JOULE-THOMSON EFFECT AND ADIABATIC CHANGE IN DEGENERATE GAS.[1]

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(*Read November* 6, 1937.)

Summary.

An expression for the Joule-Thomson effect in a degenerate gas is derived. A degenerate gas is heated after a Joule-Thomson expansion, the rise in temperature for a given fall in pressure being greater the greater the degree of degeneracy of the gas. The case of adiabatic change is also discussed for comparison. An adiabatic expansion lowers the temperature, but the degree of degeneracy (and also the degree of non-degeneracy in the case of a non-degenerate gas) remains constant during an adiabatic change.

Introduction.

During recent years there have been numerous physical as well as astrophysical applications of the Fermi-Dirac statistics. The researches beginning with Fowler (1) and later developed extensively by others, and particularly by Milne whose work has been of far-reaching importance, have established the fact that the essential features of the internal constitution of the white dwarf stars—stars characterized by a comparatively small luminosity, high effective temperature and an abnormally large mean density of the order of a million times that of water—can be explained by an application of the theory of degenerate electron gas—degenerate in the sense of Fermi-Dirac statistics. Kothari and Majumdar (2) have recently worked out the degree of ionization in degenerate matter in terms of its density or pressure, and have shown that the usual theory of the white dwarf stars, when incorporated with the theory of ionization, becomes applicable to the planets as well, and leads to several interesting conclusions. The most important perhaps is the prediction that there exists a maximum radius for a *cold*[2] body. The maximum value depends to a certain extent on the chemical composition of the material, but as regards

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[1] In a letter to me Professor Arthur Holmes, Professor of Geology, University of Durham (England) suggested the desirability of investigating temperature changes occurring in degenerate matter undergoing convection as it may have possible geological applications; and I desire to acknowledge that this note owes its origin to the stimulating correspondence with Professor Holmes.

[2] The word *cold* is used here in a technical sense. The planets and white dwarf stars will be referred to as *cold bodies.* In general, matter will be referred to as cold when any free-electrons present constitute a degenerate gas in the sense of Fermi-Dirac statistics.

the order of magnitude it is of the order of the radius of Jupiter, the largest planet in our solar system. Thus we see that degenerate matter forms an important constituent of stellar bodies and it is worth while therefore to study its various properties.[1] In the present paper we shall deal with the Joule-Thomson effect and adiabatic change. This will have a possible application when dealing with temperature changes due to convection currents occurring in degenerate matter inside stellar and planetary interiors. However, we shall not be concerned with these applications in the present note.

In section I, we shall derive the expression for Joule-Thomson effect. It will be seen that a degenerate gas is heated (and not cooled) when it suffers Joule-Thomson expansion, the rise in temperature for unit fall in pressure is greater the greater the degree of degeneracy [2] of the gas. After the Joule-Thomson expansion, because of the lowering of pressure (i.e. electron concentration) and the rise in temperature, the degree of degeneracy is lower than originally. A degenerate gas, if it undergoes continuous Joule-Thomson expansion, will experience a rise of temperature at a gradually diminishing rate and will ultimately become non-degenerate, and, then, as its behaviour approximates to that of classical perfect gas, any further Joule-Thomson expansion will produce (practically) no more rise in temperature.

In section II we deal with the case of adiabatic change. An adiabatic expansion always lowers the temperature. A degenerate gas undergoing adiabatic change (expansion or contraction) remains degenerate, the degree of degeneracy remaining constant during the change. Similarly a non-degenerate gas will remain non-degenerate (the degree of non-degeneracy remaining the same).

I. When a gas undergoes Joule-Thomson effect (also called throttling process)

$$d\,(E+pV) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where E is the internal energy and V the volume, both for an assembly of N particles (N being the Avogadro number), and p the pressure of the assembly. Using the well-known thermodynamic relations

$$dQ = dE + pdV = C_p dt + l'dp \quad \ldots \quad \ldots \quad \ldots \quad (2)^3$$

[1] According to some researches of Professor Milne all the stars (and not only the white dwarfs) possibly contain degenerate cores.

[2] The degree of degeneracy is measured by the value of the degeneracy-discriminant A_0,

$$A_0 = \frac{nh^3}{2(2\pi mkT)^{3/2}},$$

where n is the free electron concentration, T the temperature, m the electron mass, and k and h denote Boltzman's and Planck's constant respectively. (In this paper we neglect the effect of relativity mechanics. The expression for A_0 given above refers to the non-relativistic case.)

[3] Though hardly necessary, it may be permissible to make the following remarks about dQ in equation (2). dQ is defined as the heat flowing into the system when its

$$l' = -T\left(\frac{dv}{dT}\right)_p, \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

we obtain the standard expression for Joule-Thomson effect

$$\Delta T = \frac{T\left(\frac{dV}{dT}\right)_p - V}{C_p}\Delta p, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

where ΔT represents the fall in temperature or cooling for a fall in pressure Δp. C_p is the specific heat (per Avogadro number N) at constant pressure.

In the case of degenerate gas the internal energy to a first approximation is given by the well-known expression (3)

$$E = \frac{3}{10}\frac{h^2}{m}\left(\frac{3n}{8\pi}\right)^{2/3} N\left[1+\frac{5}{12}\left(\frac{2\pi mkT}{h^2}\right)^2\left(\frac{8\pi}{3n}\right)^{4/3}\right]$$

$$= \frac{3}{5}\epsilon^* N\left[1+\frac{5}{12}\left(\frac{4\pi}{3A_0}\right)^{4/3}\right] \quad \ldots \quad (5)$$

where A_0 is called the degeneracy-discriminant (for degeneracy $A_0 >> 1$)

$$A_0 = \frac{nh^3}{2(2\pi mkT)^{3/2}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

and ϵ^* is the maximum energy of the Fermi-Dirac distribution, i.e. ϵ^* is the maximum kinetic energy possessed by any electron in an assembly of electron-concentration n and at absolute zero of temperature,

$$\epsilon^* = \frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{2/3}. \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

We note that

$$\left(\frac{3A_0}{4\pi}\right) = \left(\frac{\epsilon^*}{\pi kT}\right)^{3/2}. \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

Let us now determine C_v and C_p, the specific heats (per Avogadro number of free electrons) at constant volume and constant pressure respectively.

We have from (5)

$$C_v = \left(\frac{dE}{dT}\right)_v = \frac{\pi^2 mk^2}{h^2}\left(\frac{8\pi}{3n}\right)^{2/3} NT = \frac{\pi^2}{2}R\frac{kT}{\epsilon^*} = \frac{\pi}{2}R\left(\frac{4\pi}{3A_0}\right)^{2/3} \quad \ldots \quad (9)$$

where $R = kN$ is the gas-constant.

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internal energy increases by dE and the external work done is pdv, p being the pressure of the system. But it is only during a reversible process that the external work equals pdv; in a non-reversible process it will be smaller. Therefore dQ does not represent the observed supply of heat, i.e. *heat actually supplied* except when the process is reversible. In a gas undergoing Joule-Thomson effect no heat is allowed to flow in or out, but that does not mean dQ is zero, for Joule-Thomson effect is non-reversible and so dQ cannot represent the observed heat supply which, however, is zero.

To determine C_p we can make use of the thermodynamic relation

$$C_p - C_v = T\left(\frac{dp}{dT}\right)_v\left(\frac{dV}{dT}\right)_p \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Noting that [1]

$$p = \frac{2}{3}\frac{E}{V} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

We at once obtain

$$\left(\frac{dp}{dT}\right)_v = \frac{2}{3}\frac{\pi^2 mk^2}{h^2}\left(\frac{8\pi}{3n}\right)^{2/3} nT = \frac{\pi kn}{3}\left(\frac{4\pi}{3A_0}\right)^{2/3} \quad \ldots \quad \ldots \quad (12)$$

$$-V\left(\frac{dp}{dV}\right)_T = \frac{1}{3}\frac{h^2}{m}\left(\frac{3n}{8\pi}\right)^{2/3} n\left[1 + \frac{1}{12}\left(\frac{2\pi mkT}{h^2}\right)^2\left(\frac{8\pi}{3n}\right)^{4/3}\right]$$

$$= \frac{2}{3}\epsilon^* n\left[1 + \frac{1}{12}\left(\frac{4\pi}{3A_0}\right)^{4/3}\right] \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

$$\left(\frac{dV}{dT}\right)_p = -\left(\frac{dp}{dT}\right)_v \Big/ \left(\frac{dp}{dV}\right)_T = \frac{V\frac{\pi}{2}\frac{k}{\epsilon^*}\left(\frac{4\pi}{3A_0}\right)^{2/3}}{1 + \frac{1}{12}\left(\frac{4\pi}{3A_0}\right)^{4/3}} = \frac{V}{2T}\frac{\left(\frac{4\pi}{3A_0}\right)^{4/3}}{1 + \frac{1}{12}\left(\frac{4\pi}{3A_0}\right)^{4/3}} \quad (14)$$

Substituting the above results in (10) we have

$$C_p - C_v = \frac{\frac{\pi}{6}\left(\frac{4\pi}{3A_0}\right)^2 R}{1 + \frac{1}{12}\left(\frac{4\pi}{3A_0}\right)^{4/3}} \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Substituting for C_v from (9) we get

$$C_p = \frac{\pi}{2} R\left(\frac{4\pi}{3A_0}\right)^{2/3}\left[\frac{1 + \frac{5}{12}\left(\frac{4\pi}{3A_0}\right)^{4/3}}{1 + \frac{1}{12}\left(\frac{4\pi}{3A_0}\right)^{4/3}}\right] \quad \ldots \quad \ldots \quad (16)$$

$$\doteqdot \frac{\pi}{2} R\left(\frac{4\pi}{3A_0}\right)^{2/3}\left[1 + \frac{1}{3}\left(\frac{4\pi}{3A_0}\right)^{4/3}\right] \ldots \quad \ldots \quad \ldots \quad (17)$$

Since for a degenerate gas $A_0 >> 1$, we notice that C_p differs very little from C_v, and this difference becomes smaller as the degree of degeneracy increases.

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[1] This is so in the *non-relativistic* case with which, as already mentioned, we are concerned in this paper. In the *relativistic* case $p = \frac{1}{3}\frac{E}{V}$.

We have now to substitute the expressions for C_p and $\left(\frac{dV}{dT}\right)_p$ given by (16) and (14) respectively in (4). We obtain after a little reduction

$$\Delta T = -\frac{2}{\pi kn}\left(\frac{3A_0}{4\pi}\right)^{2/3}\left[\frac{1-\frac{5}{12}\left(\frac{4\pi}{3A_0}\right)^{4/3}}{1+\frac{5}{12}\left(\frac{4\pi}{3A_0}\right)^{4/3}}\right]\Delta p \quad \ldots \quad \ldots \quad (18)$$

$$\doteq -\frac{2}{\pi kn}\left(\frac{3A_0}{4\pi}\right)^{2/3}\Delta p = -\frac{2}{\pi^2 kn}\left(\frac{\epsilon^*}{kT}\right)\Delta p \quad \ldots \quad \ldots \quad (19)$$

$$= -\frac{3}{8\pi^3}\left(\frac{4\pi}{15}\frac{2h^2}{m}\right)^{1/5}\frac{h^2}{mk^2}\,\frac{1}{Tp^{1/5}}$$

where n is the number of free-electrons per unit volume.

II. We now consider an adiabatic process. During an adiabatic change entropy remains constant. The entropy of a degenerate gas is given by

$$S = R\pi^2\,\frac{kTm}{h^2}\left(\frac{8\pi}{3n}\right)^{2/3}, \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

and hence during an adiabatic process $\frac{n^{2/3}}{T}$ remains constant. This ensures the constancy of the degeneracy-discriminant A_0, for

$$A_0 = \frac{nh^3}{2(2\pi mkT)^{3/2}}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

The pressure of a degenerate gas is given by

$$p = \frac{8\pi}{15}\frac{h^2}{m}\left(\frac{3n}{8\pi}\right)^{5/3} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

and, therefore, for an adiabatic process

$$\frac{p}{T^{5/2}} = \text{constant}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

Further, as

$$p = \frac{2}{3}\,\frac{E}{V},$$

and since for adiabatic change

$$TdS = dE + pdV = 0, \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

it follows that $pV^{5/3}$ remains constant.

It is somewhat significant to note that the above relations representing adiabatic change in a degenerate gas are just the same as those governing adiabatic process for non-degenerate (i.e. *classical* perfect) gas.

References.

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