Proceedings of the Indian National Science Academy (2024) 90:702–705
https://doi.org/10.1007/s43538-024-00246-5

RESEARCH PAPER

Inhomogeneous string cosmological model of Bianchi type-I in general relativity

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Received: 31 October 2023 / Accepted: 25 January 2024 / Published online: 23 February 2024
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Abstract
An investigation is conducted into a higher-dimensional maximally symmetric inhomogeneous Bianchi type-I string cosmological model. In order to arrive at a deterministic solution, we have to assume Einstein string space metric depends on two variables in all four directions, and these variables are related to the condition: a(x,t) = L b(x,t), where L is a constant, and b(x,t) = f(x).g(t) This allows us to achieve a deterministic answer. g(t) and c(x,t) = h(x).k(t). The relationship between scalar expansion metric potential and shear tension is a straightforward one. In addition to this, the physical and geometrical characteristics of the higher dimensional string model of the universe are dissected.

Keywords Cosmic string · Inhomogeneous universe · Cylindrically symmetric · Higher-dimensional universe

Introduction

Isotropy, homogeneity, and the existence of a space-time metric are presumptions made by the standard model of the universe. On the other hand, it is a well-accepted fact that there are inhomogeneities on a variety of scales. We can acquire a better understanding of the impact that these inhomogeneities have on the expanding cosmos if we investigate the various potential solutions to Einstein's equations for metrics. In particular, the function that higher-dimensional symmetric inhomogeneous space-time plays in the process of comprehending anisotropy and inhomogeneity on cosmic scales is extremely important. Higher-dimensional inhomogeneous symmetric cosmological models provide a substantial contribution to our understanding of galaxy and star formation throughout the early phases of the cosmos. Even while cosmology has appropriate explanations for some of the conceptual problems that arise from the idea of an ever-expanding universe, the Big Bang hypothesis cannot fully account for these explanations on its own. Inhomogeneous cylindrically Bianchi-type space-time with flat potential is discus by Poonia and Sharma (Poonia et al. 2021; Poonia and Sharma 2020) The inflationary scenario has emerged as a persuasive explanation for unresolved puzzles in cosmology, such as the homogeneity, isotropy, and flatness of the cosmos. This is because the scenario predicts that the universe expanded exponentially at the beginning of time. Singh (Singh and Tiwari 2008) discusses Perfect fluid Bianchi Type-I cosmological models with time-varying. The early inflationary phase, cosmic strings, and gravitational fields were all concepts that Guth presented for the first time, Letelier (Letelier 1983) gives the string cosmological explanation which paved the way for a better understanding of these phenomena. As a consequence of this, Einstein's equations provide light on the gravitational effects that are caused by cosmic strings. Sharma and Poonia (Sharma and Sharma 2016; Sharma et al. 2021a) discuss String cosmological model with a homogeneous inflationary universe with flat potential.

Inhomogeneous symmetric cylindrically has been investigated using an electromagnetic field by researchers Bali (2006); Tyagi (1989) and Pradhan (2007), In addition to this, Mohanty and Patel (Mohanty and Mishra 2002; Patel and Maharaj 1996) have hypothesized the existence of a universe that is flat, inhomogeneous, and cylindrically symmetric. Pradhan (Pradhan, et al. 2005) gives some explanation of the homogeneous Bianchi Type with Viscous Fluid Cosmological Models. In this study, we take a fresh look at the answers that have been suggested by a variety of scholars and provide our

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original discoveries. Kumawat (Kumawat et al. 2021) looking at the Bianchi-III inflationary universe with flat potential at a higher-dimensional maximally symmetric inhomogeneous Bianchi-type string cosmological model, we broaden our understanding. We assume the constraint a(x,t) = Lb(x,t) to achieve deterministic solutions. This condition is placed on the metric potential and shear, with tension exactly proportional to scalar expansion.

The following outline is the body of this paper: Following the presentation of the metric and field equations in Sect. "The Field Equation" is a discussion of the solutions that may be derived from the field equations in Sect. "Solution of Field Equation". In the final section of this paper, Sect. "Physical and Geometrical Features", we review the results and emphasize the innovative solutions that were produced from this study, contrasting them with the answers that were offered by other writers. In addition, we investigate the geometrical and physical aspects of the higher-dimensional model in-depth, illuminating the particular qualities that distinguish it from other models.

The field equation

We consider the metric

$$ds^2 = a^2(x,t)(dx^2 - dt^2) + b^2(x,t)dy^2 + c^2(x,t)dz^2 \tag{1}$$

where $a(x,t)$, $b(x,t)$ and $c(x,t)$ are depending on the variables x and t. Einstein's field equation for a cloud of the string is

$$R_i^j - \frac{1}{2}Rg_i^j = T_i^j \tag{2}$$

where

$$T_i^j = -\rho\, u_i u^j + \lambda\, x_i x^j \tag{3}$$

where $u_i \& x_i$ satisfy the conditions

$$u_i u^j = -x_i x^j = -1 \tag{4}$$

And

$$x_i u^i = 0 \tag{5}$$

The energy of the cloud string at rest is ρ

$$\rho = \rho_p + \lambda \tag{6}$$

The energy consistency of rest particles connected by string and tension consistency λ characterize the string. The condition is determined by the direction of anisotropy, the space-like unit vector, and the matter's four-velocity vector.

$$g_{ij}\, u^i u^j = -1 \tag{7}$$

Einstein's field Eq. (2), the line element (1) leads to a system of equations

$$\frac{b_{44}}{b} + \frac{c_{44}}{c} - \frac{a_4 b_4}{ab} - \frac{a_4 c_4}{ac} - \frac{a_1 b_1}{ab} - \frac{a_1 c_1}{ac} - \frac{b_1 c_1}{bc} + \frac{b_4 c_4}{ba} = \lambda a^2 \tag{8}$$

$$\frac{a_{44}}{a} + \left(\frac{a_4}{a}\right)^2 - \frac{a_{11}}{a} + \left(\frac{a_1}{a}\right)^2 + \frac{c_{44}}{c} - \frac{c_{11}}{c} = 0 \tag{9}$$

$$\frac{a_{44}}{a} + \left(\frac{a_4}{a}\right)^2 - \frac{a_{11}}{a} + \left(\frac{a_1}{a}\right)^2 + \frac{b_{44}}{b} - \frac{b_{11}}{b} = 0 \tag{10}$$

$$-\frac{b_{11}}{b} - \frac{c_{11}}{c} - \frac{a_1 b_1}{ab} + \frac{a_1 c_1}{ac} + \frac{a_4 b_4}{ab} + \frac{a_4 c_4}{ac} - \frac{b_1 c_1}{bc} + \frac{b_4 c_4}{bc} = \rho a^2 \tag{11}$$

$$\frac{b_{14}}{b} + \frac{c_{14}}{c} - \frac{a_4 b_1}{ab} - \frac{a_4 c_1}{ac} - \frac{a_1 b_4}{ab} - \frac{a_1 c_4}{ac} = 0 \tag{12}$$

where the sub-indices $_1$ and $_4$ in a, b, and c denote partial derivative w.r.t variables x and t.

Scalar expansion, shear scalar acceleration vector, and spital appropriate volume expressions.

$$\theta = \frac{1}{a}\left(\frac{a_4}{a} + \frac{b_4}{b} + \frac{c_4}{c}\right) \tag{13}$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{3}\theta^2 - \frac{1}{a^2}\left(\frac{a_4 b_4}{ab} + \frac{b_4 c_4}{bc} + \frac{c_4 a_4}{ca}\right) \tag{14}$$

$$\dot{u}_i = \left(\frac{a_1}{a}, 0, 0, 0\right) \tag{15}$$

$$V^3 = \sqrt{-g} = a^2 bc \tag{16}$$

Solution of field equation

String equations solve non-linear partial differential equations for general inhomogeneities and relativistic cosmologies. To limit inhomogeneities, we must incorporate space–time metric symmetries. We solve field equations and others with unknown variables. By Eqs. (9) and (10), we get

$$2\left(\frac{a_4}{a}\right)_4 - 2\left(\frac{a_1}{a}\right)_1 + \frac{b_{44}}{b} - \frac{b_{11}}{b} + \frac{c_{44}}{c} - \frac{c_{11}}{c} = 0 \tag{17}$$

To find a definitive solution. Let's suppose that the model's growth is directly proportional to the shear tensor with the condition

$$a(x,t) = L\, b(x,t) \tag{18}$$

where

$$b(x,t) = f(x).g(t) \text{ and } c(x,t) = h(x).k(t) \tag{19}$$

Using Eqs. (18) and (17)

$$3\left(\frac{b_{44}}{b}\right) - 2\left(\frac{b_4}{b}\right)^2 + \frac{c_{44}}{c} = 3\left(\frac{b_{11}}{b}\right) - 2\left(\frac{b_1}{b}\right)^2 + \frac{c_{11}}{c} = m \tag{20}$$

We have by Eqs. (18) and (17)

$$\frac{b_{14}}{b} + \frac{c_{14}}{c} - \frac{a_4 b_1}{ab} - \frac{a_1 c_4}{ac} - \frac{a_1 b_4}{ab} - \frac{a_4 c_1}{ac} = 0$$

$$\frac{b_{14}}{b} + \frac{c_{14}}{c} - 2\frac{b_4 b_1}{b^2} - \frac{c_1 b_4}{bc} - \frac{b_1 c_4}{bc} = 0 \tag{21}$$

By Eqs. (19) and (21)

$$\frac{h_1 k_4}{h k} - \frac{h_1 g_4}{h g} - \frac{f_1 k_4}{f k} - \frac{f_1 g_4}{f g} = 0 \tag{22}$$

$$\frac{\frac{h_1}{h}}{\frac{f_1}{f}} = \frac{\frac{g_4}{g} + \frac{k_4}{k}}{\frac{k_4}{k} - \frac{g_4}{g}} = n \tag{23}$$

$$\frac{\frac{h_1}{h}}{\frac{f_1}{f}} = n \quad \Rightarrow \quad h = \alpha f^n \tag{24}$$

$$\frac{\frac{g_4}{g} + \frac{k_4}{k}}{\frac{k_4}{k} - \frac{g_4}{g}} = n \quad \Rightarrow \quad k = \beta g^\gamma \quad \text{where } \gamma = \frac{n+1}{n-1} \tag{25}$$

where α, β and γ are integrating constant.

From Eq. (20)

$$3\left(\frac{b_{44}}{b}\right) - 2\left(\frac{b_4}{b}\right)^2 + \frac{c_{44}}{c} = m \tag{26}$$

Using $b(x,t) = f(x).g(t)$.

We obtain,

$$gg_{44} + R(g_4)^2 = Dg^2 \tag{27}$$

Which leads to

$$g = C_2^{\frac{1}{R+1}} \sinh^{\frac{1}{R+1}}(Bt + t_0) \tag{28}$$

where $B = \sqrt{D(R+1)}$.

Again, from Eq. (20)

$$3\left(\frac{b_{11}}{b}\right) - 2\left(\frac{b_1}{b}\right)^2 + \frac{c_{11}}{c} = m \tag{29}$$

We obtain,

$$ff_{11} + \gamma(f_1)^2 = Nf^2 \tag{30}$$

Which leads to

$$f(x) = C_5^{\frac{1}{\gamma+1}} \sinh^{\frac{1}{\gamma+1}}(Kx + x_0) \tag{31}$$

where $K = \sqrt{N(\gamma+1)}$.

Hence $b(x,t) = f(x)g(t)$

$$g(t) = C_2^{\frac{1}{R+1}} \sinh^{\frac{1}{R+1}}(Bt + t_0) \tag{32}$$

$$f(x) = C_5^{\frac{1}{\gamma+1}} \sinh^{\frac{1}{\gamma+1}}(Kx + x_0) \tag{33}$$

$$b(x,t) = P \sinh^{\frac{1}{1+\gamma}} KX \sinh^{\frac{1}{1+R}} BT = \frac{a}{L} \tag{34}$$

where $P = C_2^{\frac{1}{R+1}} C_5^{\frac{1}{\gamma+1}}$, $X = Kx + x_0$, $T = Bt + t_0$

$$c = Q \sinh^{\frac{n}{1+\gamma}} KX \sinh^{\frac{\delta}{1+R}} BT \tag{35}$$

$$\begin{aligned} ds^2 =& P^2 \sinh^{\frac{2}{1+\gamma}} KX \sinh^{\frac{2}{1+R}} BT(L^2(dx^2 - dt^2) + dy^2) \\ &+ (Q^2 \sinh^{\frac{2n}{1+\gamma}} KX \sinh^{\frac{2\delta}{1+R}} BT)dz^2 \end{aligned} \tag{36}$$

Physical and geometrical features

The geometrical and physical valuables are given as follows:

String tension consistency λ is

$$\begin{aligned} \lambda =& \frac{B(1+B)}{L^2 b^2 (R+1)} + \frac{1}{L^2 b^2}\left[\left(\frac{\gamma(\gamma-1)-1}{(R+1)^2} - \frac{RB^2(1+B)}{(R+1)^2}\right)\right. \\ & \left.\coth^2(Bt + t_0) - \frac{(2n+1)K^2}{(\gamma+1)^2}\coth^2(Kx + x_0)\right] \end{aligned} \tag{37}$$

The energy consistency 'ρ' is

$$\begin{aligned} \rho =& \frac{1}{L^2 b^2}\left[\frac{(1+2\delta)B^2}{(1+R)^2}(\coth(Bt + t_0))^2 + (\gamma(1-n) + n(1-n) + 1)\right. \\ & \left.\frac{K^2}{(1+\gamma)^2}(\coth(Kx + x_0))^2 + \frac{(1-n)K^2}{(\gamma+1)}\right] \end{aligned} \tag{38}$$

The expansion scalar (θ) of the model is given by

$$\theta = \frac{(2 + \beta\gamma)B\cot(Bt + t_0)}{(R+1)Lb} \tag{39}$$

The shear scalar (σ) of the model is given by

$$\sigma^2 = \left[\frac{(2+\beta\gamma)^2 - 3(\gamma+1)}{3(R+1)^2L^2b^2}\right]B^2 \coth(Bt+t_0) \tag{40}$$

The spital volume is

$$V^3 = a^2bc \tag{41}$$

From the Eqs. (39) and (40), we obtain

$$\frac{\sigma}{\theta} = \frac{3(2+\beta\gamma)}{\left[(2+\beta\gamma)^2 - 3(\gamma+1)\right]^{1/2}} = \text{Constant} \tag{42}$$

The angular velocity ω becomes zero i.e. $\omega = 0$.

Summary and review

In conclusion, Initially at T = 0 the model (36) starts expanding with a big bang at T = 0 and goes on an expansion till $T = \infty$ when θ becomes zero. When T is increased, the ratio of the expansion scalar (θ) to the shear scalar (σ) eventually reaches a finite value, suggesting that the ratio remains constant. i.e., $\frac{\sigma}{\theta} \rightarrow$ remains constant, hence, the model fails to exhibit isotropy. The analysis reveals that the finite flow is irrotational, and interestingly, as the values of parameters $T \rightarrow 0$ and $X \rightarrow 0$, the energy density $\rho \rightarrow \infty$ as $T \rightarrow \infty$ and $X \rightarrow \infty$, $\rho \rightarrow 0$ As $T \rightarrow 0$ and $X \rightarrow 0$, while the tension density increases $\lambda \rightarrow \infty$ as $T \rightarrow \infty$ and $X \rightarrow \infty$, $\lambda \rightarrow 0$ tension density decreases, it is observed that the time continues to increase, leading to the eventual disappearance of the cosmic string from existence. Moreover, the model signifies an accelerating universe, as indicated by the negative value of the deceleration parameter. This finding suggests that the expansion of the universe is not slowing down but rather speeding up. Overall, the investigated model provides insights into the dynamics of anisotropic Bianchi Type I string cosmology, highlighting the limitations regarding isotropy and the eventual fate of cosmic strings. The observed accelerating nature of the universe adds to our understanding of the overall evolution and dynamics of the cosmos. Further research and exploration in this field can contribute to a deeper comprehension of the underlying mechanisms that govern our universe's behavior.

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