ON THE THEORY OF A SPIRAL NEBULA. I.

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(Communicated by Principal G. S. Mahajani, PH.D.)

(*Read March 11, 1940.*)

1. *Introduction.*

In a series of papers on the evolution and the structure of a spiral nebula, Sir James Jeans [1] has shown that no possible explanation for the perpetual existence of the spiral arms on any consistent mechanical theory can be given.* The main part of his work is based on the hypothesis that a spiral nebula is formed from the process of cooling and contracting of a rotating spherical mass of gas ultimately forming a lenticular shape whose periphery is not a circle but an ellipse, the spiral arms being formed by the ejection of matter from the end points of the major axis of the elliptical sharp edge. The elliptical shape taken by the periphery of the lenticular mass is due to the tidal effects of other masses present in space. The points of difficulty which cannot be explained on any consistent theory are:

(i) No satisfactory explanation has been given as to why the spiral arms should have these particular shapes.[2]

(ii) The spiral arms are orbits described by the ejected matter. This is not tenable as the orbits under gravitation must be conics. If, as a particular case, equiangular spirals are obtained, the arms would in general increase in length with the time, so that the number of convolutions is very large; actually only two convolutions (perpetuated in static form ever since they came into being) exist.[3]

(iii) The persistence of equiangular spiral form for the arms of the nebulae can only be explained by a pure rotation which it is impossible to explain dynamically in terms of known forces.[4]

(iv) The law of force necessary to produce orbits whose envelope is an equiangular spiral is found to be very complicated, demanding a highly artificial distribution of matter to produce it.[5]

Concluding on these important points of difficulty arising in the theory of the spiral arms, Sir James Jeans observes, '. . . until the spiral arms have been satisfactorily explained, it is impossible to feel confidence in any

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* Besides the theory given by Sir James Jeans there are other theories given by different authors on the evolution and structure of spiral nebulae. A short account of the important ones is given at the end of this paper.

conjectures or hypotheses in connection with other features of the nebulae which seem amenable to treatment. Each failure to explain the spiral arms makes it more and more difficult to resist a suspicion that the spiral nebulae are the seat of types of forces entirely unknown to us, forces which may possibly express novel and unsuspected metric properties of space. The type of conjecture, which presents itself somewhat insistently, is that the centres of the nebulae are of the nature of "singular points", at which matter is poured into our universe from some other, and entirely extraneous, spatial dimension, so that, to a denizen of our universe, they appear as points at which matter is being continually created.' [6]

These difficulties can be overcome if we agree to the validity of the hypothesis put forward by E. W. Brown,[7] that, the spiral nebula is formed of a highly concentrated central distribution of matter along with an extension of the same distribution with a very low space density beyond the nucleus.* The two distinct regions are the lenticular mass and the space beyond this, which contains the spirals. The particles supposed to form this outer region are non-colliding. They describe orbits under gravitational forces only, the law of gravitation, in this case, being a combination of the usual Newtonian law with that of the direct distance.† The envelopes of these orbits form the spiral arms of the nebula.

It is well known that the distribution in the region is due to the matter ejected by the central ellipsoid in rotation. The particles moving along the spiral arms are projected with a velocity of projection v_p, and after being projected, they move entirely under the gravitational influence of the total mass. The motion of particles along the spiral arms, therefore, is purely a problem in Dynamics.‡

In the first part of this paper we shall consider the perturbed motion of a particle moving along the spiral arm. It will then be shown that the motion slows down as the particle proceeds along the orbit, till at last a stage is reached beyond which the particles do not proceed. This explains the finite length of a spiral arm. Moreover, for a particular system of particles

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* Here, we are taking recourse only to the law of force for the motion of the ejected particles outside the boundary of the rotating ellipsoid. We do not agree with Brown's hypothesis of superposition of periodic density on the homogeneous density distribution of the outer region and his notion of periodic encounters.

† It is shown in another communication that the metric properties of space (according to the theory of relativity) would give the corresponding law of force for quasi-Newtonian mechanics as: $f(r) = \frac{M}{r^2} + \beta r + \frac{Mh^2}{r^3}$ in the usual notation, where it is shown that the third term on the r.h.s. of this relation is negligible when compared to the other two terms.

‡ That such a theory based on purely dynamical principles with a suitable modification of Newton's law is quite logical has been borne out by the works of Vogt, Lambrecht, Armellini and others. cf. 'Fundamental Problems of Cosmogony and Newton's law', G. Armellini, *Accad. Lincei, Atti,* 28, pp. 209–215, and subsequent papers.

the slowing down might be reached after a short distance from the region of ejection. At this stage, clogging of matter takes place, and, thus, clusters are formed along the spiral arms. Approximations will be used in finding the solution of equations of the perturbed motion of a particle moving along spiral orbits. The question of the stability of the spiral arms will be fully considered. Astronomical units are chosen. The plane of the spiral arms and that of the lenticular periphery will be taken as the xy-plane. In part II the perturbation of the spiral orbits due to tidal forces will be considered at some length.

M is the mass of the central ellipsoidal distribution, m, that of each particle moving along the spiral arms. It is clear that M varies with the time.

PART I.—EQUATIONS OF MOTION ALONG THE SPIRAL ARMS IN THE STEADY STATE AND THE CRITERION OF STABILITY OF THE SPIRAL ARMS.

2. *The conditions for the ejection of matter by the rotating mass.*

Consider the forces acting on a particle of the rotating mass lying in the plane of the lenticular periphery. A centrifugal force $\omega^2 r$ is acting on the particle tending an outward motion away from the centre of the lenticular mass, where ω is the angular velocity of the particle and r is its distance from the centre. Secondly, a force of attraction is acting on the particle; this force is zero when the particle is inside the mass. This force is given by the expression $f(r)$:

$$\frac{M}{r^2}+\mu r=f(r), \qquad \ldots \qquad (1)$$

for a particle lying outside the lenticular mass. If a is the semi-major axis of the peripheral ellipse, the condition for the ejection of matter can be written as

$$\omega^2 a \geqslant \frac{M}{a^2}+\mu a,$$

that is,

$$(\omega^2-\mu) \geqslant \frac{M}{a^3} \qquad \ldots \qquad (2)$$

Here M is the mass of the central body and μ is a constant depending upon the ratio of semi-major and polar axes of the peripheral ellipse and the vertical section, and the density ρ of the central distribution of matter. If a, b, c are the semi-axes of the ellipsoid, the condition (2) can be expressed in the form

$$\frac{\omega^2}{2\pi\rho} \geqslant \frac{2bc}{3a^2}+\frac{\pi c}{2a} \qquad \ldots \qquad (3)$$

where $M=\frac{4}{3}\pi abc\rho$ and $\mu=\frac{\pi^2\rho c}{a}$. The following table gives different configurations for different values of the ratios $\frac{b}{a}$ and $\frac{c}{a}$:—

TABLE I.

$\frac{b}{a}$	$\frac{c}{a}$	$\frac{\omega^2}{2\pi\rho} \geqslant \frac{\omega^2}{2\pi\rho_0}$
1·0	1·0	for ρ_0: 2·237
0·5	0·5	0·952
0·5	0·33	0·635
0·33	0·33	0·598
0·5	0·25	0·476
0·33	0·25	0·448
0·25	0·25	0·434
0·75	0·18	0·360 *
0·5	0·167	0·327
0·75	0·15	0·311
0·25	0·167	0·290

3. *The equations of disturbed motion of the system of particles moving along the spiral arm.*

Let O be the centre of the rotating ellipsoid and $Oxyz$ a fixed co-ordinate system through O. The xy-plane is supposed to coincide with the plane of the lenticular periphery of the resulting configuration of the rotating ellipsoid. Then the cylindrical co-ordinates of a particle λ of the spiral arm are r_λ, θ_λ, z_λ such that the Cartesian co-ordinates are

$$x_\lambda = r_\lambda \cos\theta_\lambda, \quad y_\lambda = r_\lambda \sin\theta_\lambda, \quad z_\lambda = z_\lambda.$$

Then the potential at the particle λ is given by [8]

$$V_\lambda = \frac{M+m}{\left(r_\lambda^2+z_\lambda^2\right)^{\frac{1}{2}}} - \frac{1}{2}\mu\left(r_\lambda^2+z_\lambda^2\right) + \sum_\nu{}' m \left[\frac{1}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos\left(\theta_\lambda-\theta_\nu\right)+r_\nu^2+\left(z_\lambda-z_\nu\right)^2\right\}^{\frac{1}{2}}} - \frac{r_\lambda r_\nu \cos\left(\theta_\lambda-\theta_\nu\right)+z_\lambda z_\nu}{\left(r_\nu^2+z_\nu^2\right)^{\frac{3}{2}}}\right] \quad \ldots \quad (4)$$

where m is the mass of each particle moving along the spiral arm and $\sum_\nu{}'$ denotes the summation over all particles of the spiral except λ. After the particles leave the central body and follow the spiral tracks it is assumed that the system

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* Cf. $\frac{\omega^2}{2\pi\rho_0} = 0{\cdot}361$, where $\rho_0 > \rho$ which leads to the ejection of particles along the spiral arms. ρ_0 is the average critical density. The criterion of instability is given by

$$\bar{\rho} - \rho_0 > \frac{\omega_c^2}{2\pi G}.$$

of particles so moving forms an isolated system moving under gravitational influence only. The gravitational potential for this case is given by

$$\Omega = \frac{M}{r} - \frac{1}{2}\mu r^2 + S_2 \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

where S_2 is the effective part of the 'tidal potential'. In this investigation, however, S_2 is omitted as it gives only secular effects. The equations of motion of the particle λ are now given by

$$\left.\begin{aligned} \ddot{r}_\lambda - r_\lambda \dot{\theta}_\lambda^2 &= \frac{\partial V_\lambda}{\partial r_\lambda}, \\ 2\dot{r}_\lambda \dot{\theta}_\lambda + r_\lambda \ddot{\theta}_\lambda &= \frac{\partial V_\lambda}{r_\lambda \partial \theta_\lambda}, \\ \ddot{z}_\lambda &= \frac{\partial V_\lambda}{\partial z_\lambda}. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

If a is the semi-major axis of the lenticular peripheral ellipse, the initial conditions are (that is, when the particle λ is on the point of being ejected):

$$\left(r_\lambda\right)_0 = a, \quad \left(\theta_\lambda\right)_0 = \omega t + \epsilon, \quad \left(z_\lambda\right)_0 = 0,$$

where ω is the velocity of rotation of the particle λ at the point of leaving the central ellipsoid. Hence, $r_\lambda = a$, $\theta_\lambda = \omega t + \epsilon$ is an exact solution of (6) provided the following necessary and sufficient conditions are satisfied, viz.,

$$\left(\frac{\partial V_\lambda}{\partial r_\lambda}\right)_0 = -a\omega^2, \left(\frac{\partial V_\lambda}{r\partial \theta_\lambda}\right)_0 = 0, \left(\frac{\partial V_\lambda}{\partial z_\lambda}\right)_0 = 0, \quad (\lambda = 1, 2, \ldots, n).$$

Now, as the particle λ leaves the central ellipsoidal mass, let its motion be determined by

$$r_\lambda = a\left(1+\rho_\lambda\right), \quad \theta_\lambda = \left(\theta_\lambda\right)_0 + \sigma_\lambda = \omega t + \epsilon + \sigma_\lambda, \quad z_\lambda = 0.$$

Taking ρ_λ and σ_λ as quantities of the first order and neglecting squares, products and higher powers of ρ_λ and σ_λ, we get the equations of motion as*

$$a\left\{\ddot{\rho}_\lambda - 2\omega\dot{\sigma}_\lambda - \omega^2\rho_\lambda - \omega^2\right\} = \left(\frac{\partial V_\lambda}{\partial r_\lambda}\right)_0 + \sum_\nu a\rho_\nu \left\{\frac{\partial^2 V_\lambda}{\partial r_\nu \partial r_\lambda}\right\}_0 + \sum_\nu \sigma_\nu \left\{\frac{\partial^2 V_\lambda}{\partial \theta_\nu \partial r_\lambda}\right\}_0 \qquad \ldots \quad (7)$$

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* Here, we follow the well-known method of celestial mechanics for finding the differential equations for the motion of particles under the influence of a massive central body, and the work is on the same lines as that followed by Prof. Goldsbrough and Dr. C. G. Pendse for the problem of Saturn's rings.

$$a\left(\ddot{\sigma}_\lambda+2\omega\dot{\rho}_\lambda\right)=\left(\frac{\partial V_\lambda}{r_\lambda\partial\theta_\lambda}\right)_0+\sum_\nu a\rho_\nu\left\{\frac{\partial}{\partial r_\nu}\left(\frac{\partial V_\lambda}{r_\lambda\partial\theta_\lambda}\right)\right\}_0+\sum_\nu \sigma_\nu\left\{\frac{\partial}{\partial\theta_\nu}\left(\frac{\partial V_\lambda}{r_\lambda\partial\theta_\lambda}\right)\right\}_0 \quad \ldots \quad (8)^*$$

4. *The equations of the disturbed motion of the particle* λ.

We have from (4):

$$\frac{\partial V_\lambda}{\partial z_\lambda}=-\frac{(M+m)z_\lambda}{\left(r_\lambda^2+z_\lambda^2\right)^{\frac{3}{2}}}-\mu z_\lambda+\sum_\nu{}' m\left[\frac{(z_\nu-z_\lambda)}{\left\{r_\lambda^2-2r_\lambda r_\nu\cos(\theta_\lambda-\theta_\nu)+r_\nu^2+(z_\lambda-z_\nu)^2\right\}^{\frac{3}{2}}}-\frac{z_\nu}{\left(r_\nu^2+z_\nu^2\right)^{\frac{3}{2}}}\right].$$

Therefore, for the initial value $(r_\lambda)_0=a$, we have

$$\left(\frac{\partial V_\lambda}{\partial z_\lambda}\right)_0=0.$$

Moreover, for motion along the spiral arms, we must have $z_\lambda=0$ for all λ; hence, for particles moving along spiral tracks V_λ can be written as

$$V_\lambda=\frac{M+m}{r_\lambda}-\frac{1}{2}\mu r_\lambda^2+\sum_\nu{}' m\left[\frac{1}{\left\{r_\lambda^2-2r_\lambda r_\nu\cos(\theta_\lambda-\theta_\nu)+r_\nu^2\right\}^{\frac{1}{2}}}-\frac{r_\lambda\cos(\theta_\lambda-\theta_\nu)}{r_\nu^2}\right] \quad \ldots \quad (9)$$

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* For a general case, in addition to these equations, we must also consider third of the equations (6). We have taken the plane of the lenticular periphery as the xy-plane, so that, initially, we must have $(z_\lambda)_0=0$. For this to be the case, therefore, we must put

$r=(x^2+y^2+z^2)^{\frac{1}{2}}$, and

$$\frac{\partial V_\lambda}{\partial z_\lambda}=\frac{(M+m)z_\lambda}{r_\lambda^3}+\mu z_\lambda+\sum_\nu{}' m\left[\frac{(z_\nu-z_\lambda)}{\{(x_\lambda-x_\nu)^2+(y_\lambda-y_\nu)^2+(z_\lambda-z_\nu)^2\}^{\frac{3}{2}}}-\frac{z_\nu}{r_\nu^{3/2}}\right],$$

Hence, $\left(\frac{\partial V_\lambda}{\partial z_\lambda}\right)_0=0$; initially, there is no force to disturb the motion of the ejected particle from the xy-plane. The disturbing influence which we have considered in this paper—and, that is the only one which can be expected—has its effect in the xy-plane only. Therefore, even in the disturbed state we must have $z_\lambda=0$. This shows that the two dimensional nature of the spirals persists even in the state of disturbed motion. See the following section.

$$\frac{\partial V_\lambda}{\partial r_\lambda} = -\frac{(M+m)}{r_\lambda^2} - \mu r_\lambda - \sum_\nu{}' m \left[\frac{r_\lambda - r_\nu \cos(\theta_\lambda - \theta_\nu)}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos(\theta_\lambda - \theta_\nu) + r_\nu^2\right\}^{\frac{3}{2}}} + \frac{\cos(\theta_\lambda - \theta_\nu)}{r_\nu^2} \right] \quad .. \quad (10)$$

$$\frac{\partial V_\lambda}{r_\lambda \partial \theta_\lambda} = \sum_\nu{}' m \left[\frac{r_\nu}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos(\theta_\lambda - \theta_\nu) + r_\nu^2\right\}^{\frac{3}{2}}} - \frac{1}{r_\nu^2} \right] \sin(\theta_\nu - \theta_\lambda) \quad .. \quad (11)$$

In what follows we take $\lambda \neq \nu$ unless explicitly stated otherwise. The following differentiations hold good for motion along the spiral arms, viz.,

$$\frac{\partial^2 V_\lambda}{\partial r_\nu \partial r_\lambda} = -\sum_\nu{}' m \left[-\frac{\cos(\theta_\lambda - \theta_\nu)}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos(\theta_\lambda - \theta_\nu) + r_\nu^2\right\}^{\frac{3}{2}}} - \frac{2\cos(\theta_\lambda - \theta_\nu)}{r_\nu^3} - \frac{3\{r_\lambda - r_\nu \cos(\theta_\lambda - \theta_\nu)\}\{r_\nu - r_\lambda \cos(\theta_\lambda - \theta_\nu)\}}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos(\theta_\lambda - \theta_\nu) + r_\nu^2\right\}^{\frac{5}{2}}} \right] \quad .. \quad (12)$$

$$\frac{\partial^2 V_\lambda}{\partial r_\lambda^2} = \frac{2(M+m)}{r_\lambda^3} - \mu - \sum_\nu{}' m \left[\frac{1}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos(\theta_\lambda - \theta_\nu) + r_\nu^2\right\}^{\frac{3}{2}}} - \frac{3\{r_\lambda - r_\nu \cos(\theta_\lambda - \theta_\nu)\}^2}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos(\theta_\lambda - \theta_\nu) + r_\nu^2\right\}^{\frac{5}{2}}} \right] \quad .. \quad (13)$$

$$\sum_\nu \sigma_\nu \frac{\partial^2 V_\lambda}{\partial \theta_\nu \partial r_\lambda} = \sum_\nu{}' m \left[-\frac{r_\nu}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos(\theta_\lambda - \theta_\nu) + r_\nu^2\right\}^{\frac{3}{2}}} + \frac{1}{r_\nu^2} + \frac{3r_\lambda r_\nu\left\{r_\lambda - r_\nu \cos(\theta_\lambda - \theta_\nu)\right\}}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos(\theta_\lambda - \theta_\nu) + r_\nu^2\right\}^{\frac{5}{2}}} \right] \sin(\theta_\nu - \theta_\lambda)(\sigma_\nu - \sigma_\lambda) \quad .. \quad (14)$$

$$\frac{\partial}{\partial r_\nu}\left(\frac{1}{r_\lambda}\frac{\partial V_\lambda}{\partial \theta_\lambda}\right) = \sum_\nu{}' m \left[\frac{1}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos(\theta_\lambda - \theta_\nu) + r_\nu^2\right\}^{\frac{3}{2}}} + \frac{2}{r_\nu^3} - \frac{3r_\nu\left\{r_\nu - r_\lambda \cos(\theta_\lambda - \theta_\nu)\right\}}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos(\theta_\lambda - \theta_\nu) + r_\nu^2\right\}^{\frac{5}{2}}} \right] \sin(\theta_\nu - \theta_\lambda) \quad .. \quad .. \quad (15)$$

$$\frac{\partial}{\partial r_\lambda}\left(\frac{1}{r_\lambda}\frac{\partial V_\lambda}{\partial \theta_\lambda}\right) = \sum_\nu{}' m \left[-\frac{3r_\nu\left\{r_\lambda - r_\nu \cos(\theta_\lambda - \theta_\nu)\right\}}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos(\theta_\lambda - \theta_\nu) + r_\nu^2\right\}^{\frac{5}{2}}} \right] \sin(\theta_\nu - \theta_\lambda) \quad .. \quad (16)$$

and, lastly, we have

$$\sum_{\nu} \frac{\partial}{\partial\theta_\nu}\left\{\frac{1}{r_\lambda}\frac{\partial V_\lambda}{\partial\theta_\lambda}\right\}\sigma_\nu$$

$$= \sum_{\nu}{}' m \left[\frac{r_\nu}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos\left(\theta_\lambda - \theta_\nu\right) + r_\nu^2\right\}^{\frac{3}{2}}} - \frac{1}{r_\nu^2}\right]\left(\sigma_\nu - \sigma_\lambda\right)\cos\left(\theta_\nu - \theta_\lambda\right)$$

$$- \sum_{\nu}{}' m \left[\frac{3 r_\lambda r_\nu^2 \sin^2\left(\theta_\nu - \theta_\lambda\right)}{\left\{r_\lambda^2 - 2r_\lambda r_\nu \cos\left(\theta_\lambda - \theta_\nu\right) + r_\nu^2\right\}^{\frac{5}{2}}}\right]\left(\sigma_\nu - \sigma_\lambda\right) \quad \ldots \quad \ldots \quad (17)$$

Initially, we must have

$$\left(\frac{\partial V_\lambda}{\partial r_\lambda}\right)_0 = -\frac{M + mn}{a^2} - \mu a,$$

where, as before, a is the semi-major axis of the peripheral ellipse. Hence, we get

$$(\omega^2 - \mu)a^3 = M + mn \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

where n is the number of particles moving along the spiral orbit. Just before the particles are shot off to move along spiral orbits, the angular velocity is given by

$$\omega^2 a^3 = M \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

at the periphery, and, just at the moment of ejection of matter, we have the angular velocity given by (18). Here, it should be pointed out that n is the number of particles moving at a particular epoch. For instance, at the first ejection we have

$$\left(\omega_1^2 - \mu_1\right)a_1^3 = M + mn_1, \text{ say.} \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

Similarly, at the second ejection, we get

$$\left(\omega_2^2 - \mu_2\right)a_2^3 = M + mn_2 - 2mn_1 \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

because, the mass of the central rotating ellipsoidal figure is not M but $(M - 2mn_1)$, where n_1 is the number of particles of mass m each moving along one of the two spiral arms.* Again, at the third ejection, we must have

$$\left(\omega_3^2 - \mu_3\right)a_3^3 = (M - 2mn_1 - 2mn_2) + mn_3 \quad \ldots \quad \ldots \quad (22)$$

* When the central ellipsoid assumes a lenticular shape, and, when the angular velocity assumes a value which satisfies the inequality $(\omega^2 - \mu) \geqslant M/a^3$, particles are shot off from the two antipodal points of the lenticular sharp edge of the rotating central ellipsoid. During one ejection $2n$ particles are shot off, n particles moving along each of the two spiral arms. It is possible that there might be more than two spiral arms and the number of spiral arms need not necessarily be even. It must, of course, be evident that during each ejection two and only two trajectories (spirals) are possible. Again, it should be remarked that there are two possibilities about the points of ejection of the spiral arms. In the case of fixed points of ejection as advocated by Lindblad the arms represent stationary paths of ejected particles. In the alternate case the points of ejection are

and, lastly, for the rth ejection, this relation can be written as

$$\left(\omega_r^2-\mu_r\right)a_r^3 = \left(M-2m\sum_{s=1}^{r-1} n_s\right)+mn_r \quad \ldots \quad \ldots \quad (23)$$

Here, letters with numerical suffixes denote the values of the quantities represented for the particular epoch or ejection.

Let $n_1 = n_1,\quad n_2 = n_1+dn_1,\quad n_3 = n_2+dn_2 = n_1+dn_1+dn_2$

and so on. Then, the following differential equation holds good for the first ejection of particles, viz.,

$$d\left[\left(\omega_1^2-\mu_1\right)a_1^3\right] = -2mn_1+mdn_1 \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

Similarly, for the second ejection we have

$$d\left[\left(\omega_2^2-\mu_2\right)a_2^3\right] = -2mn_2+mdn_2 \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

and, finally, for the rth ejection, this differential equation is found to be

$$d\left[\left(\omega_r^2-\mu_r\right)a_r^3\right] = -2mn_r+mdn_r. \quad \ldots \quad \ldots \quad (26)$$

The relations (24), (25), (26) point out that the expression $(\omega^2-\mu)a^3$ is a decreasing function. Hence, a stage will be reached when no particles are shot off by the central body, so that, motion along the spiral tracks cannot go on indefinitely, though the spiral arms are perpetuated in space. It will not be too much to say that the spirals are formed by a finite number of ejections of particles from the central relating body. Also, it will not be a far-fetched conclusion if we say that this explains the finite length of the spiral arms.

Also, from (11) we obtain

$$\left(\frac{\partial V_\lambda}{r_\lambda \partial \theta_\lambda}\right)_0 = 0,$$

because, for all values of λ and ν we have

$$(\theta_\nu-\theta_\lambda)_0 = 0.$$

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supposed to move with the central rotating mass and the spirals are not tracks of individual particles but are asymptotic with the paths of a swarm of ejected particles. It is only the second alternative which is capable of explaining the case of more than two spiral arms. Suppose, for instance, that there have been only two ejections, so that there are three possibilities: (i) there may be four distinct spiral arms; (ii) there may be three spiral arms, because two of the spiral arms issuing from the same antipodal point may coincide; (iii) there may be only two spiral arms as each pair of spirals issuing from the same antipodal point may be parallel or coincident. Similar reasoning may be applied to explain the nunber of spiral arms pcssible for more than two ejections. But, the theory of the motion of particles along more than two spiral arms is too complicated. So, we have considered here the comparatively simple case of two spiral arms only.

Again, from the same considerations, we get

$$\sum_{\nu}\left(\frac{\partial^2 V_\lambda}{\partial r_\nu \partial r_\lambda}\right)_0 a\rho_\nu = \frac{2M}{a^2}\rho_\lambda + \frac{2m}{a^2}\sum_{\nu}\rho_\nu,$$

$$\sum_{\nu}\left(\frac{\partial^2 V_\lambda}{\partial \theta_\nu \partial r_\lambda}\right)_0 \sigma_\nu = 0,$$

$$\sum_{\nu}{}' \left\{\frac{\partial}{\partial r_\nu}\left(\frac{1}{r_\lambda}\frac{\partial V_\lambda}{\partial \theta_\lambda}\right)\right\}_0 a\rho_\nu = 0,$$

$$\sum_{\nu}\left\{\frac{\partial}{\partial \theta_\nu}\left(\frac{1}{r_\lambda}\frac{\partial V_\lambda}{\partial \theta_\lambda}\right)\right\}_0 \sigma_\nu = \frac{mn}{a^2}\sigma_\lambda - \frac{m}{a^2}\sum_{\nu}\sigma_\nu .$$

Therefore, the equations for the disturbed motion of the particle λ along the spiral arm take the form:

$$\ddot{\rho}_\lambda - 2\omega\dot{\sigma}_\lambda = \left(\frac{2M}{a^3}+\omega^2\right)\rho_\lambda + \frac{2m}{a^3}\sum_{\lambda}\rho_\lambda, \quad \ldots \quad \ldots \quad (27)$$

$$\ddot{\sigma}_\lambda + 2\omega\dot{\rho}_\lambda = \frac{mn}{a^2}\sigma_\lambda - \frac{m}{a^2}\sum_{\lambda}\sigma_\lambda . \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

It is not possible to integrate those equations as they stand; so, we shall express them in a form which will facilitate their integration. Taking sum of both the sides for all values of λ we write the result as:

$$\sum_{\lambda=1}^{n}\left[\ddot{\rho}_\lambda - 2\omega\dot{\sigma}_\lambda - \left\{\frac{2(M+mn)}{a^3}+\omega^2\right\}\rho_\lambda\right] = 0,$$

and

$$\sum_{\lambda=1}^{n}\left(\ddot{\sigma}_\lambda + 2\omega\dot{\rho}_\lambda\right) = 0.$$

Therefore, we get instead of (27) and (28) the set of differential equations, viz.,

$$\ddot{\rho}_\lambda - 2\omega\dot{\sigma}_\lambda = \left\{\frac{2(M+mn)}{a^3}+\omega^2\right\}\rho_\lambda \quad \ldots \quad \ldots \quad (29)$$

and

$$\ddot{\sigma}_\lambda + 2\omega\dot{\rho}_\lambda = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

5. *Solutions of the set of differential equations for the motion along the spiral arms.*

Remembering the relation (18) we get the typical pair of the set of differential equations (29) and (30) as:

$$\ddot{\rho}_\lambda - 2\omega\dot{\sigma}_\lambda = (3\omega^2 - 2\mu)\rho_\lambda \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

and

$$\ddot{\sigma}_\lambda + 2\omega\dot{\rho}_\lambda = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

Putting $\omega'^2 = \omega^2+2\mu$, we get

$$\dot{\rho}_\lambda = a_\lambda e^{i\omega' t}+b_\lambda e^{-i\omega' t} \qquad \ldots \quad \ldots \quad \ldots \quad (33)$$

that is,

$$\rho_\lambda = c_\lambda+\frac{a_\lambda}{i\omega'}e^{i\omega' t}-\frac{b_\lambda}{i\omega'}e^{-i\omega' t} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

and
$$\sigma_\lambda = d_\lambda-\frac{3\omega^2-2\mu}{2\omega}c_\lambda t+\frac{2\omega}{\omega'^2}\left\{a_\lambda e^{i\omega' t}+b_\lambda e^{-i\omega' t}\right\} \quad \ldots \quad \ldots \quad (35)$$

Secular instability is indicated by the presence of the linear term in t present in the equation (35).

6. *Stability equations for motion along the spiral arms.*

As is usual in the theory of characteristic exponents, we make the following substitutions, viz.,

$$\rho_\lambda = a_\lambda e^{ip_\lambda t}, \quad \sigma_\lambda = b_\lambda e^{ip_\lambda t},$$

where $i = \sqrt{-1}$ and $ip_\lambda t$ is the characteristic exponent of the orbit. We get the pair of equations:

$$\left.\begin{aligned} a_\lambda p_\lambda^2+2i\omega b_\lambda p_\lambda &= -(3\omega^2-2\mu)a_\lambda, \\ b_\lambda p_\lambda^2-2i\omega a_\lambda p_\lambda &= 0. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (36)$$

Therefore,

$$p_\lambda^2 = \omega^2+2\mu = \omega'^2 \qquad \ldots \quad \ldots \quad \ldots \quad (37)$$

Here, p_λ is real if $\omega^2+2\mu > 0$ which is true. Hence the spiral orbits are stable. The constants a_λ and b_λ are connected by the relation

$$b_\lambda = \pm\frac{2i\omega}{\omega'}a_\lambda, \quad p_\lambda = \pm\omega'.$$

Initial instability:—It has been shown by Jeans and others that a rotating spherical mass of gas is unstable for certain configurations. If we put $n = 0$ and drop the suffix "λ" we get

$$p^2 = 3\omega^2-\frac{2M}{a^3}.$$

For unstable configurations p must be imaginary. Therefore, we must have

$$\omega^2 < \frac{2M}{3a^3};$$

or,
$$\frac{\omega^2}{2\pi\rho_0} < \frac{4bc}{9a^2}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

in the usual notation. So that, before the matter is ejected from the central rotating body the following inequality must be satisfied, viz.,

$$\frac{\omega^2}{2\pi\rho} > \frac{\omega^2}{2\pi\rho_0}, \text{ i.e., } \rho_0 > \rho \qquad \ldots \quad \ldots \quad \ldots \quad (39)$$

where ρ is the mean critical density.

7. *Motion of the c.g. of the system of particles moving along the spiral arms.*

If $(\xi, \eta, 0)$ be the co-ordinates of the c.g. of the system of particles moving along the spiral arms, ξ, η can be expressed as:

$$\left.\begin{aligned} \xi &= \frac{a}{n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)\cos(\omega t+\epsilon+\sigma_\lambda) \\ \eta &= \frac{a}{n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)\sin(\omega t+\epsilon+\sigma_\lambda) \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (40)$$

The equation of the spiral is given by

$$(1+\rho_\lambda) = e^{\theta_\lambda \cot\alpha}, \quad \theta_\lambda = \omega t+\epsilon+\sigma_\lambda, \quad (\lambda = 1, 2, \ldots, n) \qquad (41)$$

where α is the angle of the spiral. From the pair of equations (40) we get

$$\left.\begin{aligned} e^{-i\cot\alpha}(\xi+i\eta) &= \frac{a}{n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)^2, \\ e^{i\cot\alpha}(\xi-i\eta) &= \frac{a}{n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)^2. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (42)$$

Therefore,

$$\frac{\eta}{\xi} = \tan(\cot\alpha) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (43)$$

Hence, the c.g. of the system of particles moves along a straight line making an angle $(\cot\alpha)$ with the x-axis. Now, the co-ordinates of the c.g. of the system of particles can be re-expressed as:

$$\left.\begin{aligned} \xi &= \frac{a\cos(\cot\alpha)}{n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)^2, \\ \eta &= \frac{a\sin(\cot\alpha)}{n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)^2. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (44)$$

The motion of the c.g. of the system of particles moving along the spiral arms is given by the pair of equations

$$\left.\begin{aligned}\ddot{\xi} &= -\frac{M+2mn}{a^2n}\sum_{\lambda=1}^{n}\frac{\cos(\omega t+\epsilon+\sigma_\lambda)}{(1+\rho_\lambda)^2}-\frac{\mu a}{n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)\cos(\omega t+\epsilon+\sigma_\lambda),\\ \ddot{\eta} &= -\frac{M+2mn}{a^2n}\sum_{\lambda=1}^{n}\frac{\sin(\omega t+\epsilon+\sigma_\lambda)}{(1+\rho_\lambda)^2}-\frac{\mu a}{n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)\sin(\omega t+\epsilon+\sigma_\lambda).\end{aligned}\right\} \quad (45)$$

As before, this can be expressed as:

$$\left.\begin{aligned}\ddot{\xi} &= -\left[\frac{M+2mn}{a^2n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)^{-1}+\frac{\mu a}{n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)^2\right]\cos(\cot\alpha),\\ \ddot{\eta} &= -\left[\frac{M+2mn}{a^2n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)^{-1}+\frac{\mu a}{n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)^2\right]\sin(\cot\alpha).\end{aligned}\right\} \quad \ldots \quad (46)$$

or,

$$\left.\begin{aligned}\ddot{\xi}+\mu\xi &= -\frac{M+2mn}{a^2n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)^{-1}\cos(\cot\alpha),\\ \ddot{\eta}+\mu\eta &= -\frac{M+2mn}{a^2n}\sum_{\lambda=1}^{n}(1+\rho_\lambda)^{-1}\sin(\cot\alpha).\end{aligned}\right\} \quad \ldots \quad \ldots \quad (47)$$

8. *Reasons for finite length of the spiral arms.*

r_λ being the radius vector of the particle λ moving along the spiral arm, r_λ is not finite for the farthest particle moving along the spiral arm, if the spirals extend to infinite distance. Now, we have the usual substitution, viz.,

$$r_\lambda = a(1+\rho_\lambda).$$

Therefore, $r_\lambda \to \infty$ as $\rho_\lambda \to \infty$, if only a is finite which is a certainty. (34) gives the expression for ρ_λ, viz.,

$$\begin{aligned}\rho_\lambda &= c_\lambda+\frac{a_\lambda}{i\omega'}e^{i\omega' t}-\frac{b_\lambda}{i\omega'}e^{-i\omega' t}\\ &= c_\lambda+\frac{a_\lambda}{ip_\lambda}\left[e^{ip_\lambda t}\pm\frac{2i\omega}{p_\lambda}e^{-ip_\lambda t}\right].\end{aligned}$$

c_λ, a_λ, p_λ being finite, and, p_λ and ω are small considering astronomical distances, $\rho_\lambda \to \infty$ only when $t \to \infty$. The present age of the universe is of the order of 10^9 or 10^{10} years, which is finite. It is quite improbable that c_λ and a_λ, the constants of integration, should be infinite. Therefore, the spiral arms have finite length for t finite. Moreover, as t increases, p_λ and ω decrease. Also, it is highly probable, that,

$$p_\lambda, \quad \omega \to 0 \quad \text{as} \quad t \to \infty.$$

Hence, even for t infinite the spiral arms have a finite length.

9. *Escape of matter to infinity.*

It is quite probable that the velocity of a particle (say, particle α) moving along the spiral arm of the nebula may increase beyond the finite bounds, so that

$$\dot{\rho}_\alpha \to \infty, \text{ as, either } a_\alpha \text{ or } b_\alpha \to \infty.$$

In that case, ρ_a also tends to infinity. This extreme case, if it is a probability, accounts for the escape of matter either to infinity, or, beyond the limit of gravitational influence of the central rotating ellipsoid.

PART II.—PERTURBED MOTION OF PARTICLES ALONG THE SPIRAL ARMS.

(10) *The equations of perturbed motion of the system of particles moving along a spiral arm after N ejections.*

At the outset, we must analyze the main factors which effect this perturbed motion. It is quite obvious that there are three influences, namely, (1) the perturbations due to tidal effects determined by the tidal potential $V_T = AR_\lambda^2 \cos 2\psi_\lambda$, where A is a constant, R_λ is the distance of the particle λ from the c.g. of the distant massive bodies, and, ψ_λ is the angle between the preferential direction (the direction of the tidal force) and the x-axis. It should be remembered that, so far as r_λ, θ_λ are concerned, V_T is almost a constant. The only disturbing effect due to V_T must, therefore, be of the nature of distortion of the plane of the spiral orbits from that of the lenticular periphery. (2) Perturbations due to the attraction of the central body, and, (3) perturbations due to the attractive force of the other spiral arm. (2) and (3) have, so to say, a combined effect on the motion of particles along a spiral arm. Let us assume that there have been N ejections till the epoch t, so that, the mass of the central rotating ellipsoid is $M' = \left(M - 2m \sum_{r=1}^{N} n_r\right)$, and, the mass of a spiral arm is $m \sum_{r=1}^{N} n_r$. Hence, the total effective mass is

$$m' = M - m \sum_{r=1}^{N} n_r.$$

M', $m \sum_{r=1}^{N} n_r$ and m, the mass of particle λ, form a mutually attracting system. This case, therefore, falls under the well-known three-body problem. We are, however, concerned here with the perturbed motion of particle λ. As an approximation, we shall neglect the size of the central body and consider its mass M' to be concentrated at the centre O. Let the c.g. of the spiral arm be S, and C be a point on the straight line OS such that

$$\frac{OC}{CS} = \frac{M'}{m \sum_{r=1}^{N} n_r}.$$

Then C is the c.g. of M' and $m \sum_{r=1}^{N} n_r$. Therefore, the effective mass m' acts through C. Let r'_λ be the distance of the particle λ from C, and let the angle between r'_λ and the x-axis be θ'_λ. Then, as before, we can write

$$r'_\lambda = a(1+\rho'_\lambda), \quad \theta'_\lambda = (\theta'_\lambda)_0 + \sigma'_\lambda = \omega t + \epsilon' + \sigma'_\lambda, \quad z'_\lambda = 0.$$

The disturbing potential is, then, written as:

$$V'_\lambda = \frac{m'}{\left\{ r'^2_\lambda - 2r'_\lambda r_\lambda \cos(\theta'_\lambda - \theta_\lambda) + r^2_\lambda \right\}^{\frac{1}{2}}} - \frac{m' r_\lambda \cos(\theta'_\lambda - \theta_\lambda)}{r'^2_\lambda} + AR^2_\lambda \cos 2\psi_\lambda \quad \ldots \quad (48)$$

The resulting gravitational potential Ω_λ for the general case can be expressed by the formula, viz.,

$$\begin{aligned}\Omega_\lambda &= V_\lambda + V'_\lambda \\ &= \frac{M+m}{r_\lambda} - \tfrac{1}{2}\mu r^2_\lambda + \sum{}' m \left[\frac{1}{\left\{ r^2_\lambda - 2r_\lambda r_\nu \cos(\theta_\lambda - \theta_\nu) + r^2_\nu \right\}^{\frac{1}{2}}} \right. \\ &\quad \left. - \frac{r_\lambda \cos(\theta_\lambda - \theta_\nu)}{r^2_\nu} \right] + \cdots + \frac{m'}{\left\{ r'^2_\lambda - 2r'_\lambda r_\lambda \cos(\theta'_\lambda - \theta_\lambda) + r^2_\lambda \right\}^{\frac{1}{2}}} \\ &\quad - \frac{m' r_\lambda \cos(\theta'_\lambda - \theta_\lambda)}{r'^2_\lambda} + AR^2_\lambda \cos 2\psi_\lambda \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (49)\end{aligned}$$

Let us write

$$\frac{r_\lambda}{r'_\lambda} = \xi_\lambda, \quad (r'_\lambda)_0 = a, \quad (\theta'_\lambda)_0 = \omega t + \epsilon',$$

where $\epsilon' = \pi + \epsilon$. We have, therefore, the following initial relations, viz.,

$$\left(\frac{r_\lambda}{r'_\lambda}\right)_0 = (\xi_\lambda)_0 = 1, \quad (\eta_\lambda)_0 = (\theta'_\lambda - \theta_\lambda)_0 = \pi.$$

The following expansion is taken for granted, viz.,

$$\frac{1}{\left\{1-2\xi_\lambda \cos \eta_\lambda + \xi_\lambda^2\right\}^{\frac{1}{2}}} = A_0 + A_1 \cos \eta_\lambda + A_2 \cos 2\eta_\lambda + \ldots.$$

It should be remembered that the $A's$ are functions of ξ_λ where $0 < \xi_\lambda < 1$, and the series can be differentiated term by term with respect to ξ_λ and η_λ. The equations of the perturbed motion of particle λ moving along a spiral arm can, in general, be expressed as:

$$\left.\begin{aligned} \ddot{\rho}_\lambda - 2\omega\dot{\sigma}_\lambda &= \omega^2 r_\lambda + \frac{1}{a}\left(\frac{\partial \Omega_\lambda}{\partial r_\lambda}\right)_0 + \sum_\nu \rho_\nu \left(\frac{\partial^2 \Omega_\lambda}{\partial r_\nu\, \partial r_\lambda}\right)_0 \\ &\qquad + \frac{1}{a}\sum \sigma_\nu \left(\frac{\partial^2 \Omega_\lambda}{\partial \theta_\nu\, \partial r_\lambda}\right)_0, \\ \ddot{\sigma}_\lambda + 2\omega\dot{\rho}_\lambda &= \frac{1}{a}\left(\frac{1}{r_\lambda}\frac{\partial \Omega_\lambda}{\partial \theta_\lambda}\right)_0 + \sum_\nu \rho_\nu \left\{\frac{\partial}{\partial r_\nu}\left(\frac{1}{r_\lambda}\frac{\partial \Omega_\lambda}{\partial \theta_\lambda}\right)\right\}_0 \\ &\qquad + \frac{1}{a}\sum \sigma_\nu \left\{\frac{\partial}{\partial \theta_\nu}\left(\frac{1}{r_\lambda}\frac{\partial \Omega_\lambda}{\partial \theta_\lambda}\right)\right\}_0. \end{aligned}\right\} \quad \ldots \quad (50)$$

Carrying out all the differentiations and reductions, equations (50) take the form, viz.,

$$\left.\begin{aligned} \ddot{\rho}_\lambda - 2\omega\dot{\sigma}_\lambda &= \left(\frac{2M}{a^3} + \omega^2\right)\rho_\lambda + \frac{2m}{a^3}\sum_\lambda \rho_\lambda + \frac{m'}{a^3} \\ &\quad + \frac{m'}{a^3}\left[\frac{\partial}{\partial \xi_\lambda}\{A_0 - A_1 + A_2 - A_3 + \ldots + (-1)^p A_p + \ldots\}\right]_{\xi_\lambda = 1} \\ &\quad + \frac{m'}{a^3}\left[\frac{\partial^2}{\partial \xi_\lambda^2}\{A_0 - A_1 + A_2 - A_3 + \ldots + (-1)^p A_p + \ldots\}\right]_{\xi_\lambda = 1} \rho_\lambda \\ \ddot{\sigma}_\lambda + 2\omega\dot{\rho}_\lambda &= \left[\frac{mn}{a^2} - \frac{m'}{a^3}\{1 - A_1 + 4A_2 - 9A_3 + \ldots \right. \\ &\qquad \left. + (-1)^p p^2 A_p + \ldots\}_{\xi_\lambda = 1}\right]\sigma_\lambda - \frac{m}{a^2}\sum_\lambda \sigma_\lambda. \end{aligned}\right\} \quad (51)$$

We make the following substitutions:

$$\left[\frac{\partial}{\partial \xi_\lambda}\{A_0 - A_1 + A_2 - A_3 + \ldots + (-1)^p A_p + \ldots\}\right]_{\xi_\lambda = 1} = \text{a constant},$$

$$= P', \text{ say.}$$

Similarly,

$$\left[\frac{\partial^2}{\partial \xi_\lambda^2}\{A_0-A_1+A_2-A_3+\ldots+(-1)^p A_p+\ldots\}\right]_{\xi_\lambda=1} = Q,$$

and $\quad \{1-A_1+4A_2-9A_3+\ldots+(-1)^p p^2 A_p+\ldots\}_{\xi_\lambda=1} = R.$

Then, equations (51) can be written as:

$$\left.\begin{aligned} \ddot{\rho}_\lambda-2\omega\dot{\sigma}_\lambda &= \left(\frac{2M+m'Q}{a^3}+\omega^2\right)\rho_\lambda+\frac{2m}{a^3}\sum_\lambda \rho_\lambda+\frac{m'P}{a^3}, \\ \ddot{\sigma}_\lambda+2\omega\dot{\rho}_\lambda &= \left(\frac{mn}{a^2}-\frac{m'R}{a^3}\right)\sigma_\lambda-\frac{m}{a^2}\sum_\lambda \sigma_\lambda, \end{aligned}\right\} \quad \ldots \quad (52)$$

where $1+P' = P$.

These equations involve terms, apart from ρ_λ and σ_λ, which depend upon the time t, and, they are also non-linear in character. It is not known whether solutions (52) can be found in terms of functions which will be of any use for the problem in hand. It will be seen that when $m' = 0$, (52) reduce to equations (27) and (28).

The effect of the attractive force of m' on the spiral arm as a whole is to bring it closer to the central rotating mass, so that, the spiral arms do not necessarily form equiangular spirals. This is evident in the photographs of the spiral nebulae, an instance of which is afforded by the great nebula M 31 (N.G.C. 224) in Andromeda.

III. Conclusion and Summary.

It will not be out of place if we give by way of conclusion a very short account of some well-known theories on the formation of the spiral arms, especially those given by Jeans, Brown, Lindblad, Vogt, Lambrecht, Wellman, Jehle, Narlikar and Moghe, and Banerji, Nizamuddin and Bhatnagar.

Jeans's considerations for the spiral formation centres round the critical equation for the ejection of matter along spiral arms, viz.,

$$\frac{\omega^2}{2\pi\gamma\rho_0} = 0{\cdot}36.$$

If ρ is the mean density of matter within the ellipsoid of revolution and ρ_0 is the mean critical density, then $\rho_0 > \rho$, and ejection of matter is effected from the two antipodal points of the lenticular ellipsoid. Jeans assumes a continuous distribution of matter inside this ellipsoid which does not agree with observational evidence.

According to Brown the law of force which leads to the formation of the spiral arms is given by

$$f(r) = \frac{M}{r^2}+\mu r.$$

But his hypothesis about a superposed periodic density on the uniform density of the disc-shaped ellipsoid of revolution and his assumption about periodic encounters to explain the spiral structure seem to be more artificial than real.[10]

Lindblad's work on this problem assumes that the central body from which spirals are formed consists of a uniform distribution of stars with a condensation at the centre. The equation which governs the formation of spiral arms is:

$$\left(\frac{du}{d\theta}\right)^2 = \phi(u)+\text{constant},$$

with (i) $\phi''(u_0) > 0$, or, (ii) the nucleus has a very low mass compared to the total mass. The important points in his investigations are: (1) A disc-shaped ellipsoidal form for the central condensation is necessary for the spiral structure. (2) The antipodal points (points of ejection) are stationary in space. (3) It is only in the case of thin spiral arms that they represent the orbits of an outgoing star.[11]

Vogt and Lambrecht, independently, put forward the hypothesis that the nebular nucleus is highly concentrated and that Newtonian law of attraction holds good in addition to the force of cosmic repulsion which they suppose is directly proportional to the distance from the centre of the nucleus. A more general case has been considered by the present author in a paper on the relativistic theory of a spiral nebula, to be published shortly, in which the differential equation for motion along the spiral arms gives equiangular spirals with $\cot\alpha \geqslant 1$. In this case, cosmic repulsion has an additional effect in the form of secular distortion of the spiral orbits. That the spiral arms are two in number is not evident from Vogt's theory, while Lambrecht's theory gives that the spirals result from the encounters, and that they are not the orbits of outgoing particles but they are their loci at a particular instant.[12]

Wellman assumes that the spirals are a result of a slow expansion of the elliptical orbits which gives a sort of uniform outward motion of particles from the end points on the inner and outer elliptical orbits.[13]

Jehle deduces spiral structure with the aid of generalized wave-mechanics.[14]

Narlikar and Moghe have obtained spirals for the two-dimensional orbits in an expanding universe.[15]

Banerji, Nizamuddin and Bhatnagar have worked out the theory of spiral formation on the basis of recent investigations of Plaskett and Pearce about our galaxy. They have taken account of the effect of ionization on the motion along spiral orbits.[16] More recently Prof. A. C. Banerji has thoroughly investigated the condition for the formation of the spiral arms.[17]

In this paper, we have considered the problem of motion of particles along spiral arms, on purely dynamical principles. While considering the condition for the ejection of matter along the spiral arms we have given a theory of successive ejections of particles and shown that the number of ejections can only be finite. This is taken almost as an explanation for the finite length of the spiral arms. There is, however, a possibility of escape of matter either to infinity or beyond the gravitational influence of the central rotating ellipsoid. The spiral orbits are statically stable and dynamical instability has secular effects only. It is shown that the c.g. of the system of particles moving along the spiral arms has a motion along a straight line inclined at an angle (cot α) with the x-axis as the successive ejections go on. In part II, perturbed motion along the spiral arms due to extra-nebular disturbance is considered.

I wish to record my sincere thanks to Principal Dr. G. S. Mahajani for his keen interest in my work.

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January 14, 1940.

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