ON SYMMETRICAL SPACE WITH MINIMUM RATE OF EXPANSION.

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1. Introduction.

In a previous publication[1] it has been shown that all symmetrical spaces of the type

$$ds^2 = -e^{\lambda(r,t)}dr^2 - e^{\omega(r,t)}(d\theta^2 + \sin^2\theta d\phi^2) + dt^2 \qquad \ldots \qquad \ldots \qquad (1)$$

containing incoherent dustlike matter at rest, which can be obtained by continuous deformation (preserving the continuity of metric and density) of a Friedmann space with negligible pressure, have initial rates of expansion or contraction greater than the rate of expansion or contraction of this Friedmann space. Since in the universe with negligible pressure the percentage rate of expansion is proportional to the percentage rate of diminution of material density, the above minimum property also means that the homogeneous Friedmann space possesses the slowest rate of material attenuation (density diminution), compared to heterogeneous spaces with negligible pressure of the type (1), which momentarily have their metric and density identical with those of that Friedmann space. Here all spaces compared have initially a uniform distribution of density, and it may be suspected that the minimum property of the Friedmann space is a consequence of this uniformity. In this paper this theorem has been generalized for any initial distribution of density and metric, both of which are supposed to change in a continuous manner, without producing motion of matter, whilst the metric is always of the symmetrical type (1) and pressure negligible. In the first place, it is to be realized that for arbitrarily given metric and density, the rate of expansion need not be a minimum simultaneously at every point in space at the instant under consideration. The problem we have proposed has an interest only when such a minimum rate exists simultaneously for the whole of the space, and so we should expect that some restriction is to be put on the given metric and density. It is not difficult to see that this minimum rate will be uniform throughout space and independent of r. It is here proved that this is achieved by a uniform value of initial $\dot{\omega}$.

The bearing of this problem on Lemaitre's Cosmology is apparent and it throws some light on the minimum property of the homogeneous space.

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[1] N. R. Sen, *Zs. f. Astrophysik*, Vol. 9, p. 215 (1934).

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A formal proof is given at the end showing that the homogeneous Friedmann space is unique in this respect that it is only in this case that the time history of the space follows the path of minimum rate of expansion subject to the conditions of continuity.

2. The Field Equations.

The field equations in this case can be solved completely.[1] It can be shown that the condition that the matter on the whole is at rest gives the integral [2]

$$e^{\lambda}=\frac{e^{\omega}\cdot\omega'^2/4}{f^2(r)} \qquad \ldots \qquad (2)$$

where $f(r)$ is an arbitrary function of r, and $\omega'=(\partial\omega/\partial r)$. The other integrals are given by

$$e^{\omega}(\ddot{\omega}+\tfrac{3}{4}\dot{\omega}^2-\Lambda)+\{1-f^2(r)\}=0 \qquad \ldots \qquad (3)$$

$$e^{3\omega/2}(\tfrac{1}{2}\dot{\omega}^2-\tfrac{2}{3}\Lambda)+2e^{\omega/2}\{1-f^2(r)\}=F(r) \qquad \ldots \qquad (4)$$

where $F(r)$ is another arbitrary function of r, Λ the cosmological constant, and $\dot{\omega}=(\partial\omega/\partial t)$.

The density ρ is given by

$$8\pi\rho=\frac{e^{-3\omega/2}}{\omega'}\frac{\partial F(r)}{\partial r} \qquad \ldots \qquad (5)$$

so that

$$F(r)=8\pi\int\rho e^{3\omega/2}.\ \omega' dr+C=A(r)+C \qquad \ldots \qquad (5')$$

The metric and density at time $t=0$ being given, we take $f(r)$, $\omega_0(r)$, ρ, hence $A(r)$ to be suitably given initially. The only unknown element is C, so that *there is a single infinity of spaces of the type* (1) *possible, consistent with continuity of metric and density.* The rate of expansion at any point in space is given by the function

$$\tfrac{3}{2}\dot{\omega}_0+\frac{\dot{\omega}'_0}{\omega'_0} \qquad \ldots \qquad (6)$$

which also measures the percentage rate of decrease of density, $-(\partial \log \rho/\partial t)$ at the point. The problem is to find if, and when, (6) has a minimum at every point r for a definite value of the constant C which here is the only varying parameter.

We have taken the initial moment to be $t=0$ when $\omega_0(=\omega_{t=0})$, $f(r)$ and ρ are given. C in (5′) is an absolute constant. Even when we take $t=t_0$ to be the initial moment, ω_0 and ρ will involve t_0, but in any real case corresponding to a solution of the field equations (3) and (4), C will be an absolute constant not involving t_0, as $F(r)$ is a function of r only. The type of space

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[1] R. C. Tolman, *Proc. Nat. Acad. Science.* Vol. 20, p. 169 (1934). This paper contains the deduction of the equations of §2.

[2] R. C. Tolman, *loc. cit.*

will be completely determined by the value of $F(r)$ which again depends on the constant C. In accordance with (4) we may also say that the initial value of $\dot{\omega}$, which we call $\dot{\omega}_0$ will determine the space completely.

3. Minimum Rate of Expansion.

From equation (4) we have

$$\dot{\omega}_0^2=\tfrac{4}{3}\Lambda-4e^{-\omega_0}(1-f^2)+2e^{-3\omega_0/2}[A(r)+C]\,. \qquad (4')$$

Calculating from this the usual rate of expansion (6) we find it to be of the form

$$\frac{P+3\alpha C}{(Q+4\alpha C)^{\frac{1}{2}}} \qquad (6')$$

where

$$\left.\begin{aligned} P&=2\Lambda+\tfrac{3}{2}e^{-3\omega_0/2}\,.\,A(r)-4e^{-\omega_0}(1-f^2)+4e^{-\omega_0}\frac{ff'}{\omega'_0}+8\pi\rho \\ Q&=\tfrac{4}{3}\Lambda+2e^{-3\omega_0/2}\,.\,A(r)-4e^{-\omega_0}(1-f^2) \\ \alpha&=\tfrac{1}{2}e^{-3\omega_0/2}\,. \end{aligned}\right\} \qquad (7)$$

The first condition to which the metric is to be subjected is

$$Q+4\alpha C>0 \qquad (8)$$

which is necessary for real $\dot{\omega}_0$. Only those values of the constant C are permissible which are consistent with (8). Since we are interested in the absolute value of (6′) we may look for the extrema of

$$Z=\frac{(P+3\alpha C)^2}{Q+4\alpha C}. \qquad (6')$$

The minimum sought should be for every point in space, for varying values of C. Consider *a fixed point in space* where P, Q and α will have definite values at $t=0$. Let us first find for which value of C, Z will have an extremum at that point. Calculating the derivative we have

$$\frac{dZ}{dC}=(C-C_1)(C-C_2)\times(\text{a quantity essentially positive})$$

where

$$C_1=-P/3\alpha,\ C_2=(2P-3Q)/6\alpha\,.$$

If $C_2>C_1$

$$4P-3Q>0 \qquad (9)$$

we shall have by (8)

$$C_2>-(Q/4\alpha)>C_1 \qquad (\alpha>0)$$

and there will be actually a minimum of Z for $C=C_2$ (C_1 being forbidden). Substituting from (7) we shall then have the following value for $C_2=C_m$ for a minimum

$$-A(r)+\tfrac{4}{3}e^{\omega_0/2}(1-f^2)+\tfrac{8}{3}e^{\omega_0/2}\frac{ff'}{\omega'_0}+\tfrac{16}{3}\pi\rho e^{3\omega_0/2}=C_m\,. \qquad (10)$$

This minimum value of the rate of expansion (contraction) $\sqrt{Z}$ is

$$\frac{\sqrt{3}}{2}(4P-3Q)^{\frac{1}{2}}. \quad .. \quad .. \quad .. \quad .. \quad (11)$$

The minimum has been obtained *for a single point.* Now, if the given metric and density be such that the left hand side of (10) is an absolute constant for every point in space, and (9) is satisfied also for every point, a minimum rate of expansion or contraction will exist for the space and this minimum rate is (11). On substitution from (7), this minimum rate is found to be $\frac{3}{2}(\dot{\omega}_0)_m$ where $(\dot{\omega}_0)_m$ is the value of $\dot{\omega}_0$ in (4′) for $C=C_m$, given in (10). The expression (6) for the rate of expansion shows that for minimum rate $\dot{\omega}'_0=0$. Hence $(\dot{\omega}_0)_m$ is independent of r and so is uniform throughout the space. The minimum rate of expansion (contraction) is thus also uniform throughout space. It can also be verified by actual calculation from (4), (7) and (11) that the minimum rate is actually $\frac{3}{2}(\dot{\omega}_0)_m$ and that $(\dot{\omega}'_0)_m$ vanishes.

If $C_1>C_2$, $4P-3Q<0$, it can be shown that

$$C_1>-(Q/4\alpha)>C_2.$$

The Z curve will touch the C axis at C_1 (the value C_2 is not now permissible), so that the minimum value of Z is zero. But $\sqrt{Z}$ will not now have a turning point, it merely changes sign at zero. Hence this case does not correspond to our minimum problem. It refers to the case when the contracting space passes into an expanding space or vice versa, when the rate of expansion vanishes momentarily. The necessary condition is that $P/3\alpha$ is an absolute constant throughout space. The case $C_1=C_2$ corresponds to zero rate of expansion and does not give anything new.

The solution of the problem we have set in §1 will be this. *If the metric and density be such that at the given instant* $t=t_0$

$$4P-3Q\equiv 4\Lambda-4e^{-\omega}(1-f^2)+16e^{-\omega}\frac{ff'}{\omega'}+32\pi\rho>0 \quad .. \quad (a)$$

and

$$-A(r)+\tfrac{1}{3}e^{\omega/2}(1-f^2)+\tfrac{8}{3}e^{\omega/2}\frac{ff'}{\omega'}+\tfrac{16}{3}\pi\rho e^{3\omega/2}=C \quad .. \quad (b)$$

identically, where C *is a constant, the space* (1) *will have at that instant a minimum rate of expansion or contraction. Further the value of* $\dot{\omega}$ *in* (4′) *will then be independent of* r *in virtue of* (b). *The minimum rate of expansion or contraction will be* $\sqrt{3}/2(4\mathrm{P}-3\mathrm{Q})^{\frac{1}{2}}$ *and uniform throughout space.*

Note that every uniform rate of expansion will not correspond to the case of minimum rate, since (6) equated to $\frac{3}{2}C$ has the solution

$$\dot{\omega}_0=C+D(t_0)e^{-3\omega_0/2},$$

and only the case $D=0$ corresponds to our minimum.

In the case of the Friedmann space (with $p=0$), (a) corresponds to the relation

$$\Lambda-\frac{3e^{-g}}{R_0^{\,2}}+8\pi\rho>0,$$

which is satisfied, since from the field equations [1] we have the left hand side equal to $\frac{3}{4}\dot{g}^2$. The relation (b) in case of the Friedmann space corresponds to $C=0$ as the left hand side vanishes identically.

Further, for the Friedmann space we find

$$P=\tfrac{3}{2}\dot{g}^2,\ 4P-3Q=3\dot{g}^2.$$

We can have an example of the second case discussed above in the expanding-contracting (or vice versa) type of the Friedmann space, which after momentary rest (when $\dot{g}=0$), changes from one type to another. We find that $P/3\alpha$ is an absolute constant, namely zero, in this case. This, in fact, is an example of the limiting case $C_1=C_2$ as $4P-3Q$ vanishes.

4. Case of Friedmann Space.

The relation (b) is not, in general, an integral of the field equations so that the space with minimum rate of expansion (contraction) here has reference only to initial motion for given metric and density. It can now be shown that there is only one space for which this minimum property (subject only to the condition of continuity of metric and density) is satisfied at every instant of time, and this is the homogeneous Friedmann space. Firstly, for such a space the condition (b) will be satisfied for every instant of time, so that the time derivative of the left hand side must vanish.[2] From this using (5) and remembering that $\dot{\omega}'$ vanishes under the circumstances assumed, we have

$$(1-f^2)+\frac{2ff'}{\omega'}=0 \qquad \ldots \quad \ldots \quad \ldots \quad (12)$$

the integral of which is

$$e^{\omega}=(1-f^2)e^{c'(t)} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12')$$

Equation (10) can be now written as

$$-A(r)+\frac{2}{3}\frac{F'(r)}{\omega'}=C_m \qquad \ldots \quad \ldots \quad \ldots \quad (13)$$

which is satisfied in virtue of (5′). Further, from (4) we have by substituting from (12′)

$$\dot{\omega}^2=\tfrac{4}{3}\Lambda-4e^{-c'(t)}+2[A(r)+C]\frac{e^{-\frac{3}{2}c'(t)}}{(1-f^2)^{\frac{3}{2}}}.$$

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[1] R. C. Tolman, Relativity, Thermodynamics, Cosmology.

[2] C in a particular solution is an absolute constant.

Since $\dot{\omega}$ is independent of r we must have

$$F(r)=A(r)+C=B(1-f^2)^{\frac{3}{2}} \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

where B is a constant. Again, from (5′) we have

$$A'(r)=8\pi\rho e^{3\omega/2}\omega'$$

and substituting in this relation for $A'(r)$ from (14) and ω' from (12) we have

$$8\pi\rho=\tfrac{3}{2}Be^{-\frac{3}{2}c'(t)},$$

so that ρ is a function of t only. This really corresponds to the Friedmann space. The theorem stated above now follows.

SUMMARY.

Conditions under which a deformed Friedmann-Lemaitre model of the universe with negligible pressure but preserving spherical symmetry can possess at any instant and all throughout space the slowest rate of diminution of density have been investigated. The homogeneous model is the only one in which these conditions are satisfied at every instant.

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