LEVI-CIVITA'S FORMULÆ FOR TWO BODIES.

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1. It was shown in *Proc. Acad. Sci. India*, Vol. 4, pp. 20–23 (August 1934), that extra terms, in the equation of motion, as compared to Newton's, can be treated as being due to small perturbing forces and the changes in the elements of the orbit deduced by ordinary Dynamics. In *Proc. Nat. Acad. Sci. India*, Vol. 6, p. 280 (August 1936), the acceleration of one body relative to another body of comparable mass as well as its acceleration for actual motion in space were given.

For relative motion the potential function is

$$V=\frac{G(M+m)}{r}+\frac{G(M+m)h^2}{c^2}\cdot\frac{1}{r^3}$$

where r is the distance between the two bodies, M and m their masses, G the gravitational constant, h double the areal rate and c the velocity of light. This correctly yields the differential equation

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}+\frac{3\mu}{c^2}\cdot u^2.$$

2. Prof. Tullio Levi-Civita in the *American Journal of Mathematics* (Vol. LIX, No. 2, April 1937, pp. 225–234) has also treated the extra effect as 'first order perturbations', and following Newtonian principles deduced certain Astronomical consequences of Relativistic Two-Body Problem.

He has found that for two bodies of comparable masses, the trajectory may be considered to be that of a central force with the potential function for relative motion (in our notations) as

$$V=\frac{G(M+m)}{r}+3\left\{1-\frac{1}{2}\,\frac{Mm}{(M+m)^2}\right\}\frac{G^2(M+m)^2}{c^2\,.\,r^2}$$
$$+\frac{1}{2}\,\frac{Mm}{(M+m)^2}\cdot\frac{c^2h^2}{G^2(M+m)^2}\cdot\frac{G^3(M+m)^3}{c^4\,.\,r^3}\,.$$

He has then concluded that the first term represents the Newtonian attraction, and the other two (both of the second order) are the relativistic perturbations varying according to the inverse cube and the inverse fourth power respectively of the distance, and has then inferred that by putting one of the masses equal to zero we get

$$3\,\frac{G^2M^2}{c^2r^2}$$

for the perturbative function of the Einstenian one-centre problem (*loc. cit.*, p. 229). The angular precession (per revolution) of the periastron in the case of a double star turns out to be the same as the precision for an infinitesimal planet moving about a central mass possessing the total mass of the binary system

$$\Delta \bar{\omega} = \frac{6\pi G^2(M+m)^2}{c^2h^2}.$$

3. (i) It can however be pointed out that the extra terms do not strictly represent the Einstenian perturbation.

For an infinitesimal planet we put $m = 0$, and obtain from the above potential function the acceleration by differentiation as

$$f = -\frac{GM}{r^2} - \frac{6G^2M^2}{c^2r^3} \text{ only}$$

which is
$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2.$$

Substituting $r = \frac{1}{u}$ and therefore

$$\frac{d^2r}{dt^2} = -h^2u^2 \cdot \frac{d^2u}{d\theta^2}$$

while for planets $r^2\frac{d\theta}{dt} = h$ fairly approximately, we get

$$-h^2u^2\left[\frac{d^2u}{d\theta^2}+u\right] = -GMu^2 - \frac{6G^2M^2}{c^2}u^3$$

or
$$\frac{d^2u}{d\theta^2}+u = \frac{GM}{h^2}+\frac{6G^2M^2}{c^2h^2}\cdot u.$$

This is not at all identical with Einstein's equation in which the last term contains u^2 and not u.

Putting $\sqrt{1-\frac{6G^2M^2}{c^2h^2}} = k$; $kh = h'$ and $k\theta = \theta'$, the equation becomes

$$\frac{d^2u}{d\theta'^2}+u = \frac{GM}{h'^2},$$

which has the Newtonian and not the Einstenian form, and for which the solution is

$$u = \frac{GM}{h'^2}+A\cos(\theta'+B),$$

where A and B are constants. This is a complete ellipse revolving round the centre of force in a forward direction, which was known to Newton.

(ii) Indeed this law of gravitation can be obtained without Levi-Civita's elaborate method and directly from Eddington's *supposed* solution of Einstein's orbital equation as given in his Relativity (p. 88)

$$u = \frac{\mu}{h^2}\left[1+e\cos\left(1-\frac{3\mu^2}{h^2c^2}\right)\theta\right].$$

Treating $\left(1-\frac{3\mu^2}{h^2c^2}\right)\theta$ as a new variable, this is obviously the solution of

$$\frac{d^2u}{\left(1-\frac{3\mu^2}{h^2c^2}\right)^2 d\theta^2}+u=\frac{\mu}{h^2}.$$

Hence $\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}+\frac{6\mu^2}{c^2h^2}\cdot u$, nearly.

Eddington's solution suffers exactly from the same defect as Levi-Civita's, being really the solution of another differential equation.

4. Levi-Civita's potential function cannot yield any value for the spectral shift of light. And even the value for the deflection of light from stars obtained from Levi-Civita's formula does not tally with Einstein's value.

(i) Taking R as the perpendicular distance from the centre of the Sun on an asymptote and ψ the angle between this perpendicular and the radius vector, the deflection

$$\epsilon=\frac{1}{c^2R}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[-\frac{GM}{r^2}-\frac{6G^2M^2}{c^2r^3}\right]r^2\cos\psi\, d\psi \quad \text{where } r\cos\psi=R$$

$$=-\frac{GM}{c^2R}\left[2+\frac{3\pi GM}{c^2R}\right]$$

$$=\frac{2GM}{c^2R}\text{, nearly,}$$

which is equal to Newtonian value very nearly, and not at all to Einstein's.

(ii) The value of the advance of perihelion, however, comes out correct.

$$\Delta\,\bar{\omega}=-\int_0^{2\pi}\frac{6G^2M^2}{c^2r^3}\cdot\frac{\cos\theta}{e}\sqrt{\frac{a(1-e^2)}{\mu}}\,dt$$

where $r^2\frac{d\theta}{dt}=h$ nearly, and $\frac{1}{r}=\frac{\mu}{h^2}(1+e\cos\theta)$ nearly

$$=\frac{6\pi G^2M^2}{c^2h^2}.$$

(iii) The spectral shift of light from the edge of the Sun treated as depending on the difference of potential is given by

$$\frac{\lambda_s}{\lambda_e}=1+\frac{GM}{a}\left(1+\frac{3GM}{c^2a}\right)\text{ nearly}$$

$$=1+\frac{GM}{a}\text{, nearly, where } a \text{ is the radius of the Sun.}$$

Hence the shift remains the same as that of light from the centre.

(iv) If we proceed to a higher approximation and take

$$r^2 \frac{d\theta}{dt} = h\left(1 - \frac{2GM}{c^2 r}\right),$$

even the value for the advance of perihelion is upset. In the differential equation the second biggest term will still contain u.

5. Discarding Relativity concepts, Levi-Civita has also considered the *absolute* motion in the sky of a double star system. His conclusions are:—

(i) The acceleration of the centre of mass G of a double star lies entirely in the plane of (relative) orbit and the common plane of (absolute) orbits.

(ii) The secular acceleration of the centre of mass G of the double star is directed along the major axis towards the periastron of the principal star.

(iii) Taking the origin at the centre of the principal star and measuring α towards the periastron of the (undisturbed) elliptical orbit of the other star, and taking the component of the ordinary velocity of G as $c\alpha_1$, and denoting by $c\bar{\alpha}_1$ its secular part in terms of θ, he gets—

$$c\bar{\alpha}_1 = -\frac{1}{2}\frac{Mm}{(M+m)^2}\cdot\frac{M-m}{(M+m)}\cdot\frac{e}{(1-e^2)^{\frac{3}{2}}}\cdot\frac{G(M+m)}{c^2}\sqrt{\frac{G(M+m)}{a^3}}\,\theta.$$

He has pointed out that a difference of velocity along the apsidal line, having a component also in the line of sight, ought to be detectable eventually by spectroscopic observation of binaries for which photometric observations also are available.

6. If there were a net acceleration of the centre of gravity, then (1) it must obviously be in the common plane of the two orbits, and (2) as the principal star has a larger mass and is therefore nearer the centre of gravity, and nearest at the periastron, it is equally obvious that the residue of the acceleration would be directed along the major axis towards the periastron. But on the principle of the equality of action and reaction, there cannot be any acceleration of the centre of gravity without extraneous influences.

According to the second paper mentioned in para. 1 the respective forces in absolute space round their common centre of gravity 0 taking the masses as being concentrated at their centres are

$$-\frac{GM^3}{(M+m)^2}\cdot\frac{m}{OP^2}-\frac{3GM^5h^2}{c^2(M+m)^4}\cdot\frac{m}{OP^4}$$

and

$$+\frac{GM^3}{(M+m)^2}\cdot\frac{M}{Op^2}+\frac{3Gm^5h^2}{c^2(M+m)^4}\cdot\frac{M}{Op^4}$$

which are equal and opposite, as $M.Op = m.OP$, and therefore cancel each other.

There can therefore in such a case be no net acceleration at all.