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On the General Solutions of a Nonlinear Pseudo-Oscillator Equation and Related Quadratic Liénard Systems

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We compute explicitly the exact and general solutions of the so-called pseudo-oscillator equation $x\ddot{x}+1=0$, and of its related quadratic Liénard type equations in a straightforward and direct method. This is achieved using the generalized Sundman transformation theory introduced recently in the literature by some authors of this work.

Keywords: Nonlinear Pseudo-Oscillator Equation, Nonlocal Transformation, Exact and General Solution.

Introduction

One of the most investigated truly nonlinear oscillator equation is the singular equation

$$\ddot{x}+\frac{1}{x}=0 \tag{1}$$

It is interesting to notice that equation (1) is of physical importance (Acton and Squire, 1985; Nayfeh and Mook, 1979) and is widely studied in the literature using different exact or approximate methods to compute periodic solutions (Mickens, 2007; Beléndez et al., 2008, 2008a; Mirzabeigy et al., 2012; Xu, 2011). By analyzing the so-called pseudo- oscillator equation (1), the authors in (Gadella and Lara, 2014) found that such equation has no smooth periodic solutions (Van Gorder, 2015), but, instead, admits two exact and general non-periodic solutions. Indeed, they have considered the equation (1) and rewrote it as a planar autonomous dynamical system for which, they found the Hamiltonian and the associated vector field. By analysis of the components of this vector, they observed that the phase plane trajectories presented by the system eliminate the existence of periodic solution of (1). They have deduced that the solution of (1) $x=\varphi(t)$ is invertible and its inverse function is given by $t=\varphi^{-1}(x)=f(x)$. Using the change of variable $t=f(x)$, they have rewritten equation (1) in a form of an equation which was solved by quadrature. They have inverted the solution found and obtained the general solution of (1) given by

$$x(t)=\sqrt{\frac{2}{\pi}}\,c_1\exp\left\{-\left(erf^{-1}\left(\mp\frac{t-c_2}{c_1}\right)\right)^2\right\} \tag{1-a}$$

To find the second solution, they claimed to turn the sign of a positive constant of integration into a negative sign. However, the method used by these authors (Gadella and Lara, 2014) does not allow the explicit and direct calculation of the two general solutions. Hence, one can see that only one general solution (1-a) has been exhibited explicitly by the authors (Gadella and Lara, 2014). Moreover, in their comment (Mancas and Rosu, 2016), the authors have only succeed to exhibit the single general solution (1-a).

The goal of this paper is to show in a straightforward manner that the two exact and general solutions of equation (1) highlighted in (Gadella and Lara, 2014) can be explicitly computed and to investigate explicitly the related quadratic Liénard type equations. To do so, equation (1) is shown to be a nonlocal transformation of the well-known linear harmonic oscillator equation (section 2), so that the general solutions of (1) may be easily computed and discussed (section 3). Finally, the related quadratic Liénard type equations are investigated (section 4) and a conclusion is carried out for the work.

2- Nonlocal transformation of equations

Consider the linear harmonic oscillator equation with forcing constant term.

$$y''(\tau)+a^2y(\tau)=c \qquad (2)$$

Where, prime denotes differentiation with respect to τ, a and c are arbitrary parameters, and the nonlocal transformation (Koudahoun et al., 2019)

$$y^m(\tau)=\int g^l(x)dx \;,\; d\tau = g^l(x)\left[\int g^l(x)dx\right]^{\left(\frac{1}{m}-1\right)}dt \qquad (3)$$

l and m are arbitrary parameters. Therefore, applying the nonlocal transformation (3) to (2) leads to a general class of equations (Koudahoun et al., 2019).

$$\ddot{x}+a^2mg^l(x)\left[\int g^l(x)dx\right]^{\left(\frac{2}{m}-1\right)} - mcg^l(x)\left[\int g^l(x)dx\right]^{\left(\frac{1}{m}-1\right)}=0 \qquad (4)$$

In this way, the following lemma may be proved.

Lemma

If $g(x)=x^2$, $m=2$, $c=0$, $l=-\frac{1}{2}$, and $2a^2=1$, then equation (4) transforms into equation (1).

Proof

For $m=2$, $c=0$, (4) becomes

$$\ddot{x}+2a^2g^l(x)=0 \qquad (5)$$

which is

$$\ddot{x}+\frac{2a^2}{x}=0 \qquad (6)$$

if, $g(x)=x^2$ and $l=-\frac{1}{2}$.

Finally, setting $2a^2=1$, in (6) leads to the desired equation (1).

Now, we may compute the two exact and general solutions to equation (1).

3- Exact and general solutions to equation (1)

We, first, investigate in the following, the exact and explicit general solutions of equation (6) such that one may deduce the exact and general solutions of equation (1).

Substituting the values attributed previously to m, l and the expression of $g(x)$ in (3) yields

$$y^2(\tau)=\int\frac{1}{x}dx \;,\; d\tau=\frac{1}{x}\left[\int\frac{1}{x}dx\right]^{\frac{-1}{2}}dt \qquad (7)$$

such that the general solutions to (6) is

$$x(t)=\varepsilon\exp(y^2(\tau)) \qquad (8)$$

with, $\varepsilon=1$ if $x>0$, or $\varepsilon=-1$ if $x<0$. The solution of (2) for $c=0$, may be written as

$$y(\tau)=A_0\sin(a\tau+\alpha) \qquad (9)$$

where, A_0 and α are arbitrary constants and τ satisfies

$$\int ye^{y^2}d\tau=\varepsilon(t+k) \qquad (10)$$

that is

$$\int A_0 \sin(a\tau+\alpha) e^{A_0^2 \sin^2(a\tau+\alpha)} d\tau = \varepsilon(t+k) \quad (11)$$

and k is an integration constant.

Let us evaluate the left hand side of (11) by setting it as J.

Imposing $\theta = a\tau + \alpha$, yields

$$aJ = \int A_0 \sin\theta e^{A_0^2 \sin^2\theta} d\theta \quad (12)$$

Let

$$u = A_0 \cos\theta \quad (13)$$

then

$$d\theta = \frac{-du}{A_0 \sin\theta} \quad (14)$$

and

$$\sin^2\theta = 1 - \frac{u^2}{A_0^2} \quad (15)$$

Therefore

$$-aJ = e^{A_0^2} \int e^{-u^2} du \quad (16)$$

Knowing (Abramowitz and Stegun, 1972)

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-q^2} dq \quad (17)$$

then

$$J = \frac{-\sqrt{\pi}}{2a} e^{A_0^2} erf(u) \quad (18)$$

Replacing (18) in (11) yields

$$erf(u) = \mp \frac{2a}{\sqrt{\pi}} e^{-A_0^2} (t+k) \quad (19)$$

from which

$$u = erf^{-1}\left[\mp \frac{2a}{\sqrt{\pi}} e^{-A_0^2} (t+k)\right] \quad (20)$$

Substituting (20) in (15) leads to

$$A_0^2 \sin^2\theta = A_0^2 - \left\{erf^{-1}\left[\mp \frac{2a}{\sqrt{\pi}} e^{-A_0^2} (t+k)\right]\right\}^2 \quad (21)$$

Thus, one may rewrite the solution (8) in the form

$$x(t) = \varepsilon e^{A_0^2} \exp\left\{-\left[erf^{-1}\left[\mp \frac{2a}{\sqrt{\pi}} e^{-A_0^2} (t+k)\right]\right]^2\right\} \quad (22)$$

where, $\varepsilon = \pm 1$.

In this situation, the following theorem is proved for $a = \frac{\sqrt{2}}{2}$.

Theorem

If, $a=\frac{\sqrt{2}}{2}$ *, then the exact and general solutions of equation (6) is*

$$x(t)=\pm\sqrt{\frac{2}{\pi}}c_1\exp\left\{-\left[erf^{-1}\left[\mp\frac{(t-c_2)}{c_1}\right]\right]^2\right\}$$

(23)

where $c_1=\sqrt{\frac{\pi}{2}}e^{A_0^2}$ *and* $c_2=-k$ *.*

Equation (23) is the desired exact and general solutions of equation (1). The positive value of (23) corresponds to the solution found in (Gadella and Lara, 2014). The theory developed in this work to compute the solution of equation (1) doesn't lead to a periodic solution. Moreover, the relation between the harmonic oscillator equation and equation (1) eliminates any possibility of existence of the smooth periodic solution of equation (1). That being so, we may investigate the class of quadratic Liénard type equations related to (1).

4- General class of quadratic Liénard type equations

This section is devoted to investigate the general class of quadratic Liénard type equations (Kamke, 1977)

$$\lambda u\ddot{u}+b\dot{u}^2+\mu u^l=0 \qquad (24)$$

For, $l=0$, $b=0$ and $\frac{\mu}{\lambda}=1$, with $\lambda\neq 0$, equation (24) becomes identical to the pseudo-oscillator equation (1). The problem is now to ask whether equation (24) may include (1) as a special case and to be solved explicitly in terms of (23). It seems that such a problem has not been investigated previously in the literature. The interest is to find a new class of solutions of quadratic Liénard type equations.

Now, let $\frac{b}{\lambda}=-(q+1)$, $\frac{\mu}{\lambda}=-\frac{1}{q}$, $l=2q+2$ and $q\neq 0$. Then (24) reduces to

$$\ddot{u}-(q+1)\frac{\dot{u}^2}{u}-\frac{1}{q}u^{2q+1}=0 \qquad (25)$$

Hence, one deduces easily the following corollary from the theorem in above.

Corrolary

If the change of variable

$$u=x^{\frac{-1}{q}},\ x\neq 0 \qquad (26)$$

holds, then equation (1) is mapped into (25).

Proof

By using the change of variable (26), one may obtain

$$\frac{dx}{dt}=-qu^{-q-1}\frac{du}{dt} \qquad (27)$$

and

$$\frac{d^2x}{dt^2}=-q\left\{-(q+1)u^{-(q+1)-1}\left(\frac{du}{dt}\right)^2+u^{-(q+1)}\frac{d^2u}{dt^2}\right\} \qquad (28)$$

Substituting (28) into (1), and using (26), yields immediately (25). Therefore, the theorem is proved. In this perspective, the exact and general solutions to (25) become

$$u(t) = \left\{ \pm \sqrt{\frac{2}{\pi}} c_1 \exp\left\{ - \left[erf^{-1}\left[\mp \frac{(t-c_2)}{c_1} \right] \right]^2 \right\} \right\}^{\frac{-1}{q}} \quad (29)$$

As expected, if $q = -1$, then (25) reduces to (1) and (29) coincides with (23).

5- Conclusion

Many authors have investigated the so-called pseudo-oscillator equation with different methods. In this work, the integrability of this equation is investigated using equation transformation theory. It has been possible to compute explicitly and exactly its exact and general solutions. It has been also possible to investigate a general class of quadratic Liénard type equations related to such a pseudo-oscillator equation. So, using a simple appropriate variable change, it is shown the existence of a new class of solutions for quadratic Liénard type equations.

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