Research Paper

Physics Based Finite Element Interpolation Functions for Rotating Beams

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Rotating beams are ubiquitous members of industrial structures such as wing turbine rotors, helicopter rotors, turbomachinery, robotic systems and aerial robots. Typically, finite element analysis is used to solve the vibration problem for these structures. We show that it is possible to significantly enhance the efficiency of the finite element methods for rotating beams by creating basis functions which more closely satisfy the governing differential equation of the structure. Since the rotating beam equation cannot be solved as an exact solution, different approximate strategies are explored to improve finite element convergence, especially at higher rotating speeds where the centrifugal stiffening terms become dominant.

Keywords: Finite Element Method; Basis Functions; Shape Functions; Collocation Method; Free Vibration

Introduction

Rotating beams are critical members of many important practical structures such as helicopter rotors, propellers, wind turbines, gas turbines, steam turbines and robotic manipulators (Wang and Wereley, 2004; Ganguli et al., 1998; Banerjee, 2000). Depending on the slenderness of the beam, they can be classified as Euler-Bernoulli, Rayleigh or Timoshenko beams (Bokaian, 1990). There is a plethora of literature on the analysis of non-rotating beams. However, the governing equation or rotating beams is complicated by the presence of a centrifugal term which inserts an integral into the partial differential equation (Wright A D et al., 1982). The free vibration problem of the rotating beam provides a good platform to investigate analytical and numerical approaches towards efficient computational solution. The choice of interpolation functions plays a critical role in the convergence of finite element methods for rotating beams. The rotating beam problem provides a good platform for pedagogy in finite element methods and numerical methods. This paper reviews some of the recent research of the author and his co-workers on the rotating beam problem related to selection of shape functions.

Interpolation Functions

The PDE for Euler-Bernoulli beam bending for stiffened beams is given by Wang and Wereley, (2004)

$$\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} \right) + m \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x} \left( T \frac{\partial w}{\partial x} \right) = f_z(x,t)$$  \hspace{1cm} (1)

where $EI$ is the flexural stiffness, $m$ is the mass per unit length, $T$ is the centrifugal tension and $f_z$ is the applied force. The spatial coordinate along the blade is $x$ and $t$ represents time. Consider the free vibration problem. This yields

$$\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} \right) + m \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x} \left( T \frac{\partial w}{\partial x} \right) = 0$$  \hspace{1cm} (2)

We will show several different approaches for forming interpolation functions using this equation. This is accomplished by considering the static part of the differential equation, which is obtained by ignoring the inertial term in Eq. (2)
\[
\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} \right) - \frac{\partial}{\partial x} \left( T \frac{\partial w}{\partial x} \right) = 0 \quad (3)
\]

Note that if \( T = 0 \), the above equation yields
\[
\frac{\partial^4 w}{\partial x^4} = 0 \quad (4)
\]
for a uniform beam. The solution of this equation is \( w(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \). This cubic polynomial is typically used as the basis function for beam finite elements, resulting in the well-known Hermite Cubic for displacement and slope degrees of freedom at the ends of the finite element. The cubic function works well for non-rotating beams but not for rotating beams. This is because for rotating beams, a cubic polynomial does not satisfy Eq. (3). In general, basis functions based on the static homogenous governing differential equation of the system show better convergence behaviour.

**Stiff-String Basis Functions**

Unfortunately, Eq. (3) does not have a simple closed form solution, even for a uniform beam. This is because of the presence of the integral term containing a spatially varying tension force. Let us assume \( T = \) constant and a uniform beam, leading to,
\[
EI \frac{\partial^4 w}{\partial x^4} - T \frac{\partial^2 w}{\partial x^2} = 0 \quad (5)
\]
which is the static part of the stiff-string equation, which is used to model piano strings. Solving, we obtain
\[
w(x) = a_0 + a_1 x + a_2 e^{-Cx} + a_3 e^{Cx} \quad (6)
\]
where, \( C = \sqrt{\frac{T}{EI}} \)

This function given in Eq. (6) can be used to obtain the shape functions of a rotating beam. However, the tension in a rotating beam is not constant, as assumed in Eq. (5). However, to put this in the context of the finite element method, we can use this equation at an element level. Thus, we define
\[
C_i = \sqrt{\frac{T_i}{EI_i}} \quad \text{for finite element } i
\]
within the finite element. Solving for \( a_0, a_1, a_2 \) and \( a_3 \) in terms of the nodal displacements and slopes using the above expressions, \( w \) can be approximated by
\[
w = w_1 N_1 + w_2 N_2 + w_3 N_3 + w_4 N_4 \quad (8)
\]
where \( N_1, N_2, N_3 \) and \( N_4 \) are the stiff-string shape functions and are given as
\[
N_1 = \frac{R_1(x)}{D}, N_2 = \frac{R_2(x)}{CD}, \quad N_3 = \frac{R_3(x)}{D}, N_4 = \frac{R_4(x)}{CD} \quad (9)
\]
where \( x_i \) is the location of the left edge of the element \( i \) and \( x_{N+1} = R \) is the radius of the beam. Consider the two noded, 4 degree of freedom beam finite element shown in Fig. 1. The boundary conditions for the element of length \( l \) are given by \( w(0) = w_1, \quad \frac{dw(0)}{dx} = \theta_1 = w_2, \quad w(l) = w_3, \quad \frac{dw(l)}{dx} = \theta_2 = w_4 \). Putting Eq. (6) into the element boundary conditions yields: \( w_1 = a_0 + a_2 + a_3, \quad w_2 = a_1 - Ca_2 + Ca_3, \quad w_3 = a_0 + a_1 l + a_2 e^{-Cl} + a_3 e^{Cl} \quad \text{and } w_4 = a_1 - a_2 Ce^{-Cl} + a_3 C e^{Cl} \). Here we have dropped the subscript \( i \) in \( C \) as the entire discussion here is relevant.
where

\[
D = -4 + 2e^{CI} + 2e^{-CI} + Cle^{-CI} - Cle^{CI}
\]

\[
R_1(\bar{x}) = \left( \begin{array}{c}
-2e^{CI} - e^{-CI} - Cle^{-CI} + \\
2 + Cle^{CI} + C\bar{x}e^{-CI} - C\bar{x}e^{CI} \\
-2e^{-CI + CI} + e^{-CI} + e^{CI} - e^{CI - CI}
\end{array} \right)
\]

\[
R_2(\bar{x}) = \left( \begin{array}{c}
e^{CI} - e^{-CI} - Cle^{-CI} - Cle^{CI} \\
+ C\bar{x}e^{-CI} + C\bar{x}e^{CI} - 2C\bar{x} \\
-2e^{-CI + CI} + Cle^{-CI + CI} + \\
+ e^{CI} - e^{CI - CI}
\end{array} \right)
\]

\[
R_3(\bar{x}) = \left( \begin{array}{c}
e^{CI} + e^{-CI} - 2 + C\bar{x}e^{-CI} \\
-C\bar{x}e^{CI} - e^{-CI + CI} + e^{-CI} \\
+ e^{CI} - e^{CI - CI}
\end{array} \right)
\]

\[
R_4(\bar{x}) = \left( \begin{array}{c}
2CI - e^{CI} + e^{-CI} - 2C\bar{x} \\
+C\bar{x}e^{CI} + C\bar{x}e^{-CI} - Cle^{CI} \\
-2e^{-CI + CI} + e^{CI} + e^{CI - CI} \\
-Cle^{CI} - e^{CI - CI}
\end{array} \right)
\]

These shape functions are called the stiff-string shape functions. Fig. 2 compared these stiff-string shape functions with the conventional Hermite cubics for one element case and with a non-dimensional rotational speed of 12. Fig. 3 shows this comparison at a much higher rotation speed. Figs. 4 and 5 clearly show the good performance of the stiff-string basis functions, especially for the important fundamental mode. This effect is more significant at the higher rotation speed. Table 1 and Table 2 show the comparison of predictions of rotating beam frequencies with the published literature. It can be observed that the stiff-string basis function depend...
on the material property of the element and on the local tension level. Thus, these shape functions adjust to the physics of the problem.

The analytical limits of these stiff string basis functions as the rotation speed tends to zero is shown in Eq. (11) to become the Hermite cubics.

\[
\lim_{c \to 0} N_1 = \frac{2\bar{x}^3 - 3\bar{x}^2 l + l^3}{l^3}, \quad \lim_{c \to 0} N_2 = \frac{\bar{x}^3 - 2\bar{x}^2 l + \bar{x}l^2}{l^2}
\]

\[
\lim_{c \to 0} N_3 = -\frac{2\bar{x}^3 + 3\bar{x}^2 l}{l^3}, \quad \lim_{c \to 0} N_4 = \frac{\bar{x}^3 - \bar{x}^2 l}{l^2}
\]

As the rotation speed tends to infinity, the interpolation functions and become linear and approach zero, as given in Eq. (12).

\[
\lim_{c \to \infty} N_1 = 1 - \frac{\bar{x}}{l}, \quad \lim_{c \to \infty} N_2 = 0,
\]

\[
\lim_{c \to \infty} N_3 = \frac{\bar{x}}{l}, \quad \lim_{c \to \infty} N_4 = 0
\]

In a following work, Gunda et al proposed the hybrid stiff-string polynomial basis functions (Gunda et al., 2009),

\[
w(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 e^{-Cx} + a_5 e^{Cx}
\]

The idea here is to keep the cubic polynomial which works well for the non-rotating beam and add the stiff string basis which works well for beams at high rotating speeds. These hybrid shape functions were obtained by adding a centre node with displacement and slope degrees of freedom to the finite element as shown in Fig. 6. This additional node with two degrees of freedom allows us to determine all the six constants in Eq. (13). The expressions for the shape functions are given in Reference (Gunda et al., 2009). Fig. 7 and 8 show the new hybrid basis functions, along with stiff string, Hermite cubic and fifth order polynomial basis functions. Fig. 7 results are for an element at low rotation speed of \(\lambda = 12\) and Fig. 8 shows results for an element at high rotation speed of \(\lambda = 100\). The new basis functions adapt to rotation speed changes. Fig. 9 shows the mode shapes of the cantilever beam. It can be observed that at high rotation speeds, the beam behaves like a rotating cord. The hybrid stiff-string polynomial basis functions
addressed the issues of convergence of the stiff string basis functions which favoured the fundamental node and did not give good results for higher nodes. The hybrid basis functions performed well for all modes.

Physics Based Basis Function

Consider the static homogenous equation of the rotating beam rewritten as,

$$EI \frac{d^4w}{dx^4} - \frac{m\Omega^2}{2} d \left( \left( R^2 - x^2 \right) \frac{dw}{dx} \right) = 0 \quad (14)$$

Let $$\lambda^2 = \frac{m\Omega^2 L^4}{EI}$$

Then,

$$\frac{d^4w}{dx^4} - \frac{\lambda^2}{2L^4} d \left( \left( L^2 - x^2 \right) \frac{dw}{dx} \right) = 0 \quad (15)$$

We split the above equation into two parts and find their solution [15],

$$\frac{d^4w}{dx^4} = 0 \quad (16)$$

$$\frac{d}{dx} \left( \left( L^2 - x^2 \right) \frac{dw}{dx} \right) = 0 \quad (17)$$

This yields two equations as solutions of Eq (16) and Eq. (17),

$$W^{(1)} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \quad (18)$$

$$W^{(2)} = a \ln \left( \frac{L + x}{L - x} \right) \quad (19)$$

The basis function used to create finite element shape functions is thus,
\[ w(x) = W^{(1)} + W^{(2)} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 \ln \left( \frac{L + x}{L - x} \right) \quad (20) \]

Ganesh and Ganguli (2013) developed this approach and found that this approach worked well for rotating beams. The additional constant is addressed by forcing the error given by the basis function to zero at the element midpoint. The complete shape functions are given in Ref. Ganesh and Ganguli (2013). Figs. 10 and 11 show the shape functions. The difference between the polynomial and these physics based functions is amplified at higher rotation speeds.

**Point Collocation Method**

Here, we write the static homogenous equation as,

\[ \frac{d^4 w}{dx^4} = \frac{m \Omega^2 R^2}{2EI} \frac{d^2 w}{dx^2} + \frac{m \Omega^2 x^2}{2EI} \frac{d^2 w}{dx^2} + \frac{m \Omega^2 x}{EI} \frac{dw}{dx} = 0 \quad (21) \]

Let, \[ c = \frac{m \Omega^2}{EI} \] and \[ d = \frac{m \Omega^2 L^2}{2EI} \]

\[ \frac{d^4 w}{dx^4} - d \frac{dx^2}{dx} + cx \frac{d^2 w}{dx^2} + cx \frac{dw}{dx} = 0 \quad (22) \]

The cubic polynomial does not satisfy Eq. (22). Consider a collocation point located inside the finite element and choose a basis function as,

\[ w = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \quad (23) \]

Now, the interpolating polynomial is required to satisfy the ODE at \( x = \mu \). This yields \( a_4 \) in terms of \( a_0, a_1, a_2 \) and \( a_3 \). These collocation inspired basis functions are used to get shape functions by Sushma and Ganguli (2012). Fig. 12 and 13 show the shape functions.

**Rotor Blade Problem**

A rotor blade can be modelled as a beam with axial \( (u_e) \), lag bending \( (v) \), flap bending \( (w) \) and torsion...
degrees ($\phi$) of freedom. A simplified model with these motions can be written as

$$EAu'' + m(x + u_e + 2\dot{v} - \ddot{u}_e) = 0$$  \hspace{1cm} (24)

$$EI_z v'' - mv + m\ddot{v} + 2m\dot{u}_e - \left[ v' \int_x^l mx d\xi \right]' = 0$$  \hspace{1cm} (25)

$$EI_y w'' + m\ddot{w} - \left[ w' \int_x^l mx d\xi \right]' = 0$$  \hspace{1cm} (26)

$$GJ\phi'' - mK_m^2\phi - m\phi \left( K_{m_2}^2 - K_{m_1}^2 \right) = 0$$  \hspace{1cm} (27)

Here $EA$ is the axial stiffness, $EI_z$ is the inplane or lag bending stiffness, $EI_y$ is the out-of-plane or flap bending stiffness and $GJ$ is the torsion stiffness. Also $K_m^2$ is the radius of gyration.

To derive the shape functions based on the logic discussed in the previous sections, we remove the inertial and velocity terms to obtain the static part of the governing homogenous differential equations. These equations in dimensional form are given as:

$$EAu'' + m\Omega^2 u_e = 0$$  \hspace{1cm} (28)

$$EI_z v'' - m\Omega^2 v - \left[ v' \int_x^l m\Omega^2 x d\xi \right]' = 0$$  \hspace{1cm} (29)

$$EI_y w'' - \left[ w' \int_x^l m\Omega^2 x d\xi \right]' = 0$$  \hspace{1cm} (30)

$$GJ\phi'' - m\Omega^2 \phi \left( K_{m_2}^2 - K_{m_1}^2 \right) = 0$$  \hspace{1cm} (31)

where,

$$T = \int_x^l m\Omega^2 x d\xi$$  \hspace{1cm} (32)

The constant tension $T_i$ for the element is approximated by taking the average centrifugal tension in the element. The centrifugal tension $T_i$ for the $i^{th}$ element can be expressed as:

$$T_i = \int_{x_i}^L m_i\Omega^2 x dx$$  \hspace{1cm} (33)

Here $x_i$ is the location of the left edge of the finite element. The solutions of the differential equations (28) to (31) are obtained as:

$$u_e = C_1 \sin \left( \frac{m\Omega^2}{EA} x \right) + C_2 \cos \left( \frac{m\Omega^2}{EA} x \right)$$  \hspace{1cm} (34)

$$v = \begin{cases} 
  C_3 \exp \left[ -\sqrt{\frac{T - \sqrt{T^2 + 4EI_z m\Omega^2}}{2EI_z}} x \right] 
  + C_4 \exp \left[ \sqrt{\frac{T - \sqrt{T^2 + 4EI_z m\Omega^2}}{2EI_z}} x \right] 
  + C_5 \exp \left[ -\sqrt{\frac{T + \sqrt{T^2 + 4EI_z m\Omega^2}}{2EI_z}} x \right] 
  + C_6 \exp \left[ \sqrt{\frac{T + \sqrt{T^2 + 4EI_z m\Omega^2}}{2EI_z}} x \right]
\end{cases}$$  \hspace{1cm} (35)

$$w = C_7 + C_8 x + C_9 \exp \left[ \sqrt{\frac{T}{EI_y}} x \right] + C_{10} \exp \left[ -\sqrt{\frac{T}{EI_y}} x \right]$$  \hspace{1cm} (36)
\[ \phi = C_{11} \exp \left( \sqrt{\frac{m\Omega^2 (K_{m2}^2 - K_{m1}^2)}{GJ}} x \right) + C_{12} \exp \left( -\sqrt{\frac{m\Omega^2 (K_{m2}^2 - K_{m1}^2)}{GJ}} x \right) \]  

(37)

These solutions were used to obtain shape functions for a rotor blade by Chhabra and Ganguli (2010). Figs. 14-16 show these shape functions along with the classical polynomials. Again, these shape functions give better results at high rotation speeds, relative to the polynomials. The shape functions contain the material properties and rotation speed of the blade, allowing them to adapt to different rotation speeds and non-uniform geometries.

**Spectral Finite Element Method**

The spectral finite element method solves the free vibration problem in the frequency domain. We construct the weak form of the governing differential equation and write the total energy of a non-conservative rotating system in transverse motion is the frequency domain as,

\[
\hat{\Pi} = e^{i\omega_n t} \sum_{n=1}^{N} \left[ \frac{1}{2} E \int_0^L I(x) \left( \frac{d^2 \tilde{w}_n}{dx^2} \right)^2 dx \right. \\
+ \frac{1}{2} \int_0^L T(x) \left( \frac{d\tilde{w}_n}{dx} \right)^2 dx + \frac{1}{2} \rho \int_0^L -A(x) \alpha_n^2 \tilde{w}_n^2 dx + \frac{1}{2} \eta \int_0^L \alpha_n \tilde{w}_n^2 dx - \left[ \hat{F} \right]^T \{ \tilde{W} \} \right]
\]  

(38)

where \( \left[ \hat{F} \right]^T, \{ \tilde{W} \} \) represents the externally applied nodal force and nodal displacement vectors in frequency domain, \( \omega_n \) the circular frequency of the \( n^{th} \) sampling point, and \( \tilde{w}_n \) is the spatially dependent Fourier coefficient and \( \eta \) is the damping force per unit length, per unit velocity. According to principle of minimum potential energy in the frequency domain, we have \( \delta \Pi = 0 \).
To obtain the dynamic stiffness matrix in frequency domain, we need to substitute an interpolating function for the transverse displacement $\tilde{w}_n$ in Eq. (39). We consider two choices for the interpolating function and hence two types of elements viz: the SFER and SFEN are formulated.

**Interpolating Function for SFER**

To obtain the interpolating function for SFER, we assume the beam to be uniform and replace $T(x)$ by the maximum centrifugal force, $T_{\text{max}}$ (Thakkar and Ganguli, 2006).

$$T_{\text{max}} = \int_0^L \rho A(x) \omega^2 x dx = \frac{\rho \Omega^2 L^2}{2}$$  \hspace{1cm} (40)

This allows us to represent of the rotating beam with damping, as a constant coefficient PDE, which can be written as

$$EI \frac{d^4 w}{dx^4} - T_{\text{max}} \frac{d^2 w}{dx^2} + \rho A \frac{d^2 w}{dt^2} + \eta \frac{dw}{dt} = 0$$  \hspace{1cm} (41)

where $\eta$ is the damping force per unit length, per unit velocity. Eq. (41) is analogous to the “stiff string” equation where the tension is constant along the length. The exact solution of this equation, in frequency domain is given as

$$\hat{w}_n(x) = C_{1n} e^{-ik_{1n}x} + C_{2n} e^{-ik_{2n}x} + C_{3n} e^{ik_{3n}(L-x)} + C_{4n} e^{ik_{4n}(L-x)}$$  \hspace{1cm} (44)

But here, the wave number is different from the previous case and is given by

$$k_{pn} = \pm \sqrt{\left( \frac{\rho \Omega^2 L^2}{2EI} \right) + \left( \frac{\rho \Omega^2 L^2}{2EI} \right)^2 - 4 \left( \frac{i \omega \eta}{EI} - \frac{p \omega_n^2}{EI} \right)}$$  \hspace{1cm} (45)

Where $p$ represents the mode of wave propagation. Eq. (45) is the frequency domain interpolating function for SFEN. Vinod et al showed the advantages of using the spectral finite element method for rotating beams (Thakkar and Ganguli, 2006; Vinod et al., 2007). To compare the spectral finite element methods, the resonant peaks for a uniform cantilever beam and a hinged beam are compared in Fig. 17.

The convergence study is also shown in Table 3. The SFER has better convergence than the SFEN,
where \( T(x) \) is axial force due to centrifugal stiffening and is given by

\[
T(x) = \int_0^L \rho A(x) \Omega^2 \, dx
\]

Here \( L \) is the length of beam, \( \Omega \) is the rotational speed, \( \Omega(x, t) \) is angle of rotation of cross section, \( w(x, t) \) is the vertical displacement of beam, \( \rho \) is the density, \( E \) and \( G \) are elastic constants, \( k \) is the shear coefficient, \( A(x) \) is area of cross section, \( I(x) \) is moment of inertia of cross section. The non-dimensional constants are

\[
\nu = \frac{AL^2}{I}, \quad \beta = \frac{GAkL^2}{EI}
\]

and \( \gamma = \frac{\beta}{\nu} = \frac{Gk}{E}, \quad k^2 = \frac{\rho AS^2L^4}{EI} \). For beams with constant tension, Eqs. (46) and (47) reduce to

\[
\rho A(x) \frac{\partial^2 w(x, t)}{\partial t^2} - T \frac{\partial^2 w(x, t)}{\partial x^2} - \frac{\partial}{\partial x} \left( GA(x)k \left( \frac{\partial w(x, t)}{\partial x} - \theta(x, t) \right) \right) = 0 \quad (48)
\]

\[
\rho I(x) \frac{\partial^2 \theta(x, t)}{\partial t^2} - GA(x)k \left( \frac{\partial w(x, t)}{\partial x} - \theta(x, t) \right) - \frac{\partial}{\partial x} \left( EI(x) \frac{\partial \theta(x, t)}{\partial x} \right) = 0 \quad (49)
\]

but the difference in predicted frequencies is small.

**Violin String Basis Function**

The earlier examples considered Euler-Bernoulli rotating beams. However, short beams are better modelled as Timoshenko beams. The governing equation for free vibrations of a rotating Timoshenko beam is given by (Vinod Kumar and Ganguli, 2012).

\[
\rho A(x) \frac{\partial^2 w(x, t)}{\partial t^2} - \frac{\partial}{\partial x} \left( T(x) \frac{\partial w(x, t)}{\partial x} \right) - \frac{\partial}{\partial x} \left( GA(x)k \left( \frac{\partial w(x, t)}{\partial x} - \theta(x, t) \right) \right) = 0 \quad (46)
\]

\[
\rho I(x) \frac{\partial^2 \theta(x, t)}{\partial t^2} - GA(x)k \left( \frac{\partial w(x, t)}{\partial x} - \theta(x, t) \right) - \frac{\partial}{\partial x} \left( EI(x) \frac{\partial \theta(x, t)}{\partial x} \right) = 0
\]

where \( T(x) \) is the constant axial tension. We call these the violin string equations in an analogy to the typical Euler-Bernoulli stiff string equations. The stiff strings are studied in the acoustic analysis of piano strings. Piano strings have flexural stiffness but are typically quite long. In fact, the most prized pianos are those with the largest strings. Violins are much smaller than pianos. Timoshenko theory gives a better representation of inharmonicity of violin strings.
compared to Euler-Bernoulli beam theory (Maezawa et al., 1988). The static part of Eqs. (48) and (49) is obtained by ignoring the inertial term which leads to

$$T \frac{d^2 w}{dx^2} - \frac{d}{dx} \left( GA(x) k \left( \frac{dw}{dx} - \theta \right) \right) = 0 \quad (50)$$

$$GA(x) k \left( \frac{dw}{dx} - \theta \right) - \frac{d}{dx} \left( EI(x) \frac{d\theta}{dx} \right) = 0 \quad (51)$$

Putting Eq. (51) in Eq. (50) yields and putting $T = 0$ for a non-rotating beam.

$$\frac{d^2}{dx^2} \left( EI(x) \frac{d\theta}{dx} \right) = 0 \quad (52)$$

For a uniform beam,

$$\frac{d^3 \theta}{dx^3} = 0 \Rightarrow \theta(x) = a_0 + a_1 x + a_2 x^2 \quad (53)$$

Also, for a uniform beam, Eq. (52) and Eq. (53) give

$$\frac{d^2 w}{dx^2} = \frac{d\theta}{dx} \quad (54)$$

$$\frac{dw}{dx} - \theta + \frac{EI}{GAk} \frac{d^2 \theta}{dx^2} = 0 \quad (55)$$

We propose to use Eqs. (50) and (51) to derive new shape functions which are suitable for rotating Timoshenko beams. The presence of $T(x)$ in Eq. (1) presents a difficulty in obtaining an analytical solution. While the assumption of $T(x) = T = \text{constant}$ may appear too simplistic for the whole beam, it is reasonable if the beam is divided into several finite elements and $T$ is assumed constant within each element. Thus, the rotating beam is assumed as a sequence of violin strings for the calculation of shape functions.

Using the finite element method, the beam is divided into $N$ number of elements of equal length. The tension $T(x)$ is assumed as constant $T_i$ within the element $i$, which reduces the rotating beam equation to the violin string equation. For an $i^{th}$ element along the beam, the relationship between global and local coordinates is given by

$$x = x_i + \bar{x} \quad (57)$$

where $x_i = (i-1)l$ is the distance from the root to the left edge of element $i$, $l$ is length of the element and $\bar{x}$ is the local element coordinate.

$$T(x) = T_i = \int_x^{l} \rho A(x) \Omega^2 x dx = \sum_{j=i}^{i+N} \int_{x_j}^{x_{j+1}} m_j(\bar{x}) \Omega^2 \bar{x} d\bar{x} \quad (58)$$

The tension in the inboard elements is high and decreases for the outboard elements. Assuming $EI(x) = EI$ and $\rho A(x) = \rho A_i$ within an element as constant, we get

$$T_i \frac{d^2 w}{dx^2} + \left( GAk \left( \frac{d^2 w}{dx^2} - \frac{d\theta}{dx} \right) \right) = 0 \quad (59)$$

$$GAk \left( \frac{dw}{dx} - \theta \right) + \left( EI \frac{d^2 \theta}{dx^2} \right) = 0 \quad (60)$$

This yields

$$\frac{d^3 \theta}{dx^3} = \frac{T_i}{EI} \frac{d^2 w}{dx^2} \quad (61)$$
Note that this equation relates $\theta(x)$ and $w(x)$ is quite different from Eq. (53). Also, if $T_i = 0$, Eq. (61) yields Eq. (53) for the non-rotating beam. We remove the coupling between $w$ and $\theta$ to yield,

$$
\frac{d^4 w}{dx^4} + \frac{T_i}{EI} \left( 1 + \frac{T_i}{GAk} \right) \frac{d\theta}{dx} = 0
$$

(62)

$$
\frac{d^3 \theta}{dx^3} + \frac{T_i}{EI} \left( 1 + \frac{T_i}{GAk} \right) \frac{d\theta}{dx} = 0
$$

(63)

The solutions of which are

$$
\theta(x) = b_0 + b_1 e^{-\alpha x} + b_2 e^{\alpha x}
$$

(64)

$$
d = \sqrt{\frac{T_i}{EI} \left( 1 + \frac{T_i}{GAk} \right) \sqrt{1 + C^2 m_3}}
$$

where \( \alpha = m_3 d^2 + \frac{1}{\left(1 + C^2 m_3\right)} - 1 \).

This displacement function satisfies the static homogeneous violin string equations within the element and is used to create the violin string shape functions (Vinod and Ganguli, 2011). Fig. 19 and 20 show the displacement and rotation shape functions. Fig. 21 shows the convergence of the first five modes using polynomial and violin string basis functions.

Fig. 18: Timoshenko Beam Element

Fig. 19: Displacement shape functions at different slenderness ratios for $K = 12$

Fig. 20: Rotation shape functions at different slenderness ratios for $K = 12$ ($N = 1$)
Concluding Remarks

This paper reviews some research on the development of physics based shape functions for the finite element analysis of rotating beams. The rotating beam differential equation has two extreme limits. The first case is when the rotation speed is zero, which is the non-rotating beam. The other case is a beam rotating with very high speed, in which case the centrifugal stiffness can exceed the flexural stiffness and the beam behaves like a rotating cord or string. Various approaches to develop new shape functions were illustrated in this paper. These shape functions yield better convergence at high rotation speeds. Fast convergence also allows the creation of low order finite element models which are useful for control applications. The possibility of using spectral finite element methods for the rotating beam problem is also discussed. Furthermore, a novel approach to enrich shape functions by requiring them to satisfy the governing differential equation at selected points in the element is presented. It is also found that the rotating beam problem can be used as a pedagogical tool to illustrate the finite element method to students in engineering sciences.

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