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Research Paper

Three Dimensional Static Solutions for Simply Supported Single Layer Piezoelectric Plate

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An effort has been made in this paper to analyze a simple diaphragm supported single layer piezoelectric plate for electro-mechanical loading by using mixed semi-analytical model developed by Kant *et al.* (2007). The mathematical model consists of defining a two-point boundary value problem (BVP) governed by a set of coupled first order ordinary differential equations (ODEs). The accuracy and effectiveness of the proposed model are assessed by comparing the numerical results with available three dimensional (3D) elasticity solutions. Numerical results for different length to width ratio and for various aspect ratios have been presented for future reference.

Keywords: Semi-Analytical Method; Boundary Value Problem; Piezoelectricity; Smart Plate

Introduction

Study of smart structures consisting of composite substrate with embedded or attached piezoelectric patches has been an active area of research in recent years. The coupling effect existing between elastic and electric fields in piezoelectric material is used in various engineering applications. The direct piezo-effect is used in sensors in electromechanical transducers to infer the deformation in the material. The converse piezo-effect is used in actuators for controlling deformations and vibrations by application of appropriate electric potential difference. The simultaneous use of sensing and actuating functions of the piezoelectric materials is seen in the unmanned space structures and aircrafts for which, a very accurate analysis is essential.

The mechanics of sensing and actuating of smart materials plays very significant role in the analysis of controlled structures. A large number of theories, analytical solutions and numerical models are reported for the analysis of such structures in the technical literature. Ray *et al.* (1992, 1993) have presented three dimensional (3D) exact solutions for electro-mechanical loading, for a single piezoelectric plate in cylindrical bending and for intelligent plate under cylindrical bending. Heyliger (1994a, 1994b and 1997) has obtained exact solution for an unsymmetrical cross ply composite laminate attached with single and double layers of piezoelectric material. Extensive work has been done on response of piezoelectric material to thermal loading as well. Noteworthy contributions have come from Dube *et al.* (1996), Kapuria *et al.* (1997a, 1997b), Zhang *et al.* (2002), Sakthivel and

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Arockiarajan (2010), Moghadam *et al.*(2013), amongst many. However, obtaining analytical solutions for laminates with a large number of layers becomes tedious. Hence researchers have focused their attention on two dimensional (2D) analytical models viz.classical plate theory (CPT) (Tiersten 1969; Dimitridis *et al.* 1991; Crawley and Lazarus 1991; Wang and Rogers 1991; Lee and Moon 1989; Lee 1990), first order shear deformation theory (FOST) (Chandrashekhara and Agarwal 1993; Jonnalagadda *et al.* 1994; Detwiler *et al.* 1995; Huang and Wu 1996; Bisegna *et al.* 2001; Vel and Batra 2001; Wu *et al.* 2004) and higher order shear deformation theory (HOST) (Ray *et al.* 1994; Kim *et al.* 1998).

The motivation for the present work has come through the need of developing an accurate and computationally inexpensive approach of analysis. Thus, the semi-analytical mathematical model developed by Kant *et al.* (2007) is reformulated for displacement and stress analysis of a piezoelectric plate under mechanical and electrical load. A piezoelectric plate is formulated as a two-point BVP governed by a set of first order ODEs,

$$\frac{d}{dz}y(z) = A(z)y(z) + p(z) \tag{1}$$

in the interval $-h/2 \le z \le h/2$ with any half of the dependent variables prescribed at the edges $z = \pm h/2$. Here, $y(z)$ is an n-dimensional vector of primary variables, $A(z)$ is an n x n elastic and electric coefficients matrix and $p(z)$ is an n-dimensional vector of non-homogeneous (loading) terms. It is clearly seen that mixed and/or non-homogeneous boundary conditions are easily admitted in this formulation.

Formulation

An all-round supported piezoelectric plate is considered. The dimensions of the plate are; $a \times b$, respectively in x and y directions and thickness h is in z direction (Fig. 1). The plate is assumed to be

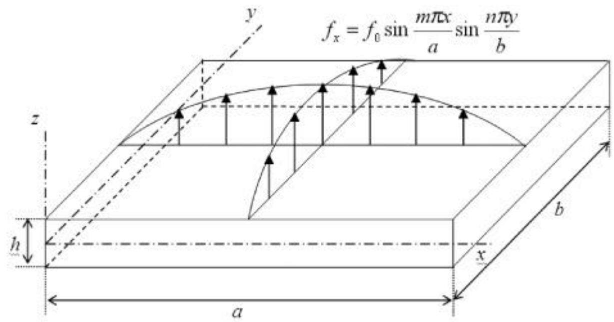

Fig. 1: Simply supported piezoelectric plate subjected to electro-mechanical load

subjected to transverse mechanical and electrical loading at the top surface. Longitudinal edges at $x = 0, a$ and $y = 0, b$ are assumed to be grounded to zero potential.

The coupled elastic and electrical field equations in piezoelectric medium given by Tirsten (1969) are;

$$\{\sigma\} = [C]\{\varepsilon\} - [\ e\ ]\{\ E\ \} \tag{2}$$

$$\{D\} = [\ e\ ]^T\{\ \varepsilon\ \} + [\ g\ ]\{\ E\ \} \tag{3}$$

In these equations, $\{\sigma\}$ is the stress vector, $\{\varepsilon\}$ is the strain vector, $\{E\}$ is the electric field intensity vector and $\{D\}$ is the electric displacement vector. Piezoelectric stress coefficients matrix $[e]$ (Cady, 1946) and dielectric constant matrix $[g]$ (Tzau and Pandita, 1987) for a commonly used material like PVDF belonging to Rhombic group, Class 7 are given by;

$$[e] = \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & e_{33} \\ 0 & 0 & 0 \\ e_{15} & 0 & 0 \\ 0 & e_{24} & 0 \end{bmatrix} \text{ and } [g] = \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{bmatrix} \tag{4}$$

Equations (2) may be expanded as;

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix}$$

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}-\begin{bmatrix}0 & 0 & e_{31}\\ 0 & 0 & e_{32}\\ 0 & 0 & e_{33}\\ 0 & 0 & 0\\ e_{15} & 0 & 0\\ 0 & e_{24} & 0\end{bmatrix}\begin{Bmatrix}-\partial\phi/\partial x\\ -\partial\phi/\partial y\\ -\partial\phi/\partial z\end{Bmatrix} \tag{5}$$

$$\begin{Bmatrix}D_x\\ D_y\\ D_z\end{Bmatrix}=\begin{bmatrix}0 & 0 & 0 & 0 & e_{15} & 0\\ 0 & 0 & 0 & 0 & 0 & e_{24}\\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0\end{bmatrix}$$

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{xz}\end{Bmatrix}-\begin{bmatrix}g_{11} & 0 & 0\\ 0 & g_{22} & 0\\ 0 & 0 & g_{33}\end{bmatrix}\begin{Bmatrix}-\partial\phi/\partial x\\ -\partial\phi/\partial y\\ -\partial\phi/\partial z\end{Bmatrix} \tag{6}$$

in which, the material stiffness coefficients C_{ij} are given by;

$$C_{11}=\frac{E_1\left(1-\nu_{23}\nu_{32}\right)}{\Delta};\ C_{12}=\frac{E_1\left(\nu_{21}+\nu_{31}\nu_{23}\right)}{\Delta}$$

$$C_{13}=\frac{E_1\left(\nu_{31}+\nu_{21}\nu_{32}\right)}{\Delta};\ C_{22}=\frac{E_2\left(1-\nu_{13}\nu_{31}\right)}{\Delta}$$

$$C_{23}=\frac{E_2\left(\nu_{32}+\nu_{12}\nu_{31}\right)}{\Delta};\ C_{33}=\frac{E_3\left(1-\nu_{12}\nu_{21}\right)}{\Delta}$$

$$C_{44}=G_{12};\ C_{55}=G_{13};\ C_{66}=G_{23} \tag{7}$$

where $\Delta=\left(1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{12}\nu_{23}\nu_{31}\right)$

Elasticity equilibrium equationsin 3D domain are;

$$\begin{aligned}&\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+B_x=0\\ &\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+B_y=0\\ &\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+B_z=0\end{aligned} \tag{8}$$

Where B_x, B_y, B_z are the body force intensities in *x*, *y*, and *z* directions respectively.

3D strain-displacement relations are;

$$\begin{aligned}&\varepsilon_x=\frac{\partial u}{\partial x} &&\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\\ &\varepsilon_y=\frac{\partial v}{\partial y} &&\gamma_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\\ &\varepsilon_z=\frac{\partial w}{\partial z} &&\gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\end{aligned} \tag{9}$$

And, 3D charge equilibrium equation (Maxwell, 1865) is;

$$\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}=0 \tag{10}$$

Equations (5), (6), (8), (9) and (10) have a total of 19 unknowns; *u*, *v*, *w*, ε_x, ε_y, ε_z, γ_{xy}, γ_{yz}, γ_{xz}, σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{xz}, D_x, D_y, D_z and ϕ in 19 equations. After some algebraic manipulation of the above sets of equations, a set of partial differential equations (PDEs) involving only eight variables called primary variables; *u*, *v*, *w*, s_z, τ_{yz}, τ_{xz}, D_z and ϕ is obtained as;

$$\frac{\partial u}{\partial z}=\left(\frac{1}{C_{55}}\right)\tau_{xz}-\left(\frac{e_{15}}{C_{55}}\right)\frac{\partial\phi}{\partial x}-\frac{\partial w}{\partial x}$$

$$\frac{\partial v}{\partial z}=\left(\frac{1}{C_{66}}\right)\tau_{yz}-\left(\frac{e_{24}}{C_{66}}\right)\frac{\partial\phi}{\partial y}-\frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial z}=\left(\frac{g_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right)\sigma_z+\left(\frac{e_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right)D_z-\left(\frac{g_{33}C_{31}+e_{33}e_{31}}{C_{33}g_{33}+e_{33}e_{33}}\right)\frac{\partial u}{\partial x}-\left(\frac{g_{33}C_{23}+e_{33}e_{32}}{C_{33}g_{33}+e_{33}e_{33}}\right)\frac{\partial v}{\partial x}$$

$$\frac{\partial \phi}{\partial z}=\left(\frac{e_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right)\sigma_z-\left(\frac{C_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right)D_z+\left(\frac{C_{33}e_{31}-e_{33}C_{31}}{C_{33}g_{33}+e_{33}e_{33}}\right)\frac{\partial u}{\partial x}+\left(\frac{C_{33}e_{32}-e_{33}C_{32}}{C_{33}g_{33}+e_{33}e_{33}}\right)\frac{\partial v}{\partial x}$$

$$\begin{aligned}\frac{\partial \tau_{xz}}{\partial z}=&-\left[C_{11}-\frac{\left(C_{13}g_{33}C_{31}+e_{33}C_{13}e_{31}\right)}{\left(C_{33}g_{33}+e_{33}e_{33}\right)}+\frac{\left(e_{31}C_{33}e_{31}-e_{31}C_{31}e_{33}\right)}{\left(C_{33}g_{33}+e_{33}e_{33}\right)}\right]\frac{\partial^2 u}{\partial x^2}\\&-\left[C_{12}-\frac{\left(C_{13}g_{33}C_{32}+e_{32}C_{13}e_{33}\right)}{\left(C_{33}g_{33}+e_{33}e_{33}\right)}+\frac{\left(e_{31}C_{33}e_{32}-e_{31}C_{32}e_{33}\right)}{\left(C_{33}g_{33}+e_{33}e_{33}\right)}+C_{44}\right]\frac{\partial^2 v}{\partial x\partial y}\\&-C_{44}\frac{\partial^2 u}{\partial y^2}-\left[\frac{C_{13}g_{33}+e_{31}e_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right]\frac{\partial \sigma_Z}{\partial x}-\left[\frac{C_{13}e_{33}-e_{31}C_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right]\frac{\partial D_z}{\partial x}+B_x\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial \tau_{yz}}{\partial z}=&-\left[C_{12}-\frac{\left(C_{32}g_{33}C_{13}+e_{33}C_{23}e_{31}\right)}{\left(C_{33}g_{33}+e_{33}e_{33}\right)}+\frac{\left(e_{32}C_{33}e_{31}-e_{32}C_{31}e_{33}\right)}{\left(C_{33}g_{33}+e_{33}e_{33}\right)}+C_{44}\right]\frac{\partial^2 u}{\partial x\partial y}\\&-\left[C_{22}-\frac{\left(C_{23}g_{33}C_{32}+e_{32}C_{23}e_{33}\right)}{\left(C_{33}g_{33}+e_{33}e_{33}\right)}+\frac{\left(e_{32}C_{33}e_{32}-e_{32}C_{32}e_{33}\right)}{\left(C_{33}g_{33}+e_{33}e_{33}\right)}\right]\frac{\partial^2 v}{\partial y^2}\\&-C_{44}\frac{\partial^2 v}{\partial x^2}-\left[\frac{C_{23}g_{33}+e_{32}e_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right]\frac{\partial \sigma_Z}{\partial y}-\left[\frac{C_{23}e_{33}-e_{32}C_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right]\frac{\partial D_z}{\partial y}+B_y\end{aligned}$$

$$\frac{\partial \sigma_z}{\partial z}=-\frac{\partial \tau_{xz}}{\partial x}-\frac{\partial \tau_{yz}}{\partial y}+B_z$$

$$\frac{\partial D_z}{\partial z}=-\left(\frac{e_{15}}{C_{55}}\right)\frac{\partial \tau_{xz}}{\partial x}-\left(\frac{e_{24}}{C_{66}}\right)\frac{\partial \tau_{yz}}{\partial y}+\left(\frac{e_{15}e_{15}}{C_{55}}+g_{11}\right)\frac{\partial^2 \phi}{\partial x^2}+\left(\frac{e_{24}e_{24}}{C_{66}}+g_{22}\right)\frac{\partial^2 \phi}{\partial y^2}$$

The displacement field and stress field over the entire domain, satisfying the boundary conditions (BCs) at $x = 0, a$ and $y = 0, b$ are expressed in the form of double Fourier series as;

$$u(x,y,z)=\sum_m\sum_n u_{mn}(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$v(x,y,z)=\sum_m\sum_n v_{mn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$w(x,y,z)=\sum_m\sum_n w_{mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\tau_{xz}(x,y,z)=\sum_m\sum_n \tau_{xzmn}(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\tau_{yz}(x,y,z)=\sum_m\sum_n \tau_{yzmn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$\sigma_z(x,y,z)=\sum_m\sum_n \sigma_{zmn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$D_z(x,y,z)=\sum_m\sum_n D_{zmn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (12)$$

The transverse mechanical load and applied electrostatic potential, assuming the longitudinal edges to be grounded with zero potential, are expressed as;

$$p(x,y,z)=\sum_m\sum_n p_{0mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\phi(x,y,z)=\sum_m\sum_n \phi_{0mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (13)$$

Substituting Equations (12) and (13) and its derivatives into Equations (11), a set of first-order ordinary differential equations (ODEs) involving primary dependent variables u, v, w, σ_z, τ_{yz}, τ_{xz}, D_z and ϕ is obtained as;

$$\frac{du_{mn}(z)}{dz}=-\left(\frac{m\pi}{a}\right)w_{mn}(z)+\left(\frac{1}{C_{55}}\right)\tau_{xzmn}(z)-\left(\frac{e_{15}}{C_{55}}\right)\left(\frac{m\pi}{a}\right)\phi_{mn}(z)$$

$$\frac{dv_{mn}(z)}{dz}=-\left(\frac{n\pi}{b}\right)w_{mn}(z)+\left(\frac{1}{C_{66}}\right)\tau_{yzmn}(z)-\left(\frac{e_{24}}{C_{66}}\right)\left(\frac{n\pi}{b}\right)\phi_{mn}(z)$$

$$\frac{dw_{mn}(z)}{dz}=\left(\frac{g_{33}C_{31}+e_{33}e_{31}}{C_{33}g_{33}+e_{33}e_{33}}\right)\left(\frac{m\pi}{a}\right)u_{mn}(z)+\left(\frac{g_{33}C_{32}+e_{33}e_{32}}{C_{33}g_{33}+e_{33}e_{33}}\right)\left(\frac{n\pi}{b}\right)v_{mn}(z)$$
$$+\left(\frac{g_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right)\sigma_{zmn}(z)+\left(\frac{e_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right)D_{zmn}(z)$$

$$\frac{d\phi_{mn}(z)}{dz}=-\left(\frac{C_{33}e_{31}-e_{33}C_{31}}{C_{33}g_{33}+e_{33}e_{33}}\right)\left(\frac{m\pi}{a}\right)u_{mn}(z)-\left(\frac{C_{33}e_{32}-e_{33}C_{32}}{C_{33}g_{33}+e_{33}e_{33}}\right)\left(\frac{n\pi}{b}\right)v_{mn}(z)$$
$$+\left(\frac{e_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right)\sigma_{zmn}(z)-\left(\frac{C_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right)D_{zmn}(z)$$

$$\frac{d\tau_{xzmn}(z)}{dz}=\left[\left(C_{11}-\frac{(C_{13}g_{33}C_{31}+e_{33}C_{13}e_{31})}{(C_{33}g_{33}+e_{33}e_{33})}+\frac{(e_{31}C_{33}e_{31}-e_{31}C_{31}e_{33})}{(C_{33}g_{33}+e_{33}e_{33})}\right)\left(\frac{m^2\pi^2}{a^2}\right)-C_{44}\left(\frac{n^2\pi^2}{b^2}\right)\right]u_{mn}(z)$$
$$-\left[C_{12}-\frac{(C_{13}g_{33}C_{32}+e_{32}C_{13}e_{33})}{(C_{33}g_{33}+e_{33}e_{33})}+\frac{(e_{31}C_{33}e_{32}-e_{31}C_{32}e_{33})}{(C_{33}g_{33}+e_{33}e_{33})}+C_{44}\right]\left(\frac{mn\pi^2}{ab}\right)v_{mn}(z)$$
$$-\left(\frac{C_{13}g_{33}+e_{31}e_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right)\left(\frac{m\pi}{a}\right)\sigma_{zmn}(z)-\left(\frac{C_{13}e_{33}-e_{31}C_{33}}{C_{33}g_{33}+e_{33}e_{33}}\right)\frac{m\pi}{a}D_{zm}(z)+B_x$$

$$\frac{d\tau_{yzmn}(z)}{dz} = \left[\left(C_{22} - \frac{(C_{23}g_{33}C_{32} + e_{33}C_{32}e_{32})}{(C_{33}g_{33} + e_{33}e_{33})} + \frac{(e_{32}C_{33}e_{32} - e_{32}C_{32}e_{33})}{(C_{33}g_{33} + e_{33}e_{33})}\right)\left(\frac{n^2\pi^2}{b^2}\right) - C_{44}\left(\frac{m^2\pi^2}{a^2}\right)\right]v_{mn}(z)$$
$$-\left[C_{12} - \frac{(C_{13}g_{33}C_{32} + e_{31}C_{23}e_{33})}{(C_{33}g_{33} + e_{33}e_{33})} + \frac{(e_{32}C_{33}e_{31} - e_{32}C_{13}e_{33})}{(C_{33}g_{33} + e_{33}e_{33})} + C_{44}\right]\left(\frac{mn\pi^2}{ab}\right)u_{mn}(z)$$
$$-\left(\frac{C_{23}g_{33} + e_{32}e_{33}}{C_{33}g_{33} + e_{33}e_{33}}\right)\left(\frac{n\pi}{b}\right)\sigma_{zmn}(z) - \left(\frac{C_{23}e_{33} - e_{32}C_{33}}{C_{33}g_{33} + e_{33}e_{33}}\right)\left(\frac{n\pi}{b}\right)D_{zmn}(z) + B_y$$

$$\frac{d\sigma_{zmn}(z)}{dz} = \left(\frac{m\pi}{a}\right)\tau_{xzmn}(z) + \left(\frac{n\pi}{b}\right)\tau_{yzmn}(z) + B_z$$

$$\frac{dD_{zmn}(z)}{dz} = \left(\frac{e_{15}}{C_{55}}\right)\left(\frac{m\pi}{a}\right)\tau_{xzmn}(z) + \left(\frac{e_{24}}{C_{66}}\right)\left(\frac{n\pi}{b}\right)\tau_{yzmn}(z)$$
$$-\left[\left(\frac{e_{15}e_{15}}{C_{55}} + g_{11}\right)\left(\frac{m^2\pi^2}{a^2}\right) + \left(\frac{e_{24}e_{24}}{C_{66}} + g_{22}\right)\left(\frac{m^2\pi^2}{a^2}\right)\right]\phi_{mn}(z) \tag{14}$$

Equations (14) represent the governing two-point BVP in the domain $-h/2 \leq z \leq h/2$, with stress components known at the top and bottom surfaces of the plate. Solutions of above equations are obtained using numerical integration in thickness direction. The approach to solving BVPs in equations (14) is by first transforming these into a set of initial value problems (IVPs) - one non-homogeneous and *n/2* homogeneous, then a linear combination of one non-homogeneous and *n/2* homogeneous solutions is obtained so as to satisfy the BCs at $z = \pm h/2$, leading to a system of *n/2* linear algebraic equations. The *n/2* unknowns X_1, X_2, X_3 (Table 1) at $z = -h/2$ are then determined. A final numerical integration gives the desired values. Availability of efficient and accurate ODE numerical integrators for IVPs helps in computing reliable values of the primary variables through the thickness.

Secondary variables may be expressed in terms of primary variables as;

$$\sigma_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{-C_{11}\left(\frac{m\pi}{a}\right)u_{mn}(z) - C_{12}\left(\frac{n\pi}{b}\right)v_{mn}(z) + C_{13}\frac{dw_{mn}(z)}{dz} + e_{31}\frac{d\phi_{mn}(z)}{dz}\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
$$\sigma_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{-C_{12}\left(\frac{m\pi}{a}\right)u_{mn}(z) - C_{22}\left(\frac{n\pi}{b}\right)v_{mn}(z) + C_{23}\frac{dw_{mn}(z)}{dz} + e_{32}\frac{d\phi_{mn}(z)}{dz}\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
$$\tau_{xy} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{C_{44}\left(\frac{n\pi}{b}\right)u_{mn}(z) + C_{44}\left(\frac{m\pi}{a}\right)v_{mn}(z)\right\}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$
$$D_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{e_{15}\frac{du_{mn}(z)}{dz} + e_{15}\left(\frac{m\pi}{a}\right)w_{mn}(z) - g_{11}\left(\frac{m\pi}{a}\right)\phi_{mn}(z)\right\}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{15}$$
$$D_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{e_{24}\frac{dv_{mn}(z)}{dz} + e_{24}\left(\frac{n\pi}{b}\right)w_{mn}(z) - g_{22}\left(\frac{n\pi}{b}\right)\phi_{mn}(z)\right\}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

Numerical Investigation

For the validation of the semi-analytical formulation, numerical investigation is carried out on a simple diaphragm supported, single layer, square PVDF plate with aspect ratio *a/h* = 10. Material properties are depicted in Table 2. Two cases of loading are considered as below;

1. Plate is subjected to mechanical load $p(x,y) = p_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$ with p_0 = 1 at the top surface with no electric potential applied at top and bottom faces. Such a plate is referred to as 'sensory plate'.

2. Plate is subjected to electric potential $\phi(x,y) = \phi_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$ with ϕ_0 = 1 at the top surface with no electric load at the bottom face and top and bottom faces are traction free. Longitudinal edges of the plate are held at zero potential. Such a plate is referred to as 'actuating plate'.

The results obtained in both the cases are compared with the exact solutions given by Heyliger *et al.* (1994b). The through-thickness variation in displacements and stresses are shown in Figs. 2 to 5 for sensory plate and in Figs. 6 to 9 for actuating plate. Also, the numerical values of these quantities at salient points are given in Table 3. The solutions obtained from semi-analytical formulation are in good agreement with the exact solutions, which shows the accuracy of the present development.

Additional parametric investigation has been carried out to study the effect of various parameters on the transverse deflection *w* of the mid-plane of the plate. First, *a/b* ratio of the plate is varied keeping mechanical and electrical loading intensities constant. It is observed that in both sensory as well as actuating plate, *w* initially increases rapidly (Fig. 10) with increase in *a/b* and around *a/b* = 4, converges to a constant value. There onwards, increase in *a/b* ratio has no effect on transverse deflection *w*.

Further, for a square sensory plate, aspect ratio *s* = *a/h* is continuously increased and variation of *w* under constant mechanical load is observed (Fig.

Table 1: Transformation of BVP into IVP

Integration	Starting edge; $z = -h/2$								Final edge; $z = +h/2$								Load term
	u	v	w	ϕ	τ_{xz}	τ_{yz}	σ_z	D_z	u	v	w	ϕ	τ_{xz}	τ_{yz}	σ_z	D_z	
1	0 (assumed)	0 (assumed)	0 (assumed)	0 (known)	0 (known)	0 (known)	0 (known)	0 (assumed)	Y_{11}	Y_{21}	Y_{31}	Y_{41}	Y_{51}	Y_{61}	Y_{71}	Y_{81}	Include
2	1 (unity)	0 (assumed)	0 (assumed)	0	0	0	0	0 (assumed)	Y_{12}	Y_{22}	Y_{32}	Y_{42}	Y_{52}	Y_{62}	Y_{72}	Y_{82}	Exclude
3	0 (assumed)	1 (unity)	0 (assumed)	0	0	0	0	0 (assumed)	Y_{13}	Y_{23}	Y_{33}	Y_{43}	Y_{53}	Y_{63}	Y_{73}	Y_{83}	Exclude
4	0 (assumed)	0 (assumed)	1 (unity)	0	0	0	0	0 (assumed)	Y_{14}	Y_{24}	Y_{34}	Y_{44}	Y_{54}	Y_{64}	Y_{74}	Y_{84}	Exclude
5	0 (assumed)	0 (assumed)	0 (assumed)	0	0	0	0	1 (unity)	Y_{15}	Y_{25}	Y_{35}	Y_{45}	Y_{55}	Y_{65}	Y_{75}	Y_{85}	Exclude
Final	X_1	X_2	X_3	Known	Known	Known	Known	X_4	u_T	v_T	w_T	$f(x, y)$	0	0	$p(x, y)$	Dz_T	Include

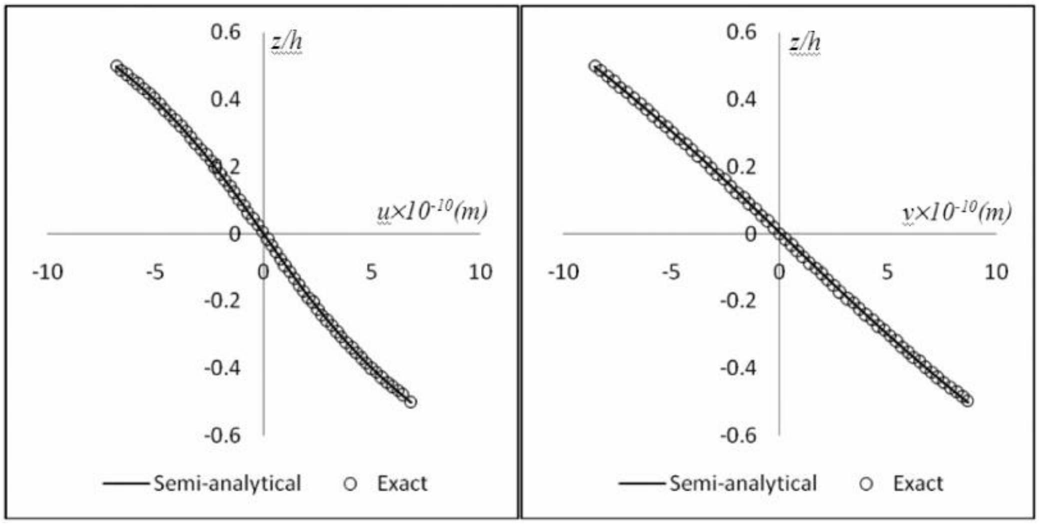

Fig. 2: Through thickness variation in a PVDF sensory plate in (*a*) in-plane displacement *u*, (*b*) in-plane displacement *v*

Table 2: Properties of PVDF (Heylinger *et al*. 1994)

E_1	237 *GPa*
E_2	23.2*GPa*
E_3	10.5*GPa*
G_{44}	2.15*GPa*
G_{55}	4.4*GPa*
G_{66}	6.43*GPa*
ν_{12}	0.154
ν_{13}	0.178
ν_{23}	0.177
e_{24}	–0.01 C/m^2
e_{31}	–0.13 C/m^2
e_{32}	–0.14 C/m^2
e_{33}	–0.28 C/m^2
$\varepsilon_{11}/\varepsilon_0$	12.5
$\varepsilon_{22}/\varepsilon_0$	11.98
$\varepsilon_{33}/\varepsilon_0$	11.98

11(*a*)). It is seen that for thick and moderately thick sensory plates, transverse deflection increases slowly and non-linearly. However, for thin and very thin plates i.e. beyond *s* = 80, transverse deflection increases rapidly and linearly. Transverse displacement (*w*) for the actuating plate under constant electric potential initially increases rapidly in thick plates as shown in Fig. 11(b) while for moderately thick and thin actuating plates it remains almost constant.

Parametric studies have been performed to study the effect of gradual incremental mechanical loading on transverse deflection (*w*) for sensory plate for various aspect ratios (*a/h*) and reported in Fig. 12(*a*). The effect of gradual incremental electric load on transverse deflection (*w*) for actuating plate for various aspect ratios (*a/h*) has been studied and depicted in Fig. 12(*b*). These figures show that the transverse deflection (*w*) varies linearly both in sensory and actuating plate with mechanical as well as electric load for all aspect ratios.

Concluding Remarks

A simple and efficient semi-analytical formulation for the analysis of simply supported single layer piezoelectric plate has been presented in this paper. Formulation consists of a two-point BVP governed by a set of linear coupled first-order ODEs with the assumption of trigonometric variation of all variables along in-plane direction. The formulation is free from any simplifying assumptions in the thickness direction.The stresses and displacements are found simultaneously and with same degree of accuracy, which is a unique feature of the present model. The results obtained by present formulation are in good agreement with 3D elasticity solutions given by Heylinger *et al.* (1994). Also, additional parametric study for different length to width ratios, for various aspect ratios and for variable mechanical and electric loading conditions have been presented, which could be useful forfuture reference.

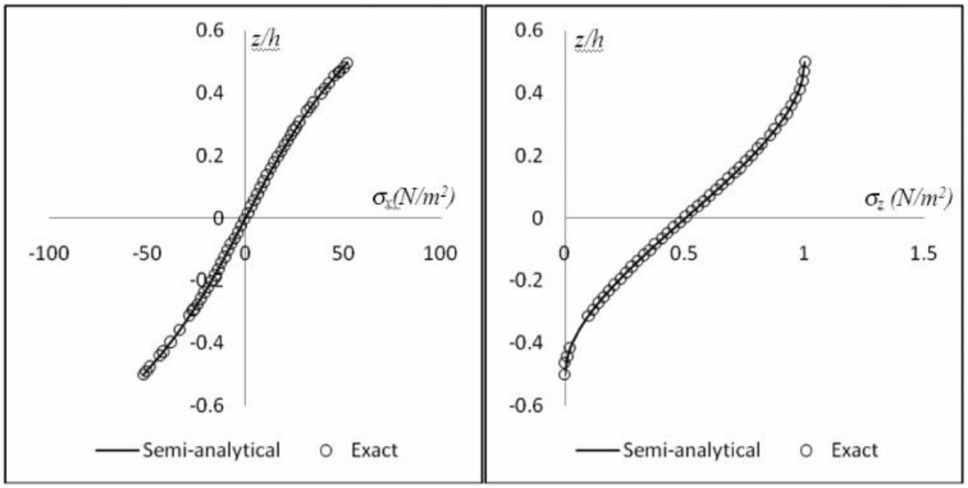

Fig. 3: Through thickness variation in a PVDF sensory plate in (*a*) in-plane normal stress σ_x, (*b*) transverse normal stress σ_z

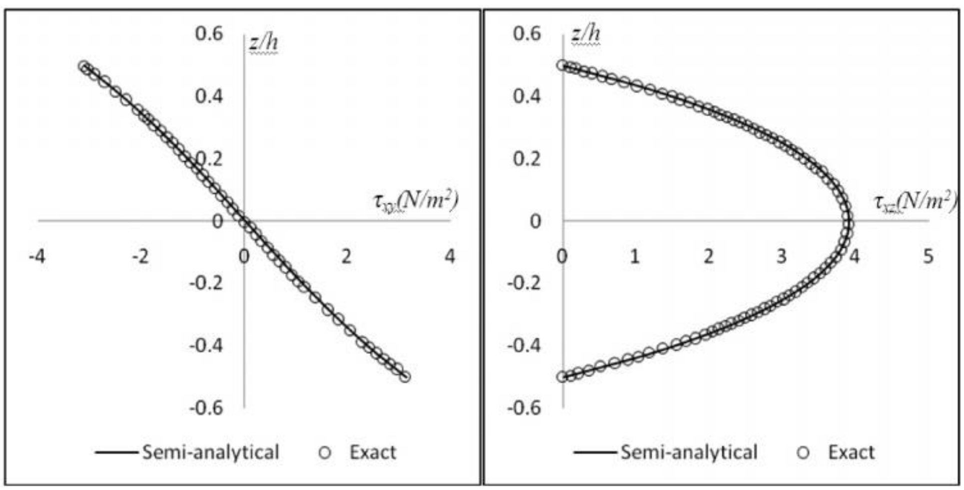

Fig. 4: Through thickness variation in a PVDF sensory plate in (*a*) in-plane shear stress τ_{xy}, (*b*) transverse shear stress τ_{xz}

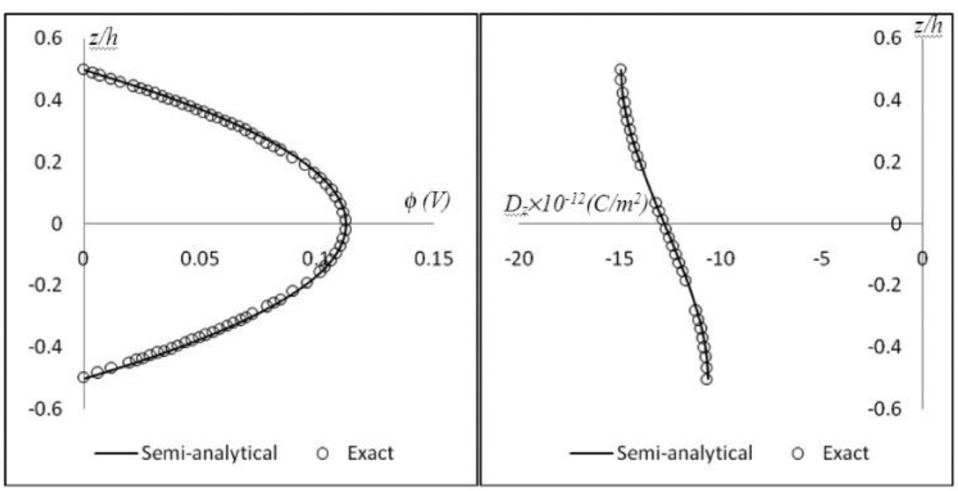

Fig. 5: Through thickness variation in a PVDF sensory plate in (*a*) induced electric potential ϕ, (*b*) transverse electric displacement D_z

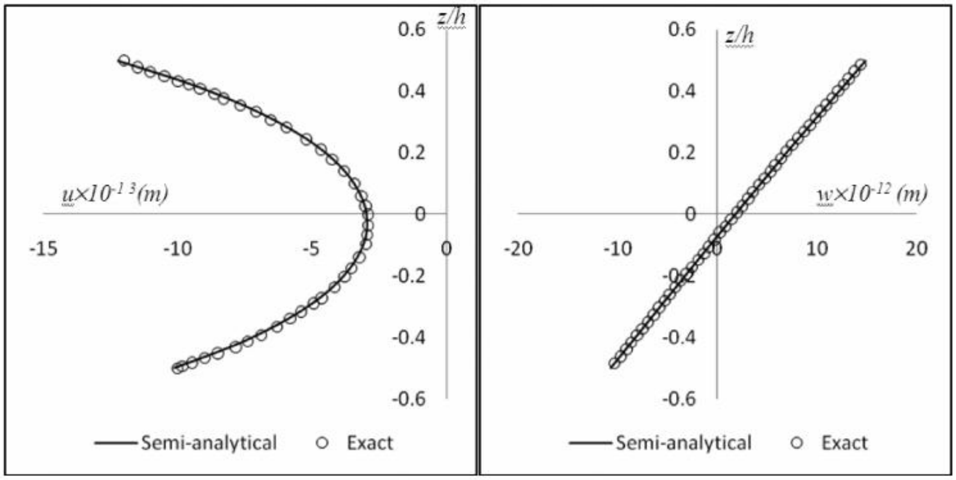

Fig. 6: Through thickness variation in a PVDF actuating plate in (*a*) in-plane displacement *u*, (*b*) transverse displacement *w*

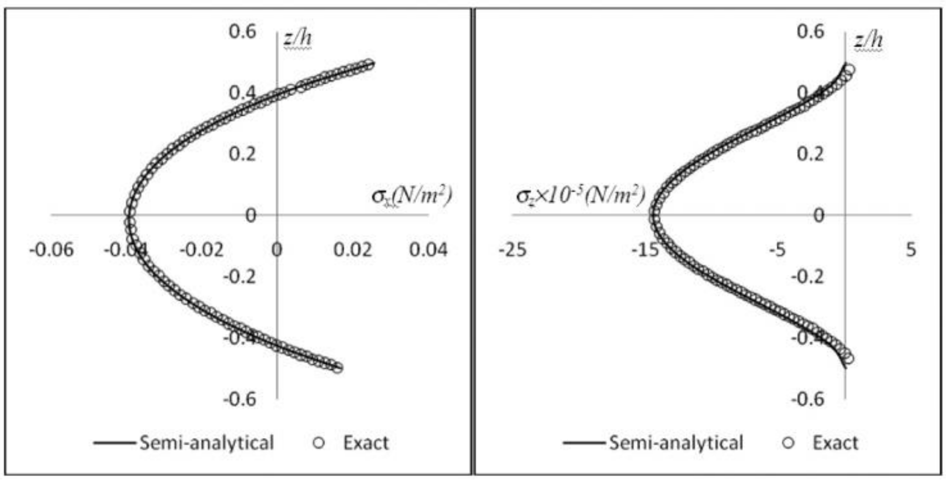

Fig. 7: Through thickness variation in a PVDF actuating plate in (*a*) in-plane normal stress σ_x, (*b*) transverse normal stress σ_z

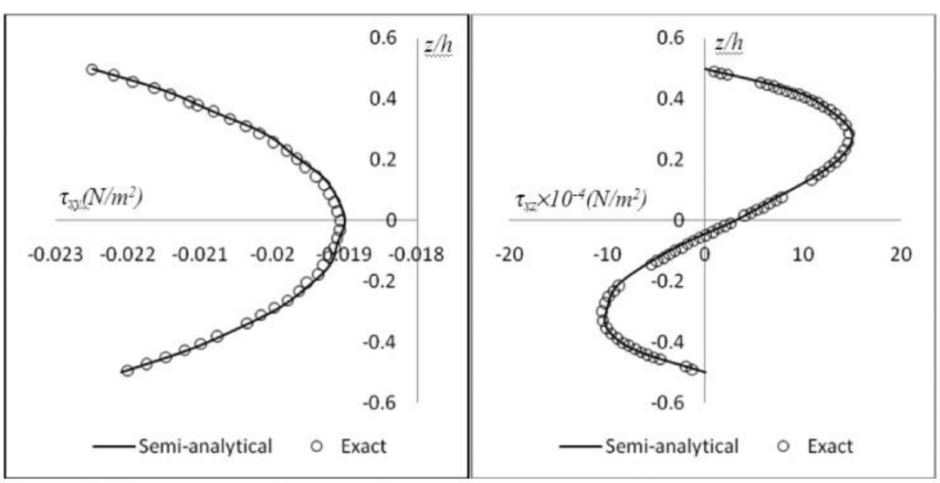

Fig. 8: Through thickness variation in a PVDF actuating plate in (*a*) in-plane shear stress τ_{xy}, (*b*) transverse shear stress τ_{xz}

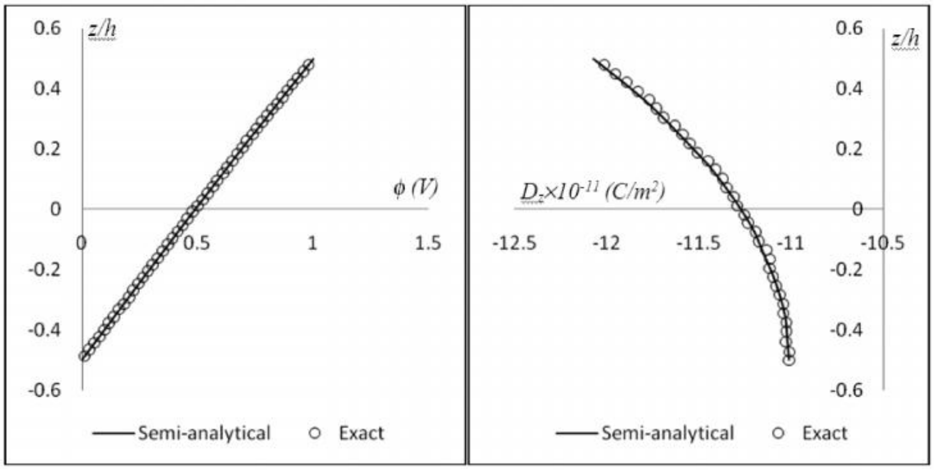

Fig. 9: Through thickness variation in a PVDF actuating plate in (*a*) applied electric potential ϕ, (*b*) transverse electric displacement D_z

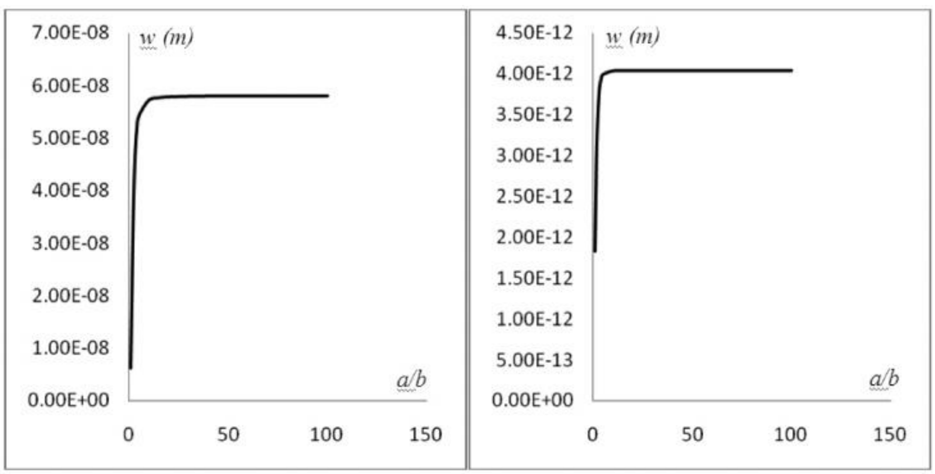

Fig. 10: Variation in mid-plane transverse deflection *w* for various *a/b* ratios of (*a*) PVDF sensory plate, (*b*) PVDF actuating plate

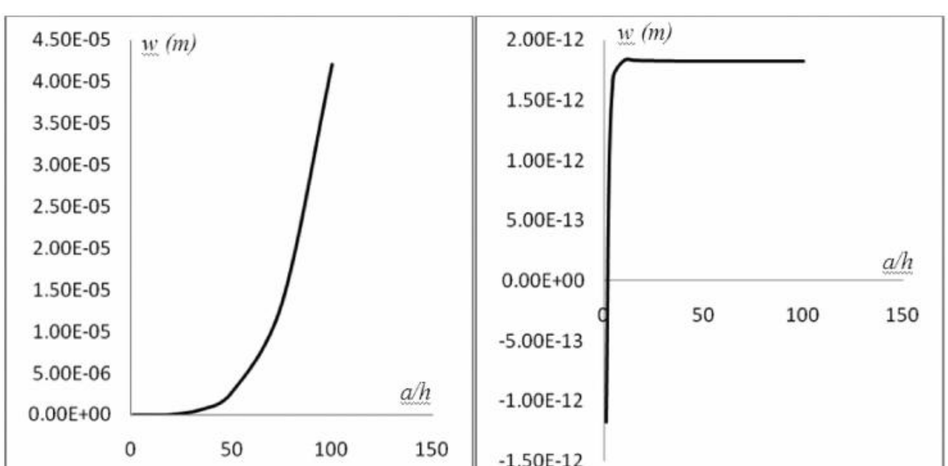

Fig. 11: Variation in mid-plane transverse deflection *w* for various *a/h* ratios for (*a*) PVDF sensory plate, (*b*) PVDF actuating plate

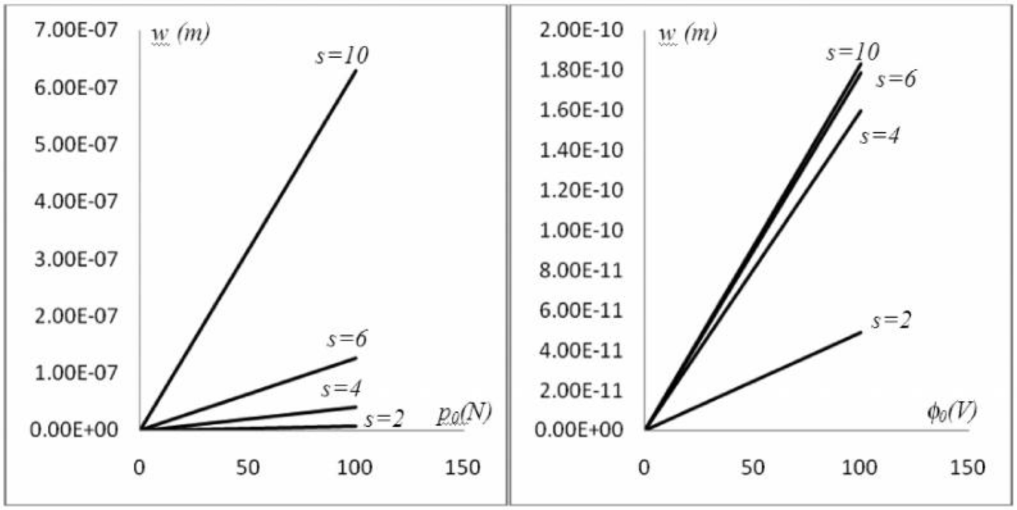

Fig. 12: Variation in transverse deflection *w*in PVDF square plate for various aspect ratios due to variation in (a) applied mechanical load intensity, (b) applied electric potential intensity

Table 3: Displacements and Stresses in Simply Supported Square PVDF Plate

Quantity	Position along thickness	Sensory plate	Actuating plate
u(m)	–0.5*h*	6.839E-10	–1.015E-12
	0	0.015E-10	–2.99E-13
	0.5*h*	–6.843E-10	–1.224E-12
v(m)	–0.5*h*	8.725E-10	–9.925E-12
	0	0.098E-10	–9.121E-12
	0.5*h*	–8.564E-10	–9.908E-12
w(m)	–0.5*h*	62.48E-10	–1.066E-11
	0	62.82E-10	0.0183E-10
	0.5*h*	62.95E-10	0.1493E-10
ϕ *(V)*	0	0.1126	0.4884
σ_x (N/m^2)	–0.5*h*	–52.0549	0.0167
	0	–0.0196	–0.0393
	0.5*h*	52.2683	0.0256
$\sigma_y(N/m^2)$	–0.5*h*	–7.185	–0.0133
	0	0.0162	–0.0221
	0.5*h*	7.2475	–0.0215
$\sigma_z(N/m^2)$	0	0.4997	–0.0001443
$\tau_{xy}(N/m^2)$	–0.5*h*	3.144	–0.0221
	0	0.0229	–0.019
	0.5*h*	–3.1123	–0.0225
$\tau_{xz}(N/m^2)$	0	3.9034	0.0003
$\tau_{yz}(N/m^2)$	0	0.7782	–0.0003056
$D_z(C/m^2)$	–0.5*h*	–1.066E-11	–1.101E-10
	0	–1.28E-11	–1.127E-10
	0.5*h*	–1.493E-11	–1.207E-10

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