ON THE PROPAGATION OF ELECTROMAGNETIC WAVES THROUGH THE EARTH'S ATMOSPHERE. (Paper 1).

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Introduction.

The introduction to this paper has already been given in another with the same heading published in these *Proceedings* (Vol. III, p. 359, 1937) henceforth called paper 2. The present paper deals with the derivation of the equations given in § 2, pp. 363 and 364 of paper 2. There only a bare statement of the equations was given; here the exact procedure of their derivation is given. The programme of these series of papers may be defined as the wave-treatment of the problem of propagation dealing with questions of polarization, reflection, oblique propagation, and absorption of the waves. We first give a fuller description of the notation employed.

Notation.

The notation used in this field of investigation differs so widely from one author to another that the reading of papers by different authors is attended with a certain amount of difficulty. It is desirable that a system of international notations be agreed upon. In this paper, an attempt has been made to use a system of symbols which may be acceptable to the different schools of investigators. We have, as far as possible, adhered to the symbols used by S. K. Mitra, which were mostly adopted from the writings of Appleton and his school. In some points we have found it necessary to deviate from Mitra's symbols. An explanation of the system of symbols is therefore given at the beginning.

E	..	Electric Field Intensity with components	.. E_x, E_y, E_z.
D	..	Electric Displacement Vector ..	.. D_x, D_y, D_z.
H	..	Magnetic Field Vector ..	.. H_x, H_y, H_z.
B	..	Magnetic Polarization.	
P	..	Polarization	.. P_x, P_y, P_z.
h	..	Earth's magnetic field components	.. h_x, h_y, h_z.

(Mitra has used H for this quantity. But it produces a certain amount of confusion with the magnetic field vector, hence we have used 'h' to denote the earth's field.)

p_h .. Larmor Frequency $\frac{eh}{mc}$.

p_x, p_y, p_z Components of Larmor Frequency.

p .. Pulsatance of the electromagnetic wave.

N .. Number of electrons, or ions per c.c. at any height. Whenever necessary the subscript 'e' for electron, 'i' for ion is affixed to N. Thus N_e denotes number of electrons. But usually the suffix is not used.

$p_0^2 = \frac{4\pi Ne^2}{m}$ either for electrons or ions.

ν .. collisional frequency, i.e. number of collisions made by an ion or electron in unit time.

$(\omega_x, \omega_y, \omega_z) = \frac{1}{p}(p_x, p_y, p_z)$, ω is the resultant value of $(\omega_x\ \omega_y\ \omega_z)$

$$= \frac{1}{p}(p_h) = \frac{eh}{mcp}.$$

$$r = \frac{p_0^2}{p^2} = \frac{4\pi Ne^2}{mp^2}.$$

$$\beta = 1 - \frac{i\nu}{p} = 1 - i\delta.$$

q = Complex Refractive Index.

$$= \mu - i\chi.$$

§ 1.

The Fundamental Equations.

The fundamental equations for the propagation of electromagnetic waves are :—

$$\left.\begin{aligned} \text{Curl } H &= \frac{1}{c}\dot{D} \\ \text{Curl } E &= -\frac{1}{c}\dot{B} \\ \text{Div } D &= 4\pi\rho \\ \text{Div } B &= 0 \\ D = kE &= E + 4\pi P \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

For the present case, we take $\mu = 1$ so that

$$B = \mu H = H.$$

From equation (1.1) it can be deduced in the usual way that

$$\left.\begin{aligned} \nabla^2 H &= \frac{1}{c^2}\,\ddot{H} - \frac{4\pi}{c}\,\text{Curl}\,\dot{P} \\ \text{and}\qquad \nabla^2 E &= \frac{1}{c^2}\,\ddot{E} + \frac{4\pi}{c^2}\,\ddot{P} - 4\pi\,\text{grad div}\,P \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1.2)$$

We have now to express P in terms of E.

Let us suppose that on account of the e.m. field of the radio wave the charged particles suffer the displacement ξ, η, ζ. The equations of motion of the charged particles are given by

$$\left.\begin{aligned} m\ddot{\xi} &= eE_x - g\dot{\xi} + \frac{e}{c}\,(\dot{\eta}h_z - \dot{\zeta}h_y) + aP_x \\ m\ddot{\eta} &= eE_y - g\dot{\eta} + \frac{e}{c}\,(\dot{\zeta}\,h_x - \dot{\xi}h_z) + aP_y \\ m\ddot{\zeta} &= eE_z - g\dot{\zeta} + \frac{e}{c}\,(\dot{\xi}h_y - \dot{\eta}h_x) + aP_z \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1.3)$$

Here $-g(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ represents the frictional force due to collisions.

The third term.. $\frac{e}{c}\,(v \times h)$ represents the deflecting force due to the earth's magnetic field.

We have further $P = Ne\,(\xi, \eta, \zeta)$, where N is the number of electrically charged particles per unit volume.

The last term in (1.3) represents the action of the polarization forces. It is now usual to take $a = 0$.

We shall replace (ξ, η, ζ) by $\frac{1}{Ne}\,(P_x, P_y, P_z)$ throughout (1.3), and let us further suppose that P is proportional to e^{ipt}. Then the equations (1.3) reduce to

$$-\frac{mp^2}{Ne}\,P_x = eE_x - \frac{igp}{Ne}\,P_x + \frac{ip}{Nc}\,(P \times h)_x$$

and two other similar equations.

The form of the equations can be much simplified if we make the following substitutions :—

$$\left.\begin{aligned} -\frac{mp^2}{Ne^2} &= -\frac{4\pi p^2}{p_0^2} = -\frac{4\pi}{r} \\ \frac{gp}{Ne^2} &= \frac{g}{m}\,\frac{p}{Ne^2/m} = \frac{4\pi}{r}\,\frac{\nu}{p} \\ \frac{ph}{eNc} &= \frac{eh}{mc}\,\frac{4\pi p}{4\pi Ne^2/m} = \frac{4\pi p p_h}{p_0^2} = \frac{4\pi\omega}{r} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1.4)$$

Then the equations take the form

$$\left.\begin{aligned} \beta P_x + i\omega_z P_y - i\omega_y P_z &= -\frac{r}{4\pi} E_x \\ -i\omega_z P_x + \beta P_y + i\omega_x P_z &= -\frac{r}{4\pi} E_y \\ i\omega_y P_x - i\omega_x P_y + \beta P_z &= -\frac{r}{4\pi} E_z \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1.5)$$

These equations can be easily solved by the usual algebraic methods. We can put

$$\left.\begin{aligned} \frac{P_x}{A} &= \Delta_{11}E_x + \Delta_{21}E_y + \Delta_{31}E_z \\ \frac{P_y}{A} &= \Delta_{12}E_x + \Delta_{22}E_y + \Delta_{32}E_z \\ \frac{P_z}{A} &= \Delta_{13}E_x + \Delta_{23}E_y + \Delta_{33}E_z \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1.6)$$

where $$A = \frac{r}{4\pi\beta}\,\frac{1}{\beta^2-\omega^2}.$$

Δ_{rs}, etc. are the subdeterminants of the determinant formed by the coefficients of the quantities P_x, P_y, P_z in equations (1.5).

It can be easily shown that

$$\left.\begin{aligned} \Delta_{11} &= \omega_x^2-\beta^2 & \Delta_{21} &= \omega_x\omega_y + i\beta\omega_z & \Delta_{12} &= \omega_x\omega_y - i\beta\omega_z \\ \Delta_{22} &= \omega_y^2-\beta^2 & \Delta_{32} &= \omega_z\omega_y + i\beta\omega_x & \Delta_{23} &= \omega_z\omega_y - i\beta\omega_x \\ \Delta_{33} &= \omega_z^2-\beta^2 & \Delta_{13} &= \omega_x\omega_z + i\beta\omega_y & \Delta_{31} &= \omega_x\omega_z - i\beta\omega_y \end{aligned}\right\} \quad (1.7)$$

These results will be utilized later.

§ 2.

Propagation along the z-axis.

We shall first consider the propagation of the rays along the z-axis. Then the second set of equations (1.2) reduces to the three equations :—

$$\left.\begin{aligned} \frac{d^2E_x}{dz^2} - \frac{1}{c^2}\frac{d^2E_x}{dt^2} &= \frac{4\pi}{c^2}\frac{d^2P_x}{dt^2} \\ \frac{d^2E_y}{dz^2} - \frac{1}{c^2}\frac{d^2E_y}{dt^2} &= \frac{4\pi}{c^2}\frac{d^2P_y}{dt^2} \\ \frac{d^2E_z}{dz^2} - \frac{1}{c^2}\frac{d^2E_z}{dt^2} &= \frac{4\pi}{c^2}\frac{d^2P_z}{dt^2} - 4\pi\frac{d^2P_z}{dz^2} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.1)$$

for the terms arising out of Grad Div P are now simplified as

$$\frac{\partial}{\partial x}(\text{Div } P) = 0,\ \frac{\partial}{\partial y}(\text{Div } P) = 0 \text{ and } \frac{\partial}{\partial z}(\text{Div } P) = \frac{d^2P_z}{dz^2}.$$

Further ∇^2 reduces to $\frac{d^2}{dz^2}$.

The third of equations (2.1) is

$$\left(\frac{d^2}{dz^2}-\frac{1}{c^2}\frac{d^2}{dt^2}\right)(E_z+4\pi P_z) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

Now from the condition Div $D = 0$, we have

$$\frac{\partial}{\partial z}\,[E_z+4\pi P_z] = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

From (2.2) and (2.3), we have

$$D_z = E_z+4\pi P_z = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

As according to (1.6), P_z is a linear function of E_x, E_y, E_z, equation (2.4) enables us to express E_z in terms of E_x and E_y. We have

$$E_z+4\pi A\ (\Delta_{13}E_x+\Delta_{23}E_y+\Delta_{33}E_z) = 0$$

or

$$E_z = -\frac{\Delta_{13}E_x+\Delta_{23}E_y}{\left(\Delta_{33}+\frac{1}{4\pi A}\right)} \quad \ldots \quad \ldots \quad \ldots \quad (2.5)$$

The equations (2.1) can now be put in the form :—

$$\left.\begin{aligned} \frac{d^2E_x}{dz^2} &= \frac{K_{11}}{c^2}\frac{d^2E_x}{dt^2}+\frac{K_{12}}{c^2}\frac{d^2E_y}{dt^2} \\ \frac{d^2E_y}{dz^2} &= \frac{K_{21}}{c^2}\frac{d^2E_x}{dt^2}+\frac{K_{22}}{c^2}\frac{d^2E_y}{dt^2} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.6)$$

where

$$K_{11} = 1+4\pi A\left\{\Delta_{11}-\frac{\Delta_{13}\Delta_{31}}{\Delta_{33}+\frac{1}{4\pi A}}\right\}$$

$$K_{12} = \quad 4\pi A\left\{\Delta_{21}-\frac{\Delta_{31}\Delta_{23}}{\Delta_{33}+\frac{1}{4\pi A}}\right\}$$

$$K_{21} = \quad 4\pi A\left\{\Delta_{12}-\frac{\Delta_{32}\Delta_{13}}{\Delta_{33}+\frac{1}{4\pi A}}\right\}$$

$$K_{22} = 1+4\pi A\left\{\Delta_{22}-\frac{\Delta_{23}\Delta_{32}}{\Delta_{33}+\frac{1}{4\pi A}}\right\} \quad \ldots \quad \ldots \quad (2.7)$$

It is easily seen that

$$\left.\begin{aligned} D_x &= K_{11}E_x+K_{12}E_y \\ D_y &= K_{21}E_x+K_{22}E_y \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (2.8)$$

Equations for the propagation of the magnetic vector.

Instead of taking (1.2) we take the equation (1.1). We have then

$$\nabla^2 H = -\frac{1}{c}\,\text{curl}\,\dot{D} \qquad \ldots \quad \ldots \quad \ldots \quad (2.9)$$

$\left(\text{for curl curl } H = -\nabla^2 H = \frac{1}{c}\,\text{curl}\,\dot{D}\right)$.

These equations reduce to

$$\frac{d^2H_x}{dz^2} = \frac{1}{c}\frac{d\dot{D}_y}{dz},\quad \frac{d^2H_y}{dz^2} = -\frac{1}{c}\frac{d\dot{D}_x}{dz} \qquad \ldots \quad \ldots \quad (2.10)$$

Now from (2.8) we have

$$\left.\begin{aligned} \dot{D}_x &= K_{11}\dot{E}_x+K_{12}\dot{E}_y \\ \dot{D}_y &= K_{21}\dot{E}_x+K_{22}\dot{E}_y \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (2.11)$$

since the quantities K are not functions of time.

Further from the second equation of (1.1), we have

$$\text{curl}\,E = \left(-\frac{dE_y}{dz},\ \frac{dE_x}{dz},\ 0\right) = -\frac{1}{c}\left(\frac{dH_x}{dt},\ \frac{dH_y}{dt},\ 0\right) \qquad \ldots \quad (2.12)$$

Applying these conditions to (2.10) and (2.11) we have, when the quantities K do not vary with z,

$$\left.\begin{aligned} \frac{d^2H_x}{dz^2} &= \frac{K_{22}}{c^2}\frac{d^2H_x}{dt^2} - \frac{K_{21}}{c^2}\frac{d^2H_y}{dt^2} \\ \frac{d^2H_y}{dz^2} &= -\frac{K_{12}}{c^2}\frac{d^2H_x}{dt^2} + \frac{K_{11}}{c^2}\frac{d^2H_y}{dt^2} \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (2.13)$$

Also from (2.12) and the first equation of (1.1) we have, utilizing (2.8),

$$\left.\begin{aligned} \frac{dE_x}{dz} &= -\frac{1}{c}\frac{dH_y}{dt},\quad \frac{dE_y}{dz} = \frac{1}{c}\frac{dH_x}{dt} \\ \frac{dH_x}{dz} &= \frac{K_{21}}{c}\frac{dE_x}{dt} + \frac{K_{22}}{c}\frac{dE_y}{dt} \\ \frac{dH_y}{dz} &= -\frac{K_{11}}{c}\frac{dE_x}{dt} - \frac{K_{12}}{c}\frac{dE_y}{dt} \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (2.14)$$

when $p_y = 0$, the last two reduces to

$$\left.\begin{aligned} \frac{dH_x}{dz} &= \frac{iL}{c}\frac{dE_x}{dt} + \frac{K_{22}}{c}\frac{dE_y}{dt} \\ \frac{dH_y}{dz} &= -\frac{K_{11}}{c}\frac{dE_x}{dt} + \frac{iL}{c}\frac{dE_y}{dt} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.14')$$

where $$L = -r(\beta - r)\omega_z / C'.$$

We can now calculate the quantities K from the relations (2.7). First let us suppose that the collision frequency ν can be neglected. We have then

$$4\pi A = \frac{r}{1-\omega^2}$$

$$\Delta_{33} + \frac{1}{4\pi A} = \frac{\{(1-\omega^2) - r(1-\omega_z^2)\}}{r} = \frac{C}{r}$$

where $$C = 1 - \omega^2 - r(1-\omega_z^2)$$

$$\left.\begin{aligned} \Delta_{11} - \frac{\Delta_{13}\Delta_{31}}{\Delta_{33} + \frac{1}{4\pi A}} &= \frac{(1-\omega^2)(r+\omega_x^2-1)}{C} \\ \Delta_{22} - \frac{\Delta_{23}\Delta_{32}}{\Delta_{33} + \frac{1}{4\pi A}} &= \frac{(1-\omega^2)(r+\omega_y^2-1)}{C} \\ \Delta_{21} - \frac{\Delta_{31}\Delta_{23}}{\Delta_{33} + \frac{1}{4\pi A}} &= \frac{(1-\omega^2)\{\omega_x\omega_y + i\omega_z(1-r)\}}{C} \\ \Delta_{12} - \frac{\Delta_{13}\Delta_{32}}{\Delta_{33} + \frac{1}{4\pi A}} &= \frac{(1-\omega^2)\{\omega_x\omega_y - i\omega_z(1-r)\}}{C} \end{aligned}\right\} \quad \ldots \quad (2.15)$$

From these expressions, we obtain the following values for the K's.

$$\left.\begin{aligned} K_{11} &= 1 + r\frac{\omega_x^2 - 1 + r}{C} \\ K_{22} &= 1 + r\frac{\omega_y^2 - 1 + r}{C} \\ K_{12} &= r\frac{\omega_x\omega_y + i(1-r)\omega_z}{C} \\ K_{21} &= r\frac{\omega_x\omega_y - i(1-r)\omega_z}{C} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2.16)$$

The expressions are considerably simplified if we put $p_y = \omega_y = 0$. This means that we are taking the magnetic meridian as our $(x-z)$-plane.

We have now

$$\left.\begin{aligned} K_{11} &= 1+r\,\frac{\omega_x^2-1+r}{C} = \frac{(1-r)(1-r-\omega^2)}{C} \\ K_{22} &= 1+r\,\frac{r-1}{C} \\ K_{12} &= -K_{21} = -iL \text{ where } L = -\frac{r(1-r)\omega_z}{C} \end{aligned}\right\} \quad .. \quad (2\cdot17)$$

When collisions are taken into account, it can be proved after some work that we have now the following relations :

$$\Delta_{33}+\frac{1}{4\pi A} = \frac{1}{r}\left\{\beta(\beta^2-\omega^2)-r(\beta^2-\omega_z^2)\right\} = \frac{C'}{r}$$

where $$C' = \beta(\beta^2-\omega^2)-r(\beta^2-\omega_z^2) \quad .. \quad .. \quad .. \quad (2.18)$$

$$\left.\begin{aligned} \Delta_{11}-\frac{\Delta_{13}\Delta_{31}}{\Delta_{33}+\frac{1}{4\pi A}} &= \frac{\beta(\beta^2-\omega^2)\{r\beta+\omega_x^2-\beta^2\}}{C'} \\ \Delta_{22}-\frac{\Delta_{23}\Delta_{32}}{\Delta_{33}+\frac{1}{4\pi A}} &= \frac{\beta(\beta^2-\omega^2)(r\beta+\omega_y^2-\beta^2)}{C'} \\ \Delta_{21}-\frac{\Delta_{31}\Delta_{23}}{\Delta_{33}+\frac{1}{4\pi A}} &= \frac{\beta(\beta^2-\omega^2)\{\omega_x\omega_y+i\omega_z(\beta-r)\}}{C'} \\ \Delta_{12}-\frac{\Delta_{13}\Delta_{32}}{\Delta_{33}+\frac{1}{4\pi A}} &= \frac{\beta(\beta^2-\omega^2)\{\omega_x\omega_y-i\omega_z(\beta-r)\}}{C'} \end{aligned}\right\} \quad .. \quad (2\cdot19)$$

It is easy to see that the relations (2.15) are deducible from (2.19) when collisions are neglected, i.e. $\beta = 1$.

We can now write out values of K's from the above relations. We have

$$\left.\begin{aligned} K_{11} &= 1-r\,\frac{\beta^2-r\beta-\omega_x^2}{C'} \\ K_{22} &= 1-r\,\frac{\beta^2-r\beta-\omega_y^2}{C'} \\ K_{12} &= \frac{r\{\omega_x\omega_y+i\omega_z(\beta-r)\}}{C'} \\ K_{21} &= \frac{r\{\omega_x\omega_y-i\omega_z(\beta-r)\}}{C'} \end{aligned}\right\} \quad .. \quad .. \quad (2.20)$$

Considerable simplification is introduced by putting $p_y = \omega_y = 0$. We have now

$$\left.\begin{aligned} K_{11} &= 1 - r\frac{\beta^2 - r\beta - \omega_x^2}{C'} = \frac{(r-\beta)(\omega^2 + r\beta - \beta^2)}{C'} \\ K_{22} &= 1 - r\frac{\beta^2 - r\beta}{C'} \\ -K_{12} &= K_{21} = iL, \\ \text{where } L &= -r(\beta - r)\omega_z/C'. \end{aligned}\right\} \quad \ldots \quad (2.21)$$

When $\beta = 1$ (no collision), these expressions reduce to (2.17).

§ 3. Solution of the Fundamental Equations.

The rigorous solution of the fundamental equations presents great difficulty, since the quantity $p_0^2 = \dfrac{4\pi Ne^2}{m}$ is not a constant, but varies with height. Let us first treat p_0^2 as a constant, and see what result is obtained.

Let us put

$$(E_x, E_y, H_x, H_y) = (A_1, B_1, C_1, D_1)\, e^{i\phi} \quad \ldots \quad \ldots \quad (3.1)$$

where $\phi = p\left(t \mp \dfrac{\mu z}{c}\right)$; the minus sign holds for the outgoing wave, the plus sign for the reflected wave. 'μ' is the refractive index, which we have to find out. When we substitute (3.1) in (2.14) for the outgoing wave, we have the following relation amongst the constants A_1, B_1, C_1, D_1:

$$\left.\begin{aligned} &\mu A_1 = D_1, \quad \mu B_1 = -C_1 \\ &\mu C_1 = -iLA_1 - K_{22}B_1, \quad \mu D_1 = K_{11}A_1 - iLB_1 \end{aligned}\right\} \quad \ldots \quad (3.2)$$

From these equations, or directly from (2.14), we obtain

$$\left.\begin{aligned} (\mu^2 - K_{11})A_1 + iLB_1 &= 0 \\ -iLA_1 + (\mu^2 - K_{22})B_1 &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

or for the magnetic vectors

$$\left.\begin{aligned} (\mu^2 - K_{11})D_1 + iLC_1 &= 0 \\ -iLD_1 + (\mu^2 - K_{22})C_1 &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3.3')$$

i.e. μ^2 is given by the roots of the quadratic equation

$$(\mu^2 - K_{11})(\mu^2 - K_{22}) - L^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

Let us put

$$\frac{K_{11} - K_{22}}{2L} = f.$$

Then it can be easily shown after a little work that the two values of μ are given by

$$\left.\begin{aligned} \mu_1^2 &= K_{11} - Lf\left(1 - \sqrt{1 + 1/f^2}\right) = K_{11} - L\rho_1 \\ \mu_2^2 &= K_{11} - Lf\left(1 + \sqrt{1 + 1/f^2}\right) = K_{11} - L\rho_2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3.5)$$

where

$$\rho_1 = f\left(1-\sqrt{1+1/f^2}\right), \quad \rho_2 = f\left(1+\sqrt{1+1/f^2}\right) \qquad \text{.. (3.6)}$$

ρ_1 and ρ_2 are the roots of the equation,

$$\rho^2 - \frac{K_{11}-K_{22}}{L}\rho - 1 = 0.$$

From equations (3.3) and (3.5) we have

$$\frac{B_1}{A_1} = \frac{\mu^2 - K_{11}}{-iL} = -i\rho$$

or $B_1 = -i\rho A_1$, $C_1 = i\rho\mu A_1$, $D_1 = \mu A_1$.

Hence we have from (3.1),

$$\left.\begin{aligned} E_x &= A_1 \cos\phi, \quad E_y = A_1\rho \sin\phi \\ H_x &= -\mu A_1\rho \sin\phi, \quad H_y = \mu A_1 \cos\phi \end{aligned}\right\} \qquad \text{.. .. (3·7)}$$

These equations show that the two waves are elliptically polarized. We have

$$\left.\begin{aligned} E_x^2 + \frac{E_y^2}{\rho^2} &= A_1^2 \\ \frac{H_x^2}{\rho^2} + H_y^2 &= \mu^2 A_1^2 \end{aligned}\right\} \qquad \text{.. .. .. (3.8)}$$

The ratio of the axes of the ellipses are—

(*a*) for the electric vector x-axis : y-axis $= 1 : \rho$

(*b*) for the magnetic vector x-axis : y-axis $= \rho : 1$.

The sense of rotation is given by equations (3.7) and the sign of ρ can be taken only after we have discussed the values of ρ_1 and ρ_2.

Let us now find out the special characteristics of the two waves. It is necessary now that the signs be properly taken.

In the northern hemisphere, the north-seeking (positive) magnetic pole points downwards as shown in the figure. Let 'h' denote the absolute value of the magnetic field. Then if 'δ' be the dip-angle, we have

$$h_x = h\cos\delta, \quad h_z = -h\sin\delta.$$

Now $\dfrac{eh}{mcp} = \omega$ is itself negative, because 'e' is negative. We have therefore

$$\omega_x = -\omega\cos\delta, \quad \omega_z = \omega\sin\delta$$

where ω is the quantity $\left|\dfrac{eh}{mcp}\right|$.

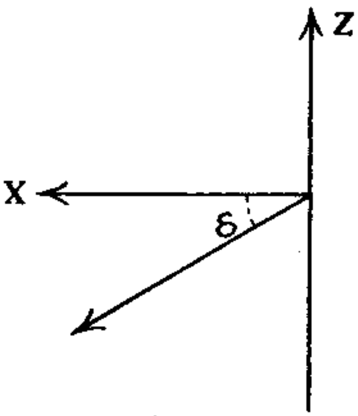

FIG. 1.

In the southern hemisphere, the dip is upwards for the positive pole, hence we have simply to substitute $(-\delta)$ for δ in the above expression. With the notation used here it is easy to see that

$$\left.\begin{aligned} K_{11} &= \frac{(1-r)(1-r-\omega^2)}{C} = \frac{t(t+\omega^2)}{C}, \quad t = r-1 \\ L &= -\frac{r(1-r)\,\omega_z}{C} = \frac{t(1+t)\,\omega \sin\delta}{C} \\ f &= \frac{K_{11}-K_{22}}{2L} = \frac{\omega\cos^2\delta}{2t\sin\delta} \end{aligned}\right\} \quad \ldots \quad (3.9)$$

We have now

$$\left.\begin{aligned} \mu_1^2 &= K_{11}-L\rho_1 = K_{11}-Lf\,(1-\sqrt{1+1/f^2}) \\ C\mu_1^2 &= t(t+\omega^2)-\frac{(1+t)}{2}\omega^2\cos^2\delta\left\{1-\sqrt{1+\frac{4t^2\sin^2\delta}{\omega^2\cos^4\delta}}\right\} \\ C\mu_2^2 &= t(t+\omega^2)-\frac{(1+t)}{2}\omega^2\cos^2\delta\left\{1+\sqrt{1+\frac{4t^2}{\omega^2}\frac{\sin^2\delta}{\cos^4\delta}}\right\} \end{aligned}\right\} \quad \ldots \quad (3.10)$$

The expressions for μ_1^2, μ_2^2 are independent of the sign of 'δ' and hence hold for both hemispheres. We further observe that

(*a*) $\mu_1^2 = 0$ when $t = 0$.

Hence μ_1 represents the conventional ordinary wave, for $t = 0$ is equivalent to the condition

$$p_0^2 = p^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.11)$$

(*b*) To prove that $\mu_2^2 = 0$ when $t^2 = \omega^2$.

In this case, we have

$$\sqrt{1+\frac{4t^2\sin^2\delta}{\omega^2\cos^4\delta}} = 1+\frac{2\sin^2\delta}{\cos^2\delta}$$

$$\therefore \quad 1+\sqrt{1+1/f^2} = 2\sec^2\delta, \text{ and}$$

$$\begin{aligned} C^1\mu_2^2 &= t(t+\omega^2)-(1+t)\omega^2 \\ &= t^2-\omega^2 \\ &= 0. \end{aligned}$$

This condition gives us that $\mu_2 = 0$ when $t = \pm\omega$

or
$$\frac{p_0^2}{p^2} = 1\pm\frac{p_h}{p} \text{ or } p_0^2 = p^2 \pm pp_h \quad \ldots \quad \ldots \quad (3.12)$$

These are the conditions of reflection for the extraordinary wave. μ_2 therefore represents the extraordinary wave.

Limiting cases :—

(1) When $\delta = 0$ (Magnetic Equator—Transverse Case).

We can show from (3.10) that

$$\mu_1^2 = -t = 1-r, \; \mu_2^2 = -\frac{t^2-\omega^2}{t+\omega^2}.$$

The $(\mu_1^2, 1-r)$ curve is the straight line representing the ordinary wave in

Mitra's Report (1935). The μ_2^2, r curve resolves into two curves on either side of the $(\mu_1^2, 1-r)$ line.

(2) When $\delta = \pm\frac{\pi}{2}$ (Magnetic Poles—Longitudinal case).

Now $\cos\delta = 0$, $\sin\delta = \pm 1$, and we easily see from (3.10) that

$$\mu_1^2 = \frac{t+\omega}{\omega-1}, \qquad \mu_2^2 = \frac{\omega-t}{1+\omega}$$

$$= \frac{1-r-\omega}{1-\omega} \qquad = \frac{\omega+1-r}{\omega+1}$$

$$= 1-\frac{r}{1-\omega} \qquad = 1-\frac{r}{1+\omega}.$$

So the (μ_1^2, r) (μ_2^2, r) curves reduce to straight lines, which are reproduced in Mitra's Report, p. 142.

§ 4. Polarization.

As mentioned in § 3, the polarization factors are given by

$$\rho_1 = f\left[1-\sqrt{1+1/f^2}\right], \quad \rho_2 = f\left[1+\sqrt{1+1/f^2}\right] \quad \ldots \quad (4.1)$$

since $f = \frac{\omega^2\cos^2\delta}{2t\sin\delta}$ and $t = r-1$, $r = \frac{p_0^2}{p^2}$, f varies with the height of the point at which the wave is being considered, t varies from -1 at the ground to zero at $r = 1$. We need consider only the polarization of the ground wave. We have then

$$t = -1$$

$$f = -\frac{\omega\cos^2\delta}{2\sin\delta} \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

For the o-wave.

(*a*) Northern hemisphere

$$\rho_1 = -\frac{\omega\cos^2\delta}{2\sin\delta}\left[1-\sqrt{1+\frac{4\sin^2\delta}{\omega^2\cos^4\delta}}\right] \quad \ldots \quad \ldots \quad (4.3)$$

Southern hemisphere, now 'δ' must be changed to '$-\delta$'; we have then

$$\rho_1 = \frac{\omega\cos^2\delta}{2\sin\delta}\left[1-\sqrt{1+\frac{4\sin^2\delta}{\omega^2\cos^4\delta}}\right] \quad \ldots \quad \ldots \quad (4.4)$$

For the x-wave.

(*b*) Northern hemisphere

$$\rho_2 = -\frac{\omega\cos^2\delta}{2\sin\delta}\left[1+\sqrt{1+\frac{4\sin^2\delta}{\omega^2\cos^4\delta}}\right] \quad \ldots \quad \ldots \quad (4.5)$$

Southern hemisphere

$$\rho_2 = \frac{\omega \cos^2 \delta}{2 \sin \delta}\left[1+\sqrt{1+\frac{4 \sin^2 \delta}{\omega^2 \cos^4 \delta}}\right] \quad .. \quad .. \quad (4.6)$$

In the table given below, we have calculated values of ρ_1, ρ_2 for a number of selected stations, for $\lambda = 100$ meters. They are also given under fig. 2.

TABLE I.

Place	δ	h	$r=\frac{eh}{mcp}$	$f=-\frac{r\cos^2\delta}{2\sin\delta}$	Ordinary ρ_1	Extra-ordinary ρ_2	Re-marks
N. Pole ..	90°	..	..	..	1	−1	
Lerwick ..	72° 42′	·4884	·4509	−·0208	·9794	−1·0210	
Slough ..	66° 54′	·4702	·4419	−·0370	·9633	−1·0373	
Allahabad ..	46°	·5182	·487	−·1653	·8487	−1·1793	
Bombay ..	25° 30′	·4135	·3710	−·3613	·7017	−1·4213	
Huancayo ..	2° 3′	·2963	·2693	−3·757	·129	−7·643	
North of Equator	0	..	..	$-\infty$	0	$-\infty$	
South of Equator	0	..	..	$+\infty$	0	$+\infty$	
La Quiaca ..	−12° 21′	·2684	·2523	·5631	−·5849	1·711	
Pilar ..	−25° 55′	·2732	·2576	·2387	−·7896	1·267	
Batavia ..	−32° 26′	·4369	·4121	·2752	−·7619	1·312	
Watheroo ..	−64° 19′	·2757	·259	·0271	−·9732	1·0274	
Melbourne ..	..	..	..	..	−·9569	1·0451	
S. Pole ..	−90°	..	..	..	−1	1	

EXPERIMENTAL CONFIRMATION.

These results have been experimentally confirmed. Berkner states that at Huancayo: The ordinary ray is polarized with its electric vector along the magnetic north-south. Table (1) shows that

$$E_x\text{-axis} : E_y\text{-axis} = 1 : \cdot 129$$

i.e. the electric vector is mainly along the x-axis, i.e. magnetic north-south.

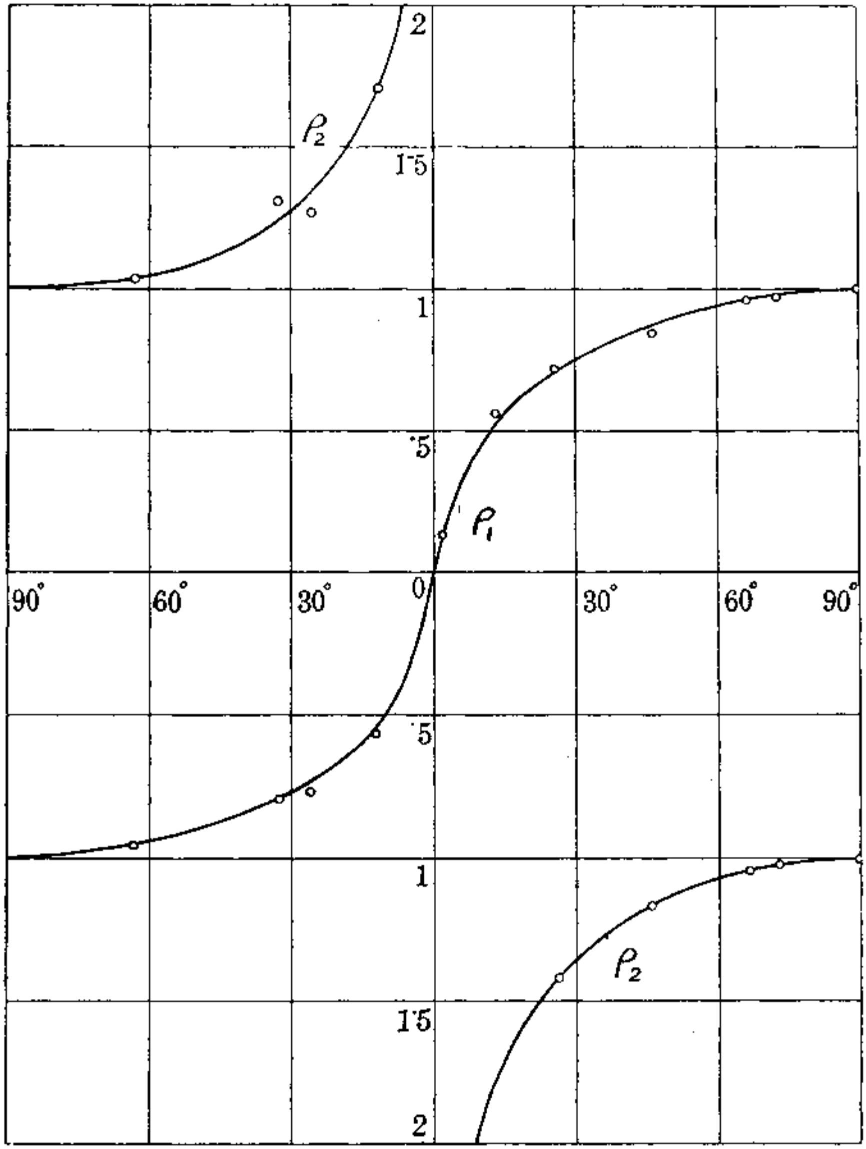

FIG. 2.

The extraordinary ray is polarized with its electric vector along the magnetic east-west. Table (1) shows that

$$E_x\text{-axis} : E_y\text{-axis} = 1 : 7{\cdot}643.$$

The variation of polarization for the *o*- and *x*-waves with latitude are shown in fig. 2.

Conclusion.

It is shown that if the complex refractive index be regarded as constant we get the same conditions for reflection and polarization of the radio-waves for vertical propagation as was obtained by Appleton. But the refractive indices vary with height, hence the treatment given here should be replaced by a wave-treatment. A simple case of wave-treatment has already been published in paper 2.

References.

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