ABSORPTION AND REFLECTION OF RADIO WAVES AT OBLIQUE INCIDENCE AND THEIR RELATIONSHIP WITH VERTICAL INCIDENCE PHENOMENA.

By K. B. MATHUR, *Physics Department, Allahabad University.*

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INTRODUCTION.

The success of short wave transmission over long distances is entirely due to the presence of ionized regions in the upper atmosphere, which are responsible for reflecting the signals without much absorption. Greater and greater use is being made of short waves for broadcasting of sound and pictures. For commercial undertakings it is vital to be able to select a frequency which will serve the largest area. Hence the paramount duty of a radio engineer is to find out the most suitable frequency to suit his purpose. However, such a frequency depends entirely upon the conditions existing in the ionosphere which differ from place to place with the season of the year, and hour of the day. Thus we find that the ionospheric data enables the radio engineer to predict the coverage of the projected station, and is thus of the greatest practical importance. It is extremely tedious to collect data for oblique propagation of radio waves, compared to such data for vertical propagation. Much of the radio engineer's work is simplified if he is able to use the vertical incidence results for calculating the absorption and reflection for oblique propagation. Martyn was the first to show that for flat earth it was possible to get oblique incidence data from the data obtained for vertical incidence. Smith (1937) further showed that it was possible to determine graphically the maximum usable frequency for oblique propagation from vertical incidence (virtual height-frequency) curves. Millington (1938) gave a modification for a curved earth of Martyn's theorem and showed that the results hold for both equivalent frequency and absorption for a thin layer. Smith (1938) has further extended the work and has removed the restriction of the thin layer. He has deduced the relation only for equivalent frequency assuming linear distribution of electrons but has neglected absorption.

In the present paper, relations have been deduced for absorption as well as for equivalent frequency for the linear and also for the exponential gradient of ionization in a thick layer. The earth's magnetic field has been neglected, and the gradient of electron density has been taken to be everywhere vertical and similar throughout.

1. Absorption.

Fig. 1 shows the transmitter T and the receiver R, separated by a distance D. The sky wave leaves the ground at an angle α and enters the ionosphere at A, ϕ_1 being the angle of incidence.

ϕ = the angle the wave normal makes with the vertical at the altitude z above A.

h = height of the ionosphere.

R = radius of the earth.

z_0 = the maximum height of penetration above the lower boundary of the ionosphere.

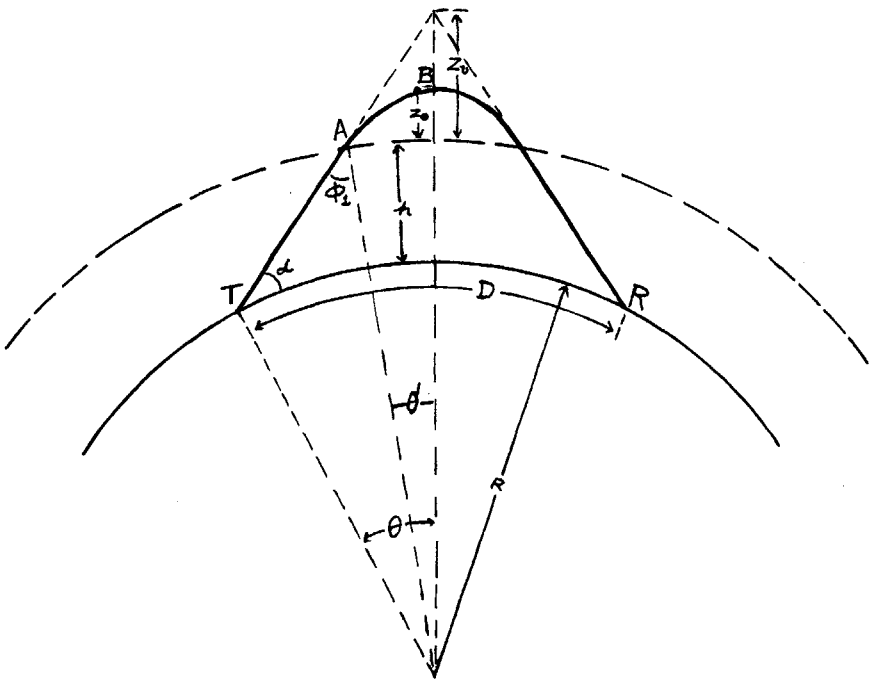

Fig. 1.

z_v = the height of the equivalent triangle above the lower boundary.

ϕ_0 = half the angle at the vertex of the equivalent triangle.

z_v' = the vertical height measured at vertical incidence.

f' = actual wave frequency at oblique propagation.

f = the equivalent frequency of f' reflected at vertical incidence from the same height z.

$$f_0 = \sqrt{\frac{Ne^2}{m\pi}}.$$

N = number of electrons per c.c.

e = charge of an electron.

m = mass of an electron.

μ_0' = the value of the refractive index μ' where the reflection takes place.

t = time taken by the sky wave to travel from T to R.

In order to avoid repetition the following equations—which have been used in the present calculations—are reproduced from Smith's paper (1938).

$$\sqrt{1-\mu_0'^2} = \cos\phi_1 \sqrt{1+\frac{2z_0}{R+h}\tan^2\phi_1} \quad .. \quad .. \quad .. \quad (1)$$

$$= \cos\phi_0 \sqrt{1-\frac{2(z_v-z_0)}{R+h}\tan^2\phi_0} \quad .. \quad .. \quad .. \quad (1a)$$

$$f = f'\sqrt{1-\mu_0'^2} = f'\cos\phi_1 \sqrt{1+\frac{2z_0}{R+h}\tan^2\phi_1} \quad .. \quad .. \quad .. \quad (2)$$

$$\theta' = \frac{\sin\phi_1}{(R+h)\sqrt{1-\mu_0'^2}} \int_0^{z_0} \frac{dz}{\sqrt{1-f_0^2/f^2}} (1+B)\left(1-\frac{2z}{R+h}\right) .. \quad (3)$$

where
$$1+B = \frac{1}{\sqrt{1-A\left(\frac{f^2}{f^2-f_0^2}\right)}} \quad .. \quad .. \quad .. \quad .. \quad (3a)$$

and
$$1-A = \frac{1-\frac{\sin^2\phi_1}{\left(1+\frac{z}{R+h}\right)^2}}{1-\mu_0'^2} = \frac{1+\frac{2z}{R+h}\tan^2\phi_1}{1+\frac{2z_0}{R+h}\tan^2\phi_1} \quad .. \quad .. \quad (3b)$$

$$\theta-\theta' = \cot\phi_1 - \sqrt{\cot^2\phi_1 - \frac{2h}{R}} \quad .. \quad .. \quad .. \quad .. \quad (4)$$

$$D = 2R\theta \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (5)$$

$$z_v = (R+h)[\sin\theta' \cot(\theta-\theta') - 1 + \cos\theta'] \quad .. \quad .. \quad (6)$$

$$z_v' = \int_0^{z_0} \frac{dz}{\sqrt{1-f_0^2/f^2}} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (7)$$

$$\int_A^B \frac{ds}{\mu} = \int_0^{z_0} \frac{f\,dz}{\sqrt{1-\mu_0'^2}\sqrt{f^2(1-A)-f_0^2}} = \int_0^{z_0} \frac{f'\,dz}{\sqrt{f'\left\{1-\frac{\sin^2\phi_1}{\left(1-\frac{z}{R+h}\right)^2}\right\}-f_0^2}} \quad (8)$$

$$t = \frac{2}{c}\left[\frac{R\sin(\theta-\theta')}{\sin\phi_1} + \int_A^B \frac{ds}{\mu}\right] \quad .. \quad .. \quad .. \quad .. \quad (9)$$

$$= \frac{2}{c}\left[\frac{R\sin(\theta-\theta')}{\sin\phi_1} + \frac{1}{\sqrt{1-\mu_0'^2}}\int_0^{z_0}\frac{f\,dz}{\sqrt{f^2(1-A)-f_0^2}}\right] \quad .. \quad (9a)$$

$$= \frac{2}{c}\left[\frac{R\sin(\theta-\theta')}{\sin\phi_1} + \frac{f'}{f}\int_0^{z_0}\frac{dz}{\sqrt{1-f_0^2/f^2}}(1+B)\right] \quad .. \quad .. \quad (9b)$$

where θ is half the angle the total ray path subtends at the centre of the earth, while θ' is half the angle subtended at the earth's centre by the ray path inside the ionosphere.

The absorption coefficient per unit path of the ray is given by

$$k = \frac{\nu}{2c\mu}\frac{f_0^2}{f^2} \quad .. \quad .. \quad .. \quad .. \quad (10)$$

where the collisional frequency of the electrons, ν, is a function of z.

Using equations (2), (3a) and (8), the total absorption for oblique propagation is

$$\int_{\text{ob.}} k\,ds = 2\int_0^{z_0}\frac{\nu}{2c}\frac{f_0^2}{f'^2}\frac{f'\,dz}{\sqrt{f'^2\left\{1-\frac{\sin^2\phi_1}{\left(1-\frac{z}{R+h}\right)^2}\right\}-f_0^2}} \quad .. \quad (11)$$

$$= \int_0^{z_0}\frac{\nu}{c}\frac{f_0^2}{f}\frac{\sqrt{1-\mu_0'^2}\,dz}{\sqrt{\frac{f^2}{1-\mu_0'^2}\left\{1-\frac{\sin^2\phi_1}{\left(1-\frac{z}{R+h}\right)^2}\right\}-f_0^2}} \quad .. \quad (11a)$$

$$= \frac{\cos\phi_1\sqrt{1+\frac{2z_0\tan^2\phi_1}{R+h}}}{cf^2}\int_0^{z_0}\frac{\nu f_0^2(1+B)dz}{\sqrt{1-f_0^2/f^2}} \quad .. \quad (11b)$$

This relation is similar to that obtained by Millington (1938) who has, however, studied the case of a thin layer, and finds that even for absorption Martyn's theorem holds.

For vertical incidence $\phi_1 = 0$, and $\int_{\text{ve.}} k\,ds = \frac{1}{cf^2}\int_0^{z_0}\frac{\nu f_0^2\,dz}{\sqrt{1-f_0^2/f^2}}$ (12)

The total absorption for oblique propagation can be easily found for special cases; viz:—

(a) $z_0 << \frac{1}{2}(R+h)\cot^2\phi_1$.

This case applies to E-layer transmission for any distance and to F-layer transmission for short distances. In this case (3*a*) and (3*b*) reduce to (as proved by Smith)

$$1-A = 1+\frac{2(z-z_0)\tan^2\phi_1}{R+h} \quad .. \quad .. \quad .. \quad (13)$$

$$1+B = 1+\frac{z_0\tan^2\phi_1}{R+h}\,\frac{1-z/z_0}{1-f_0^2/f^2} \quad .. \quad .. \quad (14)$$

Hence

$$\int_{\text{ob.}} kds = \frac{\cos\phi_1\left(1+\dfrac{z_0\tan^2\phi_1}{R+h}\right)}{cf^2} \times \int_0^{z_0} \frac{\nu f^2}{\sqrt{1-\dfrac{f_0^2}{f^2}}}\left\{1+\frac{z_0\tan^2\phi_1}{R+h}\,\frac{1-\dfrac{z}{z_0}}{1-\dfrac{f_0^2}{f^2}}\right\}dz \quad (15)$$

Case I.—Linear distribution of ionic-density.

$$1-z/z_0 = 1-f_0^2/f^2$$

and hence,

$$\int_{\text{ob.}} kds = \frac{\cos\phi_1\left(1+\dfrac{z_0}{R+h}\tan^2\phi_1\right)^2}{cf^2} \int_0^{z_0} \frac{\nu f_0^2\,dz}{\sqrt{1-f_0^2/f^2}}$$

Using (12)

$$\int_{\text{ob.}} kds = \cos\phi_1\left(1+\frac{2z_0\tan^2\phi_1}{cf^2}\right)\int_{\text{ve.}} kds \quad .. \quad .. \quad (16)$$

but from (7)

$$z_v' = 2z_0$$

hence,

$$\int_{\text{ob.}} kds = \cos\phi_1\left(1+\frac{z_v'\tan^2\phi_1}{R+h}\right)\int_{\text{ve.}} kds. \quad .. \quad .. \quad (16a)$$

Case II.—Exponential distribution of ionic density.

$$f_0^2 = \gamma^2 e^{\beta z},\ f^2 = \gamma^2 e^{\beta z_0} \text{ and } \nu = \nu_1 e^{-\frac{z}{H}} \quad .. \quad .. \quad (16b)$$

where H is the height of the homogeneous atmosphere. By substituting these in equation (22), we get

$$\int_{\text{ve.}} kds = \frac{1}{cf^2}\int_0^{z_0} \frac{\nu_1\gamma^2 e^{-\frac{z}{H}} e^{\beta z}\,dz}{\sqrt{1-e^{\beta(z-z_0)}}}$$

Putting

$$e^{\beta(z-z_0)} = \cos^2\theta$$

$$\int_{ve.} kds = \frac{2\nu_1\gamma^2 e^{z_0\left(\beta-\frac{1}{H}\right)}}{\beta c f^2} \int_0^{\cos^{-1}\left(e^{-\beta z_0/2}\right)} (\cos\theta)^{\left(1-\frac{2}{\beta H}\right)} d\theta \quad .. \quad (17)$$

When $\frac{2}{\beta H} << 1$, $\int_{ve.} kds = \frac{2\nu_1\gamma^2 e^{z_0\left(\beta-\frac{1}{H}\right)}}{\beta c f^2}\sqrt{1-e^{-\beta z_0}}$ .. .. (18)

Now for oblique propagation eqn. (15) gives

$$\int_{ob.} kds = \frac{\cos\phi_1\left(1+\frac{z_0\tan^2\phi_1}{R+h}\right)}{cf^2} \int_0^{z_0} \frac{\nu_1\gamma^2 e^{-\frac{z}{H}} e^{\beta z}}{\sqrt{1-e^{\beta(z-z_0)}}} \times \left\{1-\frac{\tan^2\phi_1}{R+h}\frac{z-z_0}{1-e^{\beta(z-z_0)}}\right\} dz \quad (19)$$

Putting

$$e^{\beta(z-z_0)} = \cos^2\theta$$

$$\int_{ob.} kds = \frac{\cos\phi_1\left(1+\frac{z_0\tan^2\phi_1}{R+h}\right)}{cf^2} \int_0^{\cos^{-1}\left(e^{-\frac{\beta z_0}{2}}\right)} \frac{2\nu_1\gamma^2 e^{z_0\left(\beta-\frac{1}{H}\right)}}{\beta} (\cos\theta)^{1-\frac{2}{\beta H}} \times \left\{1-\frac{2}{\beta}\frac{\tan^2\phi_1}{(R+h)}\frac{\log\cos\theta}{\sin^2\theta}\right\} d\theta.$$

When $\frac{2}{\beta H} << 1$

$$\int_{ob.} kds = \cos\phi_1\left(1+\frac{z_0}{R+h}\tan^2\phi_1\right)\left[\int_{ve.} kds + \frac{2}{\beta}\frac{\tan^2\phi_1}{R+h}\frac{\nu_1\gamma^2 e^{z_0\left(\beta-\frac{1}{H}\right)}}{cf^2}\left(z_v' - \frac{z_0}{\sqrt{1-e^{-\beta z_0}}}\right)\right]$$

$$= \cos\phi_1\left[1+\frac{\tan^2\phi_1}{R+h}\left\{\frac{z_v'}{\sqrt{1-e^{-\beta z_0}}}+z_0\left(1-\frac{1}{1-e^{-\beta z_0}}\right)\right\}\right]\int_{ve.} kds$$

$$= \cos\phi_1\left[1+\frac{z_v'\tan^2\phi_1}{R+h}\right]\int_{ve.} kds \quad .. \quad .. \quad .. \quad .. \quad (20)$$

where z_v' is given by the eqn. (32), proved later on. Thus we find that the absorptions for linear and exponential distribution of electrons come out to be the same.

(*b*) Thin layer; $z_0 \to 0$.

Case I.—Linear distribution.

$$\int_{ob.} kds = \cos\phi_1 \int_{ve.} kds \qquad .. \qquad .. \qquad .. \quad (21)$$

as deduced by Millington, as well as Martyn.

Case II.—Exponential gradient of ionic density.

Eqn. (20) reduces to

$$\int_{ob.} kds = \cos\phi_1 \left[\int_{ve.} kds + \frac{2}{\beta} \frac{\tan^2\phi_1 \nu_1 \gamma^2}{(R+h)} \frac{z_v'}{cf^2} \right] \qquad .. \quad (22)$$

(*c*) For $z_0 \to z_v'$.

We find that the ray suffers hardly any deviation till the point of reflection is reached. The sporadic E-region can be cited as an excellent example of such reflections.

From (1*a*), we have

$$\sqrt{1-\mu_0'^2} = \cos\phi_0 \left(1 - \frac{z_v' - z_0}{R+h} \tan^2\phi_0\right)$$

For $z_v' - z_0$ vanishingly small compared with $(R+h)\cot^2\phi_0$ this reduces to

$$\sqrt{1-\mu_0'^2} = \cos\phi_0 = f/f' \qquad .. \qquad .. \qquad .. \qquad .. \quad (23)$$

and we have from (11*a*) by substituting from (23)

$$\int_{ob.} kds = \int_0^{z_0} \frac{\cos\phi_0}{cf} \frac{\nu f_0^2 \, dz}{\sqrt{f^2 \frac{\cos^2\phi_1}{\cos^2\phi_0}\left(1 + \frac{2z\tan^2\phi_1}{R+h}\right) - f_0^2}} \qquad .. \quad (24)$$

Case I.—Linear distribution of ionic density.

Put

$$f_0^2 = \gamma_1^2 z, \quad f^2 = \gamma_1^2 z_0 \quad \text{and} \quad \nu = \nu_1 e^{-\frac{z}{H}}$$

We have from (24)

$$\int_{ob.} kds = \int_0^{z_0} \frac{\cos^2\phi_0}{\cos^2\phi_1} \frac{\nu_1\gamma_1}{cf} \frac{ze^{-\frac{z}{H}} dz}{\sqrt{z_0 - z\left(\frac{\cos^2\phi_0}{\cos^2\phi_1} - \frac{2z_0\tan^2\phi_1}{R+h}\right)}} \qquad .. \quad (25)$$

Putting

$$z\left(\frac{\cos^2\phi_0}{\cos^2\phi_1}-\frac{2z_0\tan^2\phi_1}{R+h}\right)=za^2=z_0\cos^2\theta$$

$$\int_{\text{ob.}} kds=\frac{2\cos^2\phi_0\nu_1\gamma_1 z_0^{\frac{3}{2}}}{\cos\phi_1 cf^2a^4}\int_{\cos^{-1}a}^{\frac{\pi}{2}} e^{-\frac{z_0\cos^2\theta}{a^2H}}\cos^3\theta\,d\theta$$

$$=\frac{2\cos^2\phi_0\nu_1\gamma_1 z_0^{\frac{3}{2}}e^{-\frac{z_0}{2a^2H}}}{\cos\phi_1 cf^2a^4}\int_{\cos^{-1}a}^{\frac{\pi}{2}} e^{-\frac{z_0\cos 2\theta}{2a^2H}}\cos^3\theta\,d\theta \quad .. \quad (26)$$

The first term in the above integral can be expanded by Sonine's Theorem

$$e^{A\cos\psi}=I_0(A)+2\sum_{n=1}^{n=\infty} I_n(A)\cos n\psi$$

where $I_n(A)$ is the modified Bessel Coefficient given by

$$I_n(A)=\sum_{k=0}^{\infty}\frac{1}{\pi(k)\pi(n+k)}\left(\frac{A}{2}\right)^{n+2k}$$

and $\pi(k)$ is Gauss's function having the values unity for $k=0$ and $\underline{|k}$ for positive integral values of k.

The integral in eqn. (26) can be approximately solved by taking the first few terms of the expansion. The expression for the total absorption for oblique propagation is given by

$$\int_{\text{ob.}} kds=\frac{2\cos^2\phi_0\nu_1\gamma_1}{\cos\phi_1 cf^2}\,\frac{z_0^{\frac{3}{2}}}{a^4}\,e^{-\frac{z_0}{2a^2H}}\left[\left\{\frac{2}{3}-\frac{\sqrt{1-a^2}}{3}(a^2+2)\right\}\right.$$

$$-\left(\frac{z_0}{4a^2H}\right)\left\{\frac{4}{5}-\sqrt{1-a^2}\left(\frac{2}{5}a^3+\frac{4}{5}a^2+\frac{4}{5}\right)\right\}$$

$$+\left(\frac{z_0}{4a^2H}\right)^2\left\{\frac{76}{105}-\sqrt{1-a^2}\left(\frac{8}{7}a^6-\frac{3}{35}a^4+\frac{38}{105}a^2+\frac{76}{105}\right)\right\}$$

$$\left.-\left(\frac{z_0}{4a^2H}\right)^3\left\{\frac{376}{945}-\sqrt{1-a^2}\left(\frac{32}{27}a^8-\frac{176}{189}a^6+\frac{94}{315}a^4+\frac{164}{945}a^2+\frac{376}{945}\right)\right\}\right] \quad (27)$$

where $$a^2=\frac{\cos^2\phi_0}{\cos^2\phi_1}-\frac{2z_0\tan^2\phi_1}{R+h}$$

For vertical propagation $\phi_1=\phi_0=0$, hence $a=1$. If we put these values in eqn. (27) we get the same expression as obtained by Martyn (1935).

The exponential gradient of ionization has not been used for calculating the total absorption in this case ($z_0\to z_H$) since the deviating region is too thin.

2. Relation for Equivalent Frequency.

To find out relations for maximum usable frequency with distance we have to simplify eqns. (1) to (9), but then we must know the law of distribution of electron-density. Smith (1938) has assumed this as linear and obtained simplified forms of equations (3), (7), (8) and (9). However, the actual conditions in the ionosphere are well explained by assuming exponential gradient of ionization. In what follows the particular cases are discussed and results derived assuming that the number of electrons per unit volume $N \propto e^{\beta z}$

(a) $z_o << \frac{1}{2}(R+h)\cot^2\phi_1$.

In this case eqn. (1) reduces to

$$f = f' \cos\phi_1\left(1+\frac{z_o\tan^2\phi_1}{R+h}\right) \quad .. \quad .. \quad .. \quad (28)$$

and eqn. (3) becomes

$$\theta' = \frac{\tan\phi_1}{R+h}\left(1-\frac{z_o\tan^2\phi_1}{R+h}\right)\int_0^{z_0}\frac{dz}{\sqrt{1-f_0^2/f^2}}\left\{1+\frac{z_o\tan^2\phi_1}{R+h}\;\frac{1-z/z_o}{1-f_0^2/f^2}-\frac{2z}{R+h}\right\} \quad (29)$$

Now for exponential gradient of ionization, let us put

$$f_o^2 = \gamma^2 e^{\beta z} \text{ and } f^2 = \gamma^2 e^{\beta z_0} \quad .. \quad .. \quad .. \quad (30)$$

and obtain the forms of various equations. From (29), (30) we have

$$\theta' = \frac{\tan\phi_1}{R+h}\left(1-\frac{z_o\tan^2\phi_1}{R+h}\right)\int_0^{z_0}\frac{dz}{\sqrt{1-e^{\beta(z-z_0)}}}\left\{1-\frac{\tan^2\phi_1}{R+h}\;\frac{z-z_o}{1-e^{\beta(z-z_0)}}-\frac{2z}{R+h}\right\} \quad (31)$$

and (7) reduces to

$$z_v' = \int_0^{z_0}\frac{dz}{\sqrt{1-e^{\beta(z-z_0)}}}$$

$$= \frac{2}{\beta}\log\frac{1+\sqrt{1-e^{-\beta z_0}}}{e^{-\beta z_0/2}} \quad .. \quad .. \quad .. \quad (32)$$

$$= \frac{2}{\beta}\log 2 + z_o \qquad \text{since } e^{-\beta z_0} << 1 \quad .. \quad (32a)$$

(31) can also be put as

$$\theta' = \frac{\tan\phi_1}{R+h}\left[1-\frac{z_o\tan^2\phi_1}{R+h}\right]\left[z'_v-\frac{\tan^2\phi_1}{R+h}C_1-\frac{2}{R+h}C_2\right] \quad .. \quad (33)$$

where

$$C_1 = \int_0^{z_0} \frac{z - z_o}{(1-e^{\beta(z-z_0)})^{\frac{3}{2}}}\, dz$$

$$= \frac{2z_0}{\beta\sqrt{1-e^{-\beta z_0}}} - \frac{2}{\beta} z_v' + \frac{4}{\beta^2} \int_0^{\cos^{-1}\left(e^{-\frac{\beta z_0}{2}}\right)} \frac{\log(\cos\theta)}{\cos\theta}\, d\theta$$

$$C_2 = \int_0^{z_0} \frac{z\, dz}{\sqrt{1-e^{\beta(z-z_0)}}} = z'_v z_o + \frac{4}{\beta^2} \int_0^{\cos^{-1}\left(e^{-\frac{\beta z_0}{2}}\right)} \frac{\log(\cos\theta)}{\cos\theta}\, d\theta.$$

Now $$\int_0^{\cos^{-1}\left(e^{-\frac{\beta z_0}{2}}\right)} \frac{\log(\cos\theta)}{\cos\theta}\, d\theta = -\int_0^{\frac{\beta z_0}{2}} \psi(1-e^{-2\psi})^{-1/2}\, d\psi.$$

Expanding and performing the integration we have to a close approximation,

$$-\int_0^{\cos^{-1}\left(e^{-\frac{\beta z_0}{2}}\right)} \frac{\log(\cos\theta)}{\cos\theta}\, d\theta$$

$$= \frac{1}{2}\left(\frac{\beta z_o}{2}\right)^2 - \frac{e^{-\beta z_0}}{8}(\beta z_o + 1) - \frac{3}{64} e^{-2\beta z_0}\left(\beta z_o + \frac{1}{2}\right) - \frac{5}{192} e^{-3\beta z_0}\left(\beta z_o + \frac{1}{3}\right) - \frac{181}{1152} \quad (34)$$

Hence,

$$\theta' = \frac{\tan\phi_1}{R+h}\left(1 - \frac{z_o \tan^2\phi_1}{R+h}\right)\left[z'_v - \frac{\tan^2\phi_1}{R+h}\left(\frac{2z_0}{\beta(1-e^{-\beta z_0})^{1/2}} - \frac{2}{\beta} z'_v\right)\right.$$

$$\left. - \frac{2z_o z'_v}{R+h} - \left(\frac{4}{\beta^2}\frac{\tan^2\phi_1}{R+h} + \frac{4}{\beta^2}\frac{z}{R+h}\right) \int_0^{\cos^{-1}\left(e^{-\frac{\beta z_0}{2}}\right)} \frac{\log(\cos\theta)}{\cos\theta}\, d\theta\right]$$

$$= \frac{\tan\phi_1}{R+h}\left(1 - \frac{z_o \tan^2\phi_1}{R+h}\right)\left[z'_v\left(1 - \frac{2z_o}{R+h}\right) - \frac{2}{\beta}\frac{\tan^2\phi_1}{R+h}\left(\frac{z_o}{(1-e^{-\beta z_0})^{1/2}} - z'_v\right)\right.$$

$$+ \frac{4}{\beta^2}\frac{(\tan^2\phi_1 + 2)}{R+h}\left\{\frac{\beta^2 z_o^2}{8} - \frac{e^{-\beta z_0}}{8}(\beta z_o + 1) + \frac{3}{64} e^{-2\beta z_0}\left(\beta z_o + \frac{1}{2}\right)\right.$$

$$\left.\left. - \frac{5}{192} e^{-3\beta z_0}\left(\beta z_o + \frac{1}{3}\right) - \frac{181}{1152}\right\}\right] \quad .. \quad .. \quad (35)$$

Now $e^{-\beta z_0} << 1$, neglecting coefficients of $e^{-\beta z_0}$, and using (32a), θ' can be written in a more modified form as,

$$\theta' = \frac{\tan\phi_1}{R+h}\left[z'_v\left\{1+\frac{z'_v}{R+h}\left(\frac{\tan^2\phi_1}{2}-1\right)+\frac{2\tan^2\phi_1}{R+h}(1-\log 2)\right\}\right.$$
$$\left.+\frac{4}{\beta^2}\frac{\log 2}{R+h}\left\{\log 2\left(1+\frac{1}{2}\tan^2\phi_1\right)+\tan^2\phi_1\right\}\right]$$

and to a further approximation

$$\theta' = \frac{\tan\phi_1}{R+h}\left[z'_v\left\{1-\frac{z'_v}{R+h}\left(1-\frac{\tan^2\phi_1}{2}\right)\right\}\right] \quad .. \quad .. \quad (36)$$

Using eqns. (13), (14), (16a) and (28), (9b) becomes

$$t = \frac{2}{c}\left[\frac{R\sin(\theta-\theta')}{\sin\phi_1}+\frac{\left(1-\frac{z_o\tan^2\phi_1}{R+h}\right)}{\cos\phi_1}\int_0^{z_0}\frac{dz}{\sqrt{1-\frac{f_o^2}{f^2}}}\right.$$
$$\left.\left\{1-\frac{z_o\tan^2\phi_1}{R+h}\left(1-\frac{1-\frac{z}{z_o}}{1-\frac{f_o^2}{f^2}}\right)\right\}\right]$$
$$= \frac{2}{c}\left[\frac{R\sin(\theta-\theta')}{\sin\phi_1}+\frac{\left(1-\frac{z_o\tan^2\phi_1}{R+h}\right)}{\cos\phi_1}\int_0^{z_0}\frac{dz}{\sqrt{1-e^{\beta(z-z_0)}}}\right.$$
$$\left.\left\{1-\frac{z_o\tan^2\phi_1}{R+h}\frac{\tan^2\phi_1}{R+h}\frac{z-z_o}{1-e^{\beta(z-z_0)}}\right\}\right]$$
$$= \frac{2}{c}\left[\frac{R\sin(\theta-\theta')}{\sin\phi_1}+\frac{\left(1-\frac{z_o\tan^2\phi_1}{R+h}\right)}{\cos\phi_1}\left\{z'_v\left(1-\frac{z_o\tan^2\phi_1}{R+h}\right)\right.\right.$$
$$-\frac{\tan^2\phi_1}{R+h}\frac{2}{\beta}\left(\frac{z_0}{\sqrt{1-e^{-\beta z_0}}}-z'_v\right)+\frac{\tan^2\phi_1}{R+h}\frac{4}{\beta^2}\left(\frac{\beta^2 z_o^2}{8}-\frac{e^{-\beta z_0}}{8}(\beta z_o+1)\right.$$
$$\left.\left.\left.-\frac{3}{64}e^{-2\beta z_0}\left(\beta z_o+\frac{1}{2}\right)-\frac{5}{192}e^{-3\beta z_0}\left(\beta z_o+\frac{1}{3}\right)-\frac{181}{1152}\right)\right\}\right] \quad .. \quad (37)$$

Making similar simplification as has been done for θ' and using (32a) t can be written in a modified form as

$$t = \frac{2}{c}\left[\frac{R\sin(\theta-\theta')}{\sin\phi_1}+\frac{1}{\cos\phi_1}\left\{z'_v\left(1-\frac{3}{2}\frac{\tan^2\phi_1}{R+h}z'_v+\frac{2}{\beta}\frac{\tan^2\phi_1}{R+h}\log 2\right)\right.\right.$$
$$\left.\left.+\frac{2}{\beta^2}\frac{\tan^2\phi_1}{R+h}\log 2\,(2+\log 2)\right\}\right] \quad .. \quad .. \quad (38)$$

and to a further approximation

$$t = \frac{2}{c}\left[\frac{R\sin(\theta-\theta')}{\sin\phi_1} + \frac{z_v'}{\cos\phi_1}\left(1-\frac{3\tan^2\phi_1}{2}\frac{z_v'}{R+h}\right)\right] \quad .. \quad (39)$$

Relations (4), (5) and (6) will remain the same as deduced by Smith.

3. Conclusion and Discussion.

It is now well known that the intensity of the sky-wave increases for all frequencies up to a distance of about 500 kms. after which it slowly decreases—*vide* Report of committee on radio wave propagation, *P.I.R.E.* Vol. 26, p. 1193, (1938). The theory of absorption worked out by Martyn for flat earth and Millington for curved earth for thin layer both lead to the same results, *i.e.*, the absorption for oblique propagation is sec ϕ_1 times the absorption for vertical propagation. This theory is unable to explain the maximum in the signal strength observed experimentally. However, as has been done in this paper, if the absorption is calculated for a thick layer we see from eqn. (20) that it decreases upto a certain distance, after which it increases, which can explain the form of the signal intensity curve obtained experimentally. It may be pointed out that the exact distance from the transmitter where the maximum of signal intensity is observed cannot be calculated from these results, since we have assumed the condition $z_0 << \frac{1}{2}(R+h)\cot^2\phi_1$. The removal of this restriction probably enables us to calculate the exact place where the maximum of signal intensity is observed. But the calculations are too tedious, and will be dealt with later. It should be noted that our results show that the absorption for both the linear and exponential distribution of electrons comes out to be the same.

In § 2 the various relations have been deduced on the hypothesis of exponential distribution of electrons in the ionosphere. The equations for θ' and t presented in this paper are slightly different from those given by Smith (1938) for linear distribution. The value of θ' obtained for exponential distribution is smaller than that obtained for linear distribution as long as $\tan^2\phi_1 \leq \frac{2}{3}$, *i.e.*, $\phi_1 \leq 39° 16'$, while for values of ϕ_1 greater than $39° 16'$, the reverse will happen. The time taken by the ray to reach the receiver, t, depends upon two factors, viz. (1) $\frac{R\sin(\theta-\theta')}{\sin\phi_1}$ and (2) $\int_0^{z_0}\frac{ds}{\mu}$. The value of $\frac{R\sin(\theta-\theta')}{\sin\phi_1}$ depends upon θ', the variation of which has already been discussed. $\int_0^{z_0}\frac{ds}{\mu}$ will always be less for exponential distribution than for linear distribution as is evident from eqn. (39) and the one deduced by Smith (1938) on page 696, case (*b*). (There appears to be a slip in his formula for T. It should

be $T = \frac{2}{c}\left\{\frac{R \sin(\theta-\theta')}{\sin \phi_1} + \frac{z'_v}{\cos \phi_1}\right\}$). Hence for long distances the time taken by the sky-wave is smaller for exponential distribution than the time taken if the electrons are distributed linearly. The exponential distribution of electrons represents the conditions existing in the ionosphere in a better way, therefore curves plotted after Smith (1938) using the results presented here will, it is hoped, give a more accurate value for maximum usable frequency.

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Abstract.

It is extremely useful to obtain data for long-distance transmission of electromagnetic waves through the ionosphere from observations made at vertical incidence. Martyn first showed that the absorption for oblique propagation is simply related to that observed at vertical incidence with the three simplifying assumptions: (i) that the gradient of electron density is everywhere vertical and similar throughout the path of the ray, (ii) that the earth's magnetic field can be neglected and (iii) that the earth is flat. Smith further showed that the maximum usable frequency for oblique propagation could also be derived from the characteristics of $(P'-f)$ curves obtained at vertical incidence under the same assumptions as those of Martyn. Millington developed the above theorems for a curved earth and showed them to hold for a relatively thin layer. Millington's work has been further modified by Smith to allow for a thick layer assuming linear distribution of electron density. But he has only derived an expression for equivalent frequency. In the present paper, starting from Smith's theorems, relations have been deduced both for equivalent frequency as well as for absorption for the cases of linear and exponential gradient of electron density.