

$$A_3 := k \log 2 = \max\{kh(\eta_3), |\log \eta_3|, 0.16\}.$$

Previously, we established that $a < 3n$ and $m \leq n$, enabling us to set $B := 3n$. Armed with these considerations, we are now in a position to apply Theorem 2.1 and derive

$$\begin{aligned} |\Gamma_1| &> \exp\left(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot k^2(1+\log k)(1+\log 3n)(4.4k^2 \log k)(2 \log \varphi)(k \log 2)\right) \\ &> \exp(-1.06 \cdot 10^{12}(k^5 \log^2 k)(1+\log 3n)), \end{aligned}$$

where we have used the fact that $1 + \log k < 2.5 \log k$, which holds for $k \geq 2$. Comparing this lower bound with the upper bound of $|\Gamma_1|$ provided by (3.6), we arrive at

$$(m-1) \log \alpha < 1.07 \cdot 10^{12} k^5 \log^2 k (1 + \log 3n). \tag{3.7}$$

Now, let us rewrite equation (1.1) in an alternative manner to derive a second linear expression in logarithms. To achieve this, we use estimate (2.7) once again to obtain

$$(P_k(\alpha)\alpha^n + e_k(n))P_m^{(k)} = 2^a \pm 1;$$

that is,

$$g_k(\alpha)\alpha^n - \frac{2^a}{P_m^{(k)}} = \frac{\pm 1}{P_m^{(k)}} - e_k(n). \tag{3.8}$$

As previously done, by taking the absolute value and dividing both sides by $g_k(\alpha)\alpha^n$, we acquire

$$\left| \left(P_m^{(k)} g_k(\alpha)\right)^{-1} \cdot \alpha^{-n} \cdot 2^a - 1 \right| \leq \frac{3}{2\,|g_k(\alpha)|\,\alpha^n} < \frac{5.5\alpha}{\alpha^n} < \frac{14.4}{\alpha^n}. \tag{3.9}$$

Define

$$\Gamma_2 := \left(P_m^{(k)} g_k(\alpha)\right)^{-1} \cdot \alpha^{-n} \cdot 2^a - 1. \tag{3.10}$$

Let us show that $\Gamma_2 \neq 0$. In fact, $\Gamma_2 = 0$ leads to $g_k(\alpha) = 2^a \alpha^{-n}/P_m^{(k)}$. Taking norms, we get

$$N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(g_k(\alpha)) = \frac{2^{ka}}{\left(P_m^{(k)}\right)^k} N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^{-n}) = \pm \frac{2^{ka}}{\left(P_m^{(k)}\right)^k}.$$

Now, we see that

$$N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(g_k(\alpha)) = \prod_{i=1}^{k} \frac{\alpha_i - 1}{k(\alpha_i^2 - 3\alpha_i + 1) + \alpha_i^2 - 1},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the conjugates of α. The above is a fraction A/B, with A, B integers and

$$A = \prod_{i=1}^{k} (\alpha_i - 1) = (-1)^k \Psi_k(1) = (-1)^k (1 - 2 - \cdots - 1) = \pm k.$$

Hence, we get

$$\frac{k}{B} = \pm \frac{2^{ka}}{\left(P_m^{(k)}\right)^k}.$$

The exponent of 2 in the left is at most the exponent of 2 in the factorization of k, so this exponent is less than $k \leq ka$. This shows that in the right, the fraction $2^{ka}/\left(P_m^{(k)}\right)^k$ cannot be reduced (so 2^{ka} cannot be coprime to $P_m^{(k)}$). This shows that $P_m^{(k)}$ is even, which is false since $P_m^{(k)} P_n^{(k)} = 2^a \pm 1$ implies that $P_m^{(k)}$ is odd.

Now, we will apply Theorem 2.1 to Γ_2. In this application, we select the following parameters:

$$(\eta_1, b_1) := (P_m^{(k)} g_k(\alpha), -1), \quad (\eta_2, b_2) := (\alpha, -n), \quad (\eta_3, b_3) := (2, a).$$

Once again, let $\mathbb{K} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. As previously, we can choose $A_2 := 2\log\varphi$ and $A_3 := k\log 2$. Now, we still need to calculate A_1. With the help of the estimates (2.6), (3.7), and the proprieties (2.1)-(2.3), we obtain

$$\begin{aligned} h(\eta_1) &\leq h(P_m^{(k)}) + h(g_k(\alpha)) \\ &< (m-1)\log\alpha + 4k\log\varphi + k\log(k+1) \\ &< 1.08 \cdot 10^{12} k^5 \log^2 k(1+\log 3n) \end{aligned}$$

for all $k \geq 2$. On the other hand, as

$$\eta_1 := P_m^{(k)} g_k(\alpha) < \frac{\alpha^{m-1}}{2} \quad \text{and} \quad \eta_1^{-1} = \frac{1}{P_m^{(k)} g_k(\alpha)} < 1,$$

then, by (3.7), we find

$$|\log\eta_1| < (m-1)\log\alpha + \log 0.5 < 1.08 \cdot 10^{12} k^5 \log^2 k(1+\log 3n).$$

This implies that

$$\max\{kh(\eta_1), |\log\eta_1|, 0.16\} < 1.08 \times 10^{12} k^6 \log^2 k(1+\log 3n) := A_1.$$

By choosing $B := 3n$, we can apply Theorem 2.1 to Γ_2 and compare the resulting inequality with (3.9) to derive

$$n\log\alpha < 1.75 \times 10^{24} k^9 \log^3 k \log^2 n,$$

where we utilized the facts $1 + \log k < 2.5\log k$ and $1 + \log(3n) < 2.6\log n$, valid for $k \geq 2$ and $n \geq 4$. Consequently, we obtain

$$\frac{n}{\log^2 n} < 2.6 \times 10^{24} k^9 \log^3 k. \tag{3.11}$$

Applying the inequality from Lemma 2.5, with $T := 2.6 \times 10^{24} k^9 \log^3 k$, and considering the inequality $56.22 + 9\log k + 3\log\log k < 88.6\log k$ for all $k \geq 2$, we obtain

$$\begin{aligned} n &< 4(2.6 \cdot 10^{24} k^9 \log^3 k)\left(\log(2.6 \cdot 10^{24} k^9 \log^3 k)\right)^2 \\ &< (1.04 \cdot 10^{25} k^9 \log^3 k)(56.22 + 9\log k + 3\log\log k)^2 \\ &< 8.2 \cdot 10^{28} k^9 \log^5 k. \end{aligned}$$

This concludes the proof of Lemma 3.1. □

3.2 The small cases $2 \leq k \leq 630$

In this subsection, we explore the values of k within the interval [2, 630]. Define

$$\Lambda_1 := \log(\Gamma_1 + 1) = a\log 2 - (n+m)\log\alpha - 2\log(g_k(\alpha)). \tag{3.12}$$

Assume that $m \geq 10$. Utilizing the estimate (3.6) and considering the fact that $\alpha > \varphi$, it follows that $|\Gamma_1| < 0.27$. By setting $d = 0.27$ in Lemma 2.6, we obtain

$$|\Lambda_1| < \frac{-\log 0.73}{0.27} \cdot |\Gamma_1| < 23.5 \cdot \alpha^{-m}.$$

Hence, we have

$$\left| a \cdot \frac{\log 2}{\log\alpha} - (n+m) - \frac{2\log(g_k(\alpha))}{\log\alpha} \right| < 48.9 \cdot \alpha^{-m}. \tag{3.13}$$

To apply Lemma 2.2 to Λ_1, we consider the parameters

$$\gamma := \frac{\log 2}{\log \alpha}, \quad \mu := 2 - \frac{2\log(g_k(\alpha))}{\log \alpha}, \quad \text{and} \quad (A, B) := (48.9, \alpha).$$

For each $k \in [3, 630]$, we determine a suitable approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ. These convergents satisfy $q_\ell > 6M_k$ and $\varepsilon = \varepsilon(k) = ||\mu q|| - M_k||\gamma q|| > 0$, where $M_k := \lfloor 2.46 \times 10^{30} k^9 \log^5 k \rfloor$. This value serves as an upper bound for a according to Lemma 3.1. Utilizing a Mathematica program, we find that q_{114} meets the conditions of Lemma 2.2. Subsequently, we apply Lemma 2.2 to inequality (3.13). A Mathematica program indicates that $\min_{3\le k\le 630}\{\varepsilon_k\} > 0.4861$. Moreover, the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$ over all $k \in [3, 630]$ is 150.457, providing an upper bound for m as per Lemma 2.2.

For the case $k = 2$, we have $\alpha = 1 + \sqrt{2}$ and $g_2(\alpha) = \sqrt{2}$. Consequently, inequality (3.13) is transformed into

$$\left|(a-1)\cdot\frac{\log 2}{\log(1+\sqrt{2})} - (n+m)\right| < 48.9\cdot\alpha^{-m}. \tag{3.14}$$

Let

$$[a_0, a_1, a_2, \ldots] = [0, 1, 3, 1, 2, 6, 1, 2, 11, 2, 2, 1, 1, 18, \ldots]$$

be the continued fraction expression of $\dfrac{\log 2}{\log(1+\sqrt{2})}$ and let p_ℓ/q_ℓ denote its convergent. It is worth noting that in this scenario, we have $a < 2.46 \cdot 10^{30} \cdot 2^9 \cdot \log^5 2 < 2.02 \cdot 10^{32}$. By utilizing Maple, we find that

$$q_{61} < 2.02 \cdot 10^{32} < q_{62}$$

and

$$a_M := \max\{a_i : i = 1, 2, \ldots, 61\} = a_{27} = 100.$$

Thus, based on the properties of continued fractions (refer to Lemma 2.3), we deduce

$$\left|(a-1)\cdot\frac{\log 2}{\log(1+\sqrt{2})} - (n+m)\right| > \frac{1}{(a_M+2)a}.$$

Combining the above inequality with (3.14) and considering the fact that $a < 2.02 \cdot 10^{32}$, it comes that

$$\alpha^m < 48.9 \cdot 102 \cdot 2.02 \cdot 10^{32}.$$

Consequently, we derive $m < 95$. Therefore, in all cases, we have $m < 151$.

Next, we fix $m < 151$ and we consider

$$\Lambda_2 := \log(\Gamma_2 + 1) = a\log 2 - n\log\alpha - \log(P_m^{(k)} g_k(\alpha)). \tag{3.15}$$

Assuming $n \ge 10$, by the estimate (3.9) and considering that $\alpha > \varphi$, we ascertain $|\Gamma_2| < 0.12$. Applying $d = 0.12$ in Lemma 2.6, we derive

$$|\Lambda_2| < \frac{-\log 0.88}{0.12} \cdot |\Gamma_2| < 15.4 \cdot \alpha^{-n}.$$

Consequently, we obtain

$$\left|a\cdot\frac{\log 2}{\log\alpha} - n - \frac{\log(P_m^{(k)} g_k(\alpha))}{\log\alpha}\right| < 32.1\cdot\alpha^{-n}. \tag{3.16}$$

To apply Lemma 2.2 to Λ_2, we take

$$\gamma := \frac{\log 2}{\log\alpha}, \quad \mu := -\frac{\log(P_m^{(k)} g_k(\alpha))}{\log\alpha}, \quad \text{and} \quad (A, B) := (32.1, \alpha).$$

For each $k \in [2, 630]$, we find a suitable approximation of γ, along with a convergent p_ℓ/q_ℓ of the continued fraction of γ. This choice ensures that $q_\ell > 6M_k$, and $\varepsilon = \varepsilon(k) = ||\mu q|| - M_k||\gamma q|| > 0$, where $M_k := \lfloor 2.46 \times 10^{30} k^9 \log^5 k \rfloor$. This value serves as an upper bound for a according to Lemma 3.1. Utilizing a Mathematica

program, we see that q_{116} satisfies the conditions of Lemma 2.2. Subsequently, we apply Lemma 2.2 to inequality (3.16). A computational analysis with a Mathematica program indicates that $\min_{2\le k\le 630, 2\le m\le 150}\{\varepsilon_k\} > 0.4992$, and the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$ over all $k \in [2, 630]$ is 159.934. This maximum value serves as an upper bound for n according to Lemma 2.2.

Finally, we used a Mathematica program to conduct a comparative analysis between $P_n^{(k)} P_m^{(k)}$ and $2^a \pm 1$ across the parameter space defined by $k+2 \le n \le 159$, $m \le n$, and $1 \le a \le 477$, while ensuring that $a < 3n$. This investigation aimed to verify that equation (1.1) exhibits no solutions within these specified ranges.

3.3 The large cases $k > 630$

In this subsection, we investigate the scenario where $k > 630$. For $k > 630$, a straightforward verification reveals that

$$m \le n < 8.2 \cdot 10^{28} k^9 \log^5 k < \varphi^{k/2}.$$

Consequently, in accordance with Lemma 2.4, we obtain

$$g_k(\alpha)\alpha^n = \frac{\varphi^{2n}}{\varphi+2}(1+\zeta_1), \quad \text{where} \quad |\zeta_1| < \frac{4}{\varphi^{k/2}} \tag{3.17}$$

and

$$g_k(\alpha)\alpha^m = \frac{\varphi^{2m}}{\varphi+2}(1+\zeta_2), \quad \text{where} \quad |\zeta_2| < \frac{4}{\varphi^{k/2}}. \tag{3.18}$$

By substituting (3.17) and (3.18) into (3.3), we derive

$$\begin{aligned}\frac{\varphi^{2n+2m}}{(\varphi+2)^2} - 2^a = {} & \frac{\varphi^{2n+2m}}{(\varphi+2)^2}(\zeta_1+\zeta_2+\zeta_1\zeta_2) + \frac{\varphi^{2n}}{\varphi+2}e_k(m) + \frac{\varphi^{2n}}{\varphi+2}e_k(m)\zeta_1 \\ & + \frac{\varphi^{2m}}{\varphi+2}e_k(n) + \frac{\varphi^{2m}}{\varphi+2}e_k(n)\zeta_2 \pm 1 - e_k(n)e_k(m).\end{aligned}$$

Hence, we obtain

$$\left|\frac{\varphi^{2n+2m}}{(\varphi+2)^2} - 2^a\right| \le \frac{24\varphi^{2n+2m}}{(\varphi+2)^2\varphi^{k/2}} + \frac{\varphi^{2n}}{2(\varphi+2)} + \frac{\varphi^{2n}}{2(\varphi+2)\varphi^{k/2}} + \frac{\varphi^{2m}}{2(\varphi+2)} + \frac{\varphi^{2m}}{2(\varphi+2)\varphi^{k/2}} + \frac{5}{4}.$$

Therefore, the above inequality, along with the conditions $n \ge m$ and $n \ge k+2$, yields

$$\begin{aligned}|\Gamma_3| &< \frac{24}{\varphi^{k/2}} + \frac{\varphi+2}{2}\left(\frac{1}{\varphi^{2m}} + \frac{1}{\varphi^{k/2+2m}} + \frac{1}{\varphi^{2n}} + \frac{1}{\varphi^{k/2+2n}}\right) \\ &< \frac{32}{\varphi^{\min\{k/2,2m\}}},\end{aligned} \tag{3.19}$$

where

$$\Gamma_3 := 1 - (\varphi+2)^2 \cdot 2^a \cdot \varphi^{-2n-2m}.$$

We have $\Gamma_3 \neq 0$, otherwise we would end up with $\varphi^{2n+2m} = (\varphi+2)\cdot 2^a$, implying that $(\varphi+2)\cdot 2^a$ is a unit in $\mathbb{Q}(\sqrt{5})$, which is impossible. Now, we can utilize Theorem 2.1 for Γ_3, considering the following parameters:

$$(\eta_1, b_1) := (2, a), \quad (\eta_2, b_2) := (\varphi+2, 2), \quad (\eta_3, b_3) := (\varphi, -(2n+2m)).$$

It is clear that $\eta_1, \eta_2, \eta_3 \in \mathbb{K} := \mathbb{Q}(\sqrt{5})$ and so $d_{\mathbb{K}} = 2$. We choose $B := 4n$ considering that $n \ge m$ and $a < 3n$. The usual absolute logarithmic heights of these numbers are as follows: $h(\eta_1) = \log 2$, $h(\eta_2) = \dfrac{\log 5}{2}$, and $h(\eta_3) = h(\varphi) = \dfrac{\log\varphi}{2}$. Consequently, we select $A_1 := 2\log 2$, $A_2 := \log 5$, and $A_3 := \log\varphi$.

Applying Theorem 2.1 to Γ_3, we obtain

$$\begin{aligned}|\Gamma_3| &> \exp\left(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 4(1+\log 2)(1+\log 4n)(2\log 2)(\log 5)(\log \varphi)\right) \\ &> \exp(-2.92 \cdot 10^{12} \log n),\end{aligned}$$

where we have used the fact that $1 + \log 4n < 2.8 \log n$ for all $n \geq 4$. By comparing the resulting inequality with (3.19), we derive

$$\min\{k/2, 2m\} < 6.1 \cdot 10^{12} \log n.$$

According to Lemma 3.1 and considering the inequality $66.58 + 9\log k + 5\log\log k < 21 \log k$, valid for all $k > 630$, we get

$$\begin{aligned}\min\{k/2, 2m\} &< 6.1 \cdot 10^{12} \log(8.2 \cdot 10^{28} k^9 \log^5 k) \\ &< 6.1 \cdot 10^{12}(66.58 + 9\log k + 5\log\log k) \\ &< 1.3 \cdot 10^{14} \log k.\end{aligned}$$

If $\min k/2, 2m = k/2$, the inequality $k < 2.6 \cdot 10^{14} \log k$ arises. Solving this inequality and applying Lemma 3.1, we deduce that

$$k < 9.6 \cdot 10^{15} \quad \text{and} \quad n < 3.9 \cdot 10^{180}. \tag{3.20}$$

If $\min\{k/2, 2m\} = 2m$, so we obtain in this case that

$$m < 6.5 \cdot 10^{13} \log k. \tag{3.21}$$

Now, let us revisit (3.8) and express it differently

$$\frac{\varphi^{2n}}{\varphi+2} - \frac{2^a}{P_m^{(k)}} = \frac{\pm 1}{P_m^{(k)}} - e_k(n) - \frac{\varphi^{2n}\zeta_1}{\varphi+2}.$$

This yields

$$\left|\frac{\varphi^{2n}}{\varphi+2} - \frac{2^a}{P_m^{(k)}}\right| \leq \frac{4\varphi^{2n}}{(\varphi+2)\varphi^{k/2}} + \frac{3}{2}.$$

Consequently, we have

$$|\Gamma_4| \leq \frac{4}{\varphi^{k/2}} + \frac{3(\varphi+2)}{2\varphi^{2n}} < \frac{4}{\varphi^{k/2}} + \frac{3(\varphi+2)}{2\varphi^{2k+2}} < \frac{9.5}{\varphi^{k/2}}, \tag{3.22}$$

where

$$\Gamma_4 := \frac{\varphi+2}{P_m^{(k)}} \cdot 2^a \cdot \varphi^{-2n} - 1. \tag{3.23}$$

If $\Gamma_4 = 0$, then $(\varphi+2)\varphi^{2n} = P_m^{(k)}/2^a$. We take the conjugate to obtain $(\varphi+2)\varphi^{2n} = (\psi+2)\psi^{2n}$, where $\psi = (1-\sqrt{2})/2$ is the conjugate of φ. This leads to a contradiction since the left-hand side is greater than 3 in absolute value, and the right-hand side is smaller than 2 in absolute value. Therefore, $\Gamma_4 \neq 0$, and we can apply Theorem 2.1 with the following parameters:

$$(\eta_1, b_1) := ((\varphi+2)/P_m^{(k)}, 1), \quad (\eta_2, b_2) := (2, a), \quad (\eta_3, b_3) := (\varphi, 2n).$$

As previously calculated, we set $A_2 := 2\log 2$ and $A_3 := \log\varphi$. Additionally, we choose $B := 4n$. Now, let's estimate $h(\eta_1)$. Utilizing the fact that $P_m^{(k)} < \varphi^{2m-2}$ and the inequality (3.21), we get

$$h(\eta_1) = h(\varphi+2) + h(P_m^{(k)}) < \log 5 + (2m-2)\log\varphi < 3.15 \cdot 10^{13} \log k.$$

Thus, we select $A_1 := 6.3 \cdot 10^{13} \log k$. Consequently, Theorem 2.1 implies that

$$|\Gamma_4| > \exp(-1.15 \cdot 10^{26} \log^2 n), \tag{3.24}$$

where we have used the fact that $1 + \log(4n) < 2.8 \log n$ for all $n \geq 4$. Combining (3.22) and (3.24), we obtain

$$k < 4.8 \cdot 10^{26} \log^2 n.$$

Applying Lemma 3.1 and considering that $66.58 + 9\log k + 5\log\log k < 21\log k$ for all $k > 630$, we deduce

$$\begin{aligned} k &< 4.8 \cdot 10^{26} \left(\log(8.2 \cdot 10^{28} k^9 \log^5 k)\right)^2 \\ &< 4.8 \cdot 10^{26} (66.58 + 9\log k + 5\log\log k)^2 \\ &< 2.2 \cdot 10^{29} \log^2 k. \end{aligned}$$

Solving this inequality and appying Lemma 3.1, we get

$$k < 1.3 \cdot 10^{33} \quad \text{and} \quad n < 2.25 \cdot 10^{336}. \tag{3.25}$$

From (3.20) and (3.25), we can deduce that the inequalities (3.25) are consistently satisfied.

The obtained bounds are exceedingly large; thus, we will proceed to reduce them. Let us put

$$\Lambda_3 := \log(\Gamma_3 + 1) = a\log 2 - (2n + 2m)\log\varphi + 2\log(\varphi + 2). \tag{3.26}$$

Suppose that $m \geq 20$. In this case, we have $|\Gamma_3| < 0.27$. By taking $d = 0.27$ in Lemma 2.6, we get

$$|\Lambda_3| < \frac{-\log 0.73}{0.27} \cdot |\Gamma_3| < 37.3 \cdot \varphi^{-\min\{k/2, 2m\}}.$$

So, we obtain

$$\left| a \cdot \frac{\log 2}{\log\varphi} - (2n + 2m) + \frac{2\log(\varphi + 2)}{\log\varphi} \right| < 77.6 \cdot \varphi^{-\min\{k/2, 2m\}}. \tag{3.27}$$

To apply Lemma 2.2 to Λ_3, we consider

$$\gamma := \frac{\log 2}{\log\varphi}, \quad \mu := \frac{2\log(\varphi + 2)}{\log\varphi}, \quad \text{and} \quad (A, B) := (77.6, \varphi).$$

Now, the goal is to refine our bounds, which are currently too large, utilizing Lemma 2.2. Let's set $M := 6.75 \times 10^{336}$, an upper bound for a as derived from Lemma 3.1. Applying Lemma 2.2 to (3.27), we aim to obtain an upper bound for k. After a computer search with Maple, we find that q_{658} satisfies the conditions of Lemma 2.2 and provides us with the information that $\min\{k/2, 2m\} < 1632$.

Case 1: $\min\{k/2, 2m\} = k/2$. In this case, we get

$$k \leq 3264. \tag{3.28}$$

Case 2: $\min\{k/2, 2m\} = 2m$. In this case, we obtain that $m \leq 816$. Fix $2 \leq m \leq 3153$ and let's put

$$\Lambda_4 := \log(1 + \Gamma_4) = a\log 2 - 2n\log\varphi + \log((\varphi + 2)/P_m^{(k)}).$$

As $k > 630$, from (3.22), it follows that $|\Gamma_4| < 0.01$. Consequently, applying Lemma 2.6, we derive

$$|\Gamma_4| < -\frac{\log(0.99)}{0.01} \cdot |\Lambda_4| < 9.6 \cdot \varphi^{-k/2}. \tag{3.29}$$

Thus, we obtain

$$\left| a \cdot \frac{\log 2}{\log\varphi} - 2n + \frac{\log((\varphi + 2)/P_m^{(k)})}{\log\varphi} \right| < 20 \cdot \varphi^{-\frac{k}{2}}. \tag{3.30}$$

We apply Lemma 2.2 to Λ_4 with the parameters

$$\gamma := \frac{\log 2}{\log\varphi}, \quad \mu := \frac{\log((\varphi + 2)/P_m^{(k)})}{\log\varphi}, \quad M := 6.75 \cdot 10^{336} \quad \text{and} \quad (A, B) := (20, \varphi).$$

Consider the case where $2m < k/2$, which implies $m < k + 1$. In this scenario, we can substitute $P_m^{(k)}$ with F_{2m-1} in our calculations. By using Maple, we find that q_{660} meets the conditions of Lemma 2.2. Consequently, we find $k < 3284$. Applying this new bound, we find $n < 1.3 \cdot 10^{65}$. Subsequently, we apply Lemma 2.2 again with the same data, but now we set M to be $3.9 \cdot 10^{65}$. With Maple, we see that q_{136} satisfies the conditions of Lemma 2.2, leading to $k < 690$. This contradicts our initial assumption that $k > 630$.

4 Conclusion

For any integer $k \geq 2$, the sequence of the k-Pell numbers $(P_n^{(k)})_{n\geq 2-k}$ is defined by the k initial values $P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \cdots = P_0^{(k)} = 0$, $P_1^{(k)} = 1$ and such that each term afterwards is is given by the linear recurrence

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)}, \quad \text{for all } n \geq 2.$$

In this paper, we study solutions of the Diophantine equation $P_n^{(k)} P_m^{(k)} = 2^a \pm 1$, where a is nonnegative integers.

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