ON THE THEORY OF A SPIRAL NEBULA. II.

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PART I. RELATIVISTIC THEORY.

1. *Introduction.*

In a former communication the author has given a dynamical theory of the formation of spiral arms of a nebula.[1] There the law of force assumed was

$$f(r) = \frac{M}{r^2} + \mu r \qquad \ldots \qquad (1)$$

in the usual notation, and the potential turns out to be of the form, viz.,

$$V = -\frac{M}{r} + \frac{1}{2}\mu r^2 + S_2. \qquad \ldots \qquad (2)$$

The usual geodesic equations of general relativity for the motion of particles in extra-galactic nebulae suggest spiral tracks. The (u, θ) equation for such type of motion takes the form:

$$\frac{d^2u}{d\theta^2} + u = \frac{M}{h^2} + 3Mu^2 - \frac{\beta}{h^2u^3} \qquad \ldots \qquad (3)$$

where $r = u^{-1}$, $\beta = \frac{1}{3}(2\pi\rho_0 + \lambda)$ and $h = r^2\dot{\theta}$, the well-known areal constant. Integrating this differential equation we get

$$\left(\frac{du}{d\theta}\right)^2 = \phi(u) + \text{constant} \qquad \ldots \qquad (4)$$

where
$$\phi(u) = \frac{2Mu}{h^2} - u^2 + \frac{\beta}{h^2u^2} + 2Mu^3.$$

Lindblad's condition for the formation of the spiral arms is given by $\phi''(u_0) > 0$, which in this case happens to be [2]

$$12Mu_0 + \frac{6\beta}{h^2u_0^4} - 2 > 0. \qquad \ldots \qquad (5)$$

That this inequality holds good can be shown by recourse to numerical substitutions. We have

$$\dot{u}_0 = r_0^{-1} = 2\times10^{-23}\,\text{cm}^{-1}, \quad \beta = 2{\cdot}06\times10^{-54}\,\text{cm}^{-2},$$
$$h^2 = 2{\cdot}125\times10^{37}\,\text{cm}^2, \text{ and } M = 3\times10^{15}\,\text{cm}.$$

L.H.S. of (5) $= 7{\cdot}2\times10^{-7} - 2 + 3{\cdot}6353$
$= 7{\cdot}2\times10^{-7} + 1{\cdot}6353,$

which is positive. Here, the numerical values are in gravitational units.

So, if the theory of the formation of the spiral arms of a nebula is to fall in the domain of rational mechanics, it seems necessary to modify the classical Newtonian law of gravitation in conformity with the general theory of relativity. Hence, from the point of view of classical dynamics equation (3) is obtained when the gravitational potential is of the form

$$V = -\frac{M}{r}+\frac{1}{2}\beta r^2-\frac{Mh^2}{r^3}. \quad \ldots \quad (6)$$

Considering initial values we have

$$V_0 = -6\times 10^{-9}+2{\cdot}575\times 10^{-9}-1{\cdot}2\times 10^{-15} \quad \ldots \quad (7)$$

this shows that the third quantity on the r.h.s. of (7) can be neglected when compared to the first two quantities. Now, this quantity represents the term $3Mu^2$ on the r.h.s. of (3) which goes to point out that it can safely be omitted. Consequently, the term $-\frac{Mh^2}{r^3}$ in (6) can be omitted without any substantial loss to the theory. This means that the gravitational potential can simply be written as

$$V = -\frac{M}{r}+\frac{1}{2}\beta r^2 \quad \ldots \quad (8)$$

which was the main assumption in the former paper referred to above with the exception that β is replaced there by μ which is another constant. We shall now proceed to find the solution of (3) and show that it corresponds to an equiangular spiral.

2. *Solution of the orbital equation.*

Let u be expressed by the power series

$$u = \frac{1}{a}(1+b_1\theta+b_2\theta^2+\ldots). \quad \ldots \quad (9)$$

On substitution in (3) we get

$$\left.\begin{aligned}
2b_2 &= \frac{Ma}{h^2}+\frac{3M}{a}-\frac{\beta a^4}{h^2}-1,\\
6b_3 &= \left(\frac{6M}{a}+\frac{3\beta a^4}{h^2}-1\right)b_1,\\
12b_4 &= \left(\frac{6M}{a}+\frac{3\beta a^4}{h^2}-1\right)b_2+\left(\frac{3M}{a}-\frac{6\beta a^4}{h^2}\right)b_1^2,\\
20b_5 &= \left(\frac{6M}{a}+\frac{3\beta a^4}{h^2}-1\right)b_3+\left(\frac{6M}{a}-\frac{12\beta a^4}{h^2}\right)b_1b_2+\frac{10\beta a^4}{h^2}b_1^3, \text{ etc.}
\end{aligned}\right\} \quad \ldots \quad (10)$$

If this solution gives an equiangular spiral of the form

$$u = \frac{1}{a}e^{-\theta\cot\alpha} \quad \ldots \quad (11)$$

we must put $b_1 = \cot\alpha$ which gives $2b_2 = \cot^2\alpha$. But $2b_2 = 1{\cdot}7798+1{\cdot}8\times 10^{-7}$, and so $\cot\alpha = 1{\cdot}3341$. (9) and (11) tally up to the term containing θ^2 if we put cot α with a negative sign. Hence $\alpha = -36^\circ\, 52'$. In these considerations the term $3Mu^2$ on the r.h.s. of (3) is of no consequence. It can, therefore, be safely omitted.

It is explicitly assumed here that the spiral nebula is formed of a central mass-condensation and an equatorial layer of matter composed of particles some of which are of size comparable to that of a star. This outer layer consists of two spiral arms extending outwards up to a region beyond which a large number of condensations are essentially formed which compose the starry region of the further outer portion of the nebular matter. It is a plausible hypothesis that the spiral arms are the orbits of matter ejected by the immediately surrounding region of the central condensation. The relativistic theory of the formation of the spiral arms is, however, not capable of determining the number of spiral arms possible. This theory being of a very general character the number of spiral arms is of no consequence from the most general point of view. This being, more or less, a local property of the spiral nebulae, it can be determined by having recourse to classical dynamical theory or its modification.

It is tentatively assumed for the purpose of the considerations set fourth in the following pages that the boundary of the central condensation is represented by the Schwarzschild incompressible fluid sphere $r = a$. The matter beyond $r = a$ moves along spiral tracks which is a natural consequence of the geodesic motion according to the general theory of relativity. It is the purpose of the following sections to study the stability of the spiral motion. Presently, some preliminary formulae are obtained.

3. *Geodesics and the spiral tracks.*

Taking into account the influence of cosmic repulsion on the motion of the system of particles in the outlying layers the fluid beyond $r = a$ is represented by the line-element

$$ds^2 = -e^{\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + e^{\nu}dt^2 \qquad \ldots \quad \ldots \quad (12)$$

where $$e^{-\lambda} = 1 - \frac{2m}{r} - \beta r^2 = e^{\nu}, \quad \beta = \frac{2\pi\rho_0 + \Lambda}{3}, \quad \rho_0 = T. \qquad \ldots \quad (13)$$

The term in r^2 is due to cosmic repulsion. The orbits are given by the geodesic equations:

$$\frac{d^2x^{\alpha}}{ds^2} + \{\mu^{\nu}, \alpha\} \frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = 0, \; (\alpha, \mu, \nu = 1, 2, 3, 4). \qquad \ldots \quad (14)$$

Changing the independent variable from s to t, (14) can be re-written in the form

$$\frac{d^2x^{\alpha}}{dt^2} + \{\mu\nu, \alpha\} \frac{dx^{\mu}}{dt^2}\frac{dx^{\nu}}{dt^2} - \{\mu\nu, 4\} \frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt}\frac{dx^{\alpha}}{dt} = 0, \; (\alpha = 1, 2, 3). \quad \ldots \quad (15)$$

Putting $\theta = \frac{\pi}{2}, \frac{d\theta}{dt} = 0$ equations (14) for $\alpha = 1$ and $\alpha = 3$ yield with the help of (13) the relation

$$\frac{d^2r}{d\phi^2}+\left(\frac{1}{2}\lambda'-\frac{2}{r}\right)\left(\frac{dr}{d\phi}\right)^2-e^{-\lambda}\left(r-\frac{1}{2}e^{-\lambda}\lambda'\omega^{-2}\right)=0 \quad \ldots \quad (16)$$

where $\omega=\frac{d\phi}{dt}$. This is the differential equation for spiral orbits.[3]

4. *Conditions of stability.*

It has already been shown elsewhere [4] that the condition that every point of the space represented by (12) is a point of equilibrium if

$$\{44,\alpha\}=0, \quad (\alpha=1,2,3). \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

In the present case

$$\{44,1\}=\frac{1}{2}e^{\nu-\lambda}\nu'\neq 0, \text{ and } \{44,2\}=0=\{44,3\}. \quad \ldots \quad (18)$$

(18) shows that every point of space beyond $r=a$ cannot be a position of equilibrium. But it may be possible to find out some points $(r_\alpha, \theta_\alpha, \phi_\alpha)$ which are positions of equilibrium. Let one such point be (r_0, θ_0, ϕ_0). If a neighbouring point be denoted by $(r_0+\xi, \theta_0+\eta, \phi_0+\zeta)$, remembering that $\left(\frac{dr}{dt}\right)_{r=r_0}=0$, $\left(\frac{d^2r}{dt^2}\right)_{r=r_0}=0$, equations for small motion turn out to be:

$$\left.\begin{aligned}&\frac{d^2\xi}{dt^2}+\frac{d}{dt}[\lambda]\frac{d\xi}{dt}=0,\\ &\frac{d^2\eta}{dt^2}+\frac{d}{dt}[2\log r]\frac{d\eta}{dt}=0,\\ \text{and}\quad &\frac{d^2\zeta}{dt^2}+\frac{d}{dt}[2\log r+2\log\sin\theta]\frac{d\zeta}{dt}=0.\end{aligned}\right\} \quad \ldots \quad \ldots \quad (19)$$

It has been shown by Prof. Narlikar [5] that for stability it is necessary and sufficient that

$$\frac{d\lambda}{dt},\ \frac{d}{dt}[2\log r],\ \frac{d}{dt}[2\log r+2\log\sin\theta] \quad \text{be positive;}$$

that is,
$$\frac{2\left(\frac{m}{r^2}-\beta r\right)\dot r}{\left(1-\frac{2m}{r}\beta r^2\right)}>0,\ \frac{2\dot r}{r}>0,\ 2\left(\frac{\dot r}{r}+\cot\theta.\dot\theta\right)>0. \quad \ldots \quad \ldots \quad (20)$$

The last two of the inequalities (20) give

$$\dot r>0,\quad \dot r+r\cot\theta.\dot\theta>0. \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

The first inequality (20) then leads to

$$\frac{m}{r^2}-\beta r>0,\quad 1-\frac{2m}{r}-\beta r^2>0;$$

i.e.,
$$r>(3\beta)^{-\frac{1}{3}} \text{ and } r>3m. \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

Therefore, for the stability of the orbits r should lie within the range $(3m, (3\beta)^{-\frac{1}{3}})$. This range will be called hereafter the 'region of stability'.

5. *The characteristic exponents of the orbits.*

It can easily be shown that if the kinetic energy of a dynamical system be expressed in the form

$$T = \frac{1}{2} g_{\mu\nu} v^{\mu} v^{\nu}, \quad v^{\alpha} = \frac{dx^{\alpha}}{ds}, \qquad \text{(23)}$$

then the geodesics can be expressed as Lagrange equations of motion. Applying this method to the particular case under consideration the kinetic energy is given by

$$T = \frac{1}{2}\left[e^{\nu} t'^2 - e^{\nu} r'^2 - r^2 \phi'^2\right] \qquad \text{(24)}$$

where a dash denotes a differentiation with regard to s and $\theta = \frac{\pi}{2}$, $\frac{d\theta}{dt} = 0$. The Lagrange equations of motion take the form

$$\left.\begin{aligned} \frac{d}{ds}\left(\frac{\partial T}{\partial r'}\right) - \frac{\partial T}{\partial r} = 0, \\ \frac{d}{ds}\left(\frac{\partial T^{*}}{\partial \phi'}\right) - \frac{\partial T}{\partial \phi} = 0, \\ \frac{d}{ds}\left(\frac{\partial T}{\partial t'}\right) - \frac{\partial T}{\partial t} = 0. \end{aligned}\right\} \qquad \text{(25)}$$

The first integrals of the last two differential equations are respectively given by

$$r^2 \phi' = h, \qquad \text{(26)}$$

and

$$e^{\nu} t' = \alpha h, \qquad \text{(27)}$$

where h is the well-known areal constant and α is another arbitrary constant. Writing the line-element (12) in terms of differential coefficients and changing the independent variable from s to t we obtain with the help of (26) and (27)

$$e^{\lambda} \frac{d^2 r}{dt^2} - 3e^{2\lambda}\left(\frac{m}{r^2} - \beta r\right)\left(\frac{dr}{dt}\right)^2 + \left(\frac{m}{r^2} - \beta r\right) - r\left(\frac{d\phi}{dt}\right)^2 = 0. \qquad \text{(28)}$$

Also from (26) and (27) we get

$$r^2 e^{\lambda} \frac{d\phi}{dt} = \frac{1}{\alpha}. \qquad \text{(29)}$$

Differentiating (29) we have

$$r^2 \frac{d^2\phi}{dt^2} + 2r\left(1 - \frac{3m}{r}\right) e^{\lambda} \frac{dr}{dt} \frac{d\phi}{dt} = 0. \qquad \text{(30)}$$

For a circular orbit we have

$$\left(\frac{dr}{dt}\right)_{r=r_0} = 0, \quad \left(\frac{d^2 r}{dt^2}\right)_{r=r_0} = 0. \qquad \text{(31)}$$

Therefore, (28) gives us

$$r_0 \dot{\phi}_0^2 = \frac{m}{r_0^2} - \beta r_0. \qquad \text{(32)}$$

To determine the characteristic exponents of the orbit let us put

$$r = r_0+\xi, \quad \phi = \phi_0+\eta, \qquad \ldots \quad \ldots \quad \ldots \quad (33)$$

where ξ and η are very small quantities. Substituting these values in (28) and remembering the conditions (31) we get

$$\frac{d^2\xi}{dt^2}+\frac{1}{r_0^2}\left[\left(\frac{6m^2}{r_0^2}-\frac{2m}{r_0}\right)+(3\beta^2r_0^4-\beta r_0^2)\right]\xi = 0. \quad \ldots \quad \ldots \quad (34)$$

Therefore, the characteristic exponents are given by

$$\pm\frac{1}{r_0}\left[\left(\frac{2m}{r_0}-\frac{6m^2}{r_0^2}\right)+(\beta r_0^2-3\beta^2r_0^4)\right]^{\frac{1}{2}}. \quad \ldots \quad \ldots \quad (35)$$

It is a well-known result in the theory of orbits that if the characteristic exponents are imaginary the motion is stable. The condition that (35) gives imaginary values is

$$2m(r_0-3m) < \beta r_0^4(3\beta r_0^2-1). \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

(36) gives the most general condition of stability of the spiral orbits and it also determines the so-called 'region of stability'. The relations (22) are included in this general condition.

6. *Some numerical results.*

A numerical estimate as to the upper and lower limit of the possible values of r would seem worth while as it would give an exact idea as to the 'region of stability'. These results are supposed to be valid so far as the orders of magnitude are concerned. We have

m = mass of the nebula, r = range of stability,

$\odot$ = mass of the sun = $1{\cdot}5\times10^5$ cm.

(*a*) $r > 3m$:—

TABLE I.

m	r
$2\times10^9\odot$	9×10^{14} cm.
$2\times10^{10}\odot$	9×10^{15} cm.
$2\times10^{11}\odot$	9×10^{16} cm.

Table I gives the lower limit for r with an uncertainty proportional to the same in the determination of the mass of a spiral nebula.

(*b*) $r < (3\beta)^{-\frac{1}{2}}$:—

TABLE II.

β	r
10^{-54} cm^{-2}	$5{\cdot}77\times10^{26}$ cm.
10^{-55} cm^{-2}	$1{\cdot}8\times10^{27}$ cm.

Table II gives the upper limit for the range r.

The maximum range for elliptic and quasi-elliptic orbits lies between 10^{22} cm. to 10^{24} cm.[6] (order of magnitude only) which lies well within the region of stability

PART II. SOME NEW DYNAMICAL RESULTS WHICH FACILITATE NUMERICAL ESTIMATION OF THE FEATURES OF A SPIRAL NEBULA.

7. *Solution of the equations of motion in terms of a single variable.*

Having considered the theory of the formation of the spiral arms from the dynamical as well as the relativistic points of view, it is our purpose in the following pages to deduce some new formulae which will give with facility numerical estimates of some of the elements of a spiral nebula. For this purpose we make a plausible assumption that ρ_λ and σ_λ (which occur in equations (27) and (28) of the former paper [7]) can be expressed in terms of a single variable as

$$\rho_\lambda = l^\lambda \text{ and } \sigma_\lambda = k^\lambda \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

where l and k are constants to be determined later and λ is a function of time only. (27) and (28) of I are then written as

$$L^2 l^\lambda \dot{\lambda}^2 + L l^\lambda \ddot{\lambda} - 2\omega K k^\lambda \dot{\lambda} = \left[\frac{2(M-m)}{a^3} + \omega^2\right] l^\lambda + \frac{2m}{a^3} \frac{l(l^n-1)}{l-1} \quad \ldots \quad (38)$$

$$K^2 k^\lambda \dot{\lambda}^2 + K k^\lambda \ddot{\lambda} + 2\omega L l^\lambda \dot{\lambda} = \frac{m(n+1)}{a^2} k^\lambda - \frac{m}{a^2} \frac{k(k^n-1)}{(k-1)} \quad \ldots \quad \ldots \quad (39)$$

where $L = \log l$ and $K = \log k$. (38) and (39) would determine λ uniquely only if

$$k^\lambda = \frac{1}{a} l^\lambda \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

in which case we get a single differential equation of the first order, viz.,

$$2\omega\left(a+\frac{1}{a}\right)\frac{d}{dt}(l^\lambda) = \left[\frac{m(n+1)}{a^2} - \frac{3M+m(n-2)}{a^3} - \beta\right] l^\lambda - \left(\frac{m}{a^2}+\frac{2m}{a^3}\right)\frac{l(l^n-1)}{l-1} \quad \ldots \quad (41)$$

where

$$\mu = \left[\frac{m(n+1)}{a^2} - \frac{3M+m(n-2)}{a^3} - \beta\right] \Big/ 2\omega\left(a+\frac{1}{a}\right).$$

An integral of (41) is given by the relation

$$3\omega\omega_0 e^{\mu t} = \left(\frac{m}{a^2}+\frac{2m}{a^3}\right)\frac{l(l^n-1)}{l-1} - \left[\frac{m(n+1)}{a^2} - \frac{3M+m(n-2)}{a^3}\right] - \beta \quad \ldots \quad (42)$$

where $$\omega_0 = \left(\frac{M}{a^3}\right)^{\frac{1}{2}}.$$

Initially, we have the conditions $t = 0$, $\lambda = 0$, $m = 0$. Suppose $\lambda = n$ when $t = T$. Then we get an algebraic equation of the nth degree to determine λ. It is:

$$\left(\frac{3M+mn}{a^3}+\beta-\frac{mn}{a^2}\right) l^n+\left(\frac{m}{a^2}+\frac{2m}{a^3}\right)(l^{n-1}+\ldots+l)-3\omega\omega_0 e^{\mu T} = 0. \quad .. \quad (43)$$

The greatest positive root of this equation is given by the inequality, viz.,

$$\left.\begin{aligned} l_G &< 1+\left[3\omega\omega_0 e^{\mu T} \Big/ \left(\frac{3M+mn}{a^3}+\beta-\frac{mn}{a^2}\right)\right]^{\frac{1}{n}} \\ \text{or} \quad l_G &< 1+\left[3\omega\omega_0 e^{\mu T} \Big/ \left(\frac{3M+3mn-2m}{a^3}+\beta-\frac{m}{a^2}\right)\right] \end{aligned}\right\}^{8}. \quad .. \quad (44)$$

From (44) we get

$$3\omega\omega_0 e^{\mu T} \simeq \left[\frac{\left\{\dfrac{3M+3mn-2m}{a^3}+\beta-\dfrac{m}{a^2}\right\}^n}{\left\{\dfrac{3M+mn}{a^3}+\beta-\dfrac{mn}{a^2}\right\}}\right]^{\frac{1}{n-1}} \quad .. \quad .. \quad (45)$$

Also, for our purpose here, we require a positive root which is greater than unity. One such root is given by

$$l \simeq 1+\left[\frac{\dfrac{3M+3mn-2m}{a^3}+\beta-\dfrac{m}{a^2}}{\dfrac{3M+mn}{a^3}+\beta-\dfrac{mn}{a^2}}\right]^{\frac{1}{n-1}} \quad .. \quad .. \quad (46)$$

to a much closer approximation.

8. *Length of spiral arms.*

Taking the equation of the spiral as $r = ae^{\theta \cot \alpha}$ an element of length along the spiral arm is given by

$$ds = r \operatorname{cosec} \alpha \, d\theta = a \operatorname{cosec} \alpha \, e^{\theta \cot \alpha} d\theta. \quad .. \quad .. \quad (47)$$

Length of a spiral arm ranges from $\theta = 0$ to $\theta = \theta_n$, so that, we have

$$\begin{aligned} L &= a \operatorname{cosec} \alpha \int_0^{\theta_n} e^{\theta \cot \alpha} \, d\theta \\ &= \frac{a \operatorname{cosec} \alpha}{\cot \alpha}\left[e^{\theta_n \cot \alpha}-1\right] \\ &= \sec \alpha \, (r_n-a) \\ &= a \sec \alpha \, l^n \quad .. \quad .. \quad .. \quad .. \quad (48) \end{aligned}$$

where L = length of a spiral arm and $r_n = a(1+l^n)$. Substituting the value of l given by (46) we get

$$L = a\sec\alpha\left[1+\left\{\frac{\frac{3M+3mn-2m}{a^3}+\beta-\frac{m}{a^2}}{\frac{3M+mn}{a^3}+\beta-\frac{mn}{a^2}}\right\}^{\frac{1}{n-1}}\right]^n$$

$$= a\sec\alpha\left[1+n\left\{\frac{\frac{3M+3mn-2m}{a^3}+\beta-\frac{m}{a^2}}{\frac{3M+mn}{a^3}+\beta-\frac{mn}{a^2}}\right\}^{\frac{1}{n-1}}\right] \quad \ldots \quad (49)$$

The radius of the spiral arms (measured from the centre of the ellipsoidal mass) is given by

$$R = a(1+l^n) = a\left[2+n\left\{\frac{\frac{3M+3mn-2m}{a^3}+\beta-\frac{m}{a^2}}{\frac{3M+mn}{a^3}+\beta-\frac{mn}{a^2}}\right\}^{\frac{1}{n-1}}\right] \quad \ldots \quad (50)$$

So that, we can easily determine m and n as R and M are known from observations. We consider here two particular cases: (a) the nebula M31 (N.G.C. 224 in Andromeda), and (b) the nebula M33 (N.G.C. 598 in Triangulum).

(a) M31 [9] :—

$$R = 1{\cdot}97376\times10^{22}\text{ cm.} = a\left[2+n\left\{\frac{\frac{3M+3mn-2m}{a^3}+\beta-\frac{m}{a^2}}{\frac{3M+mn}{a^3}+\beta-\frac{mn}{a^2}}\right\}^{\frac{1}{n-1}}\right],$$

$a = 4{\cdot}9344\times10^{21}$ cm., $M = 3{\cdot}5\times10^9\odot$.

Therefore

$$2 = n\left[1+\frac{2m}{3M+\beta a^3+mn(1-a)}\right]_{\text{Approx.}} \quad \ldots \quad \ldots \quad (51)$$

and

$$L = 3a\sec\alpha.$$

It is found that there are two possible values of α. They are

$$\alpha = -36^\circ\,52', \quad \text{and} \quad \alpha = \pm45^\circ.$$

Then

$$\left.\begin{aligned} L_{\alpha_1} &= 3a\sec\alpha_1 &&= 1{\cdot}8504\times10^{22}\text{ cm.} \\ \text{and} \quad L_{\alpha_2} &= 3a\sec\alpha_2 &&= 1{\cdot}47\times10^{22}\text{ cm.}\end{aligned}\right\} \quad \ldots \quad (52)$$

where $L_{\alpha_1} = (L)_{\alpha=\alpha_1}$, etc.

(*b*) M33 [10]:—

$$R = 2{\cdot}734935\times10^{21}\text{ cm.} = a\left[2+n\left\{\frac{\frac{3M+3mn-2m}{a^3}+\beta-\frac{m}{a^2}}{\frac{3M+mn}{a^3}+\beta-\frac{mn}{a^2}}\right\}^{\frac{1}{n-1}}\right],$$

$a = 3{\cdot}5466\times10^{20}$ cm., $M = 1{\cdot}5\times10^{10}\odot$.

Therefore, as before, we get

$$5{\cdot}711428 = n\left[1+\frac{2m}{3M+\beta a^3+mn(1-a)}\right]_{\text{Approx.}} \quad \ldots \quad (53)$$

Then

$$L_{\alpha_1} = 6{\cdot}711428a \sec\alpha_1 = 2{\cdot}976\times10^{21}\text{ cm.}$$

and

$$L_{\alpha_2} = 6{\cdot}711428a \sec\alpha_2 = 3{\cdot}365\times10^{21}\text{ cm.} \quad \ldots \quad \ldots \quad (54)$$

The following table gives the probable values of m and n for the nebulae M31 and M33 as calculated from the equations (51) and (52) respectively. n_1 denotes the value of n for M31, and n_2 denotes the value of n for M33.

TABLE III.

m	n_1	n_2
$0{\cdot}1\times\odot$	$2{\cdot}81966\times10^{11}$	$1{\cdot}576\times10^{9}$
$0{\cdot}3\times\odot$	$8{\cdot}45898\times10^{11}$	$4{\cdot}728\times10^{9}$
$0{\cdot}5\times\odot$	$1{\cdot}40983\times10^{12}$	$7{\cdot}88\times10^{9}$
$0{\cdot}91\times\odot$	$2{\cdot}56589\times10^{12}$	$1{\cdot}434\times10^{10}$
$10\times\odot$	$2{\cdot}81966\times10^{13}$	$1{\cdot}576\times10^{11}$
$20\times\odot$	$5{\cdot}63932\times10^{13}$	$3{\cdot}152\times10^{11}$
$30\times\odot$	$8{\cdot}45898\times10^{13}$	$4{\cdot}728\times10^{11}$
$40\times\odot$	$1{\cdot}127864\times10^{14}$	$6{\cdot}304\times10^{11}$
$50\times\odot$	$1{\cdot}40983\times10^{14}$	$7{\cdot}88\times10^{11}$

It is known that the mass of a star ranges from ½ to 50 times that of the sun.[11] The stars which are formed along the spiral arms of the nebulae considered above have their masses each equal to $0{\cdot}91\odot$ where $\odot$ = mass of the sun. It should be remembered that these estimates give the order of magnitude only.

9. *Solution of the perturbation problem.*

The expression of the dynamical equations of motion of the outgoing masses along the spiral arms in terms of a single variable λ facilitates the solution of the problem of perturbed motion of the masses. For this purpose, therefore, it is necessary to evaluate the constants P, Q, R which occur in equations (52) of the last paper. This is done without much labour if expressions are found for the A's which are functions of ξ_λ only. We give below some of the values:

$$\left.\begin{aligned}
A_0 &= \frac{1}{(1+\xi_\lambda^2)^{\frac{1}{2}}}\left[1+\frac{3}{4}\cdot\frac{\xi_\lambda^2}{(1+\xi_\lambda^2)^2}+\frac{105}{64}+\frac{\xi_\lambda^4}{(1+\xi_\lambda^2)^4}+\ldots\right],\\
A_1 &= \frac{1}{(1+\xi_\lambda^2)^{\frac{1}{2}}}\left[\frac{\xi_\lambda}{(1+\xi_\lambda^2)}+\frac{15}{8}\,\frac{\xi_\lambda^3}{(1+\xi_\lambda^2)^3}+\ldots\right],\\
A_2 &= \frac{1}{(1+\xi_\lambda^2)^{\frac{1}{2}}}\left[\frac{3}{4}\cdot\frac{\xi_\lambda^2}{(1+\xi_\lambda^2)^2}+\frac{35}{16}\,\frac{\xi_\lambda^4}{(1+\xi_\lambda^2)^4}+\ldots\right],\\
A_3 &= \frac{1}{(1+\xi_\lambda^2)^{\frac{1}{2}}}\left[\frac{5}{8}\,\frac{\xi_\lambda^3}{(1+\xi_\lambda^2)^3}+\frac{315}{128}\,\frac{\xi_\lambda^5}{(1+\xi_\lambda^2)^5}+\ldots\right], \text{ etc.}
\end{aligned}\right\} \quad \ldots \quad (55)$$

Therefore,

$$P' = \left[\frac{\partial}{\partial\xi_\lambda}\left\{A_0-A_1+A_2-A_3+\ldots+(-1)^pA_p+\ldots\right\}\right]_{\xi_\lambda=1}$$
$$= -0{\cdot}843,$$

so that, $\quad 1+P' = P = 0{\cdot}157$;

$$Q = \left[\frac{\partial^2}{\partial\xi_\lambda^2}\left\{A_0-A_1+A_2-A_3+\ldots+(-1)^pA_p+\ldots\right\}\right]_{\xi_\lambda=1}$$
$$= 5{\cdot}985;$$

and $\quad R = \left\{1-A_1+4A_2-9A_3+\ldots+(-1)^pp^2A_p+\ldots\right\}_{\xi_\lambda=1}$
$$= 0{\cdot}411.$$

Suppose that the number of masses moving along each of the spiral arms is n, then, we have

$$m' = M-mn.$$

With the substitutions $\rho_\lambda = l^\lambda$ and $\sigma_\lambda = \frac{1}{a}\,l^\lambda$, the equations of perturbed motion (equations (52) of the paper referred to above) reduce to the form:

$$\left.\begin{aligned}
L^2l^\lambda\dot\lambda^2+Ll^\lambda\ddot\lambda-\frac{2\omega}{a}\,Ll^\lambda\dot\lambda &= \left\{\frac{7{\cdot}985M-m(5{\cdot}985n+2)}{a^3}+\omega^2\right\}l^\lambda\\
&\quad+\frac{2m}{a^3}\cdot\frac{l(l^n-1)}{l-1}+\frac{0{\cdot}157(M-mn)}{a^3},\\
L^2l^\lambda\dot\lambda^2+Ll^\lambda\ddot\lambda+2\omega aLl^\lambda\dot\lambda &= \left\{\frac{m(n+1)}{a^2}-\frac{0{\cdot}411(M-mn)}{a^3}\right\}l^\lambda\\
&\quad-\frac{m}{a^2}\cdot\frac{l(l^n-1)}{l-1}.
\end{aligned}\right\} \quad \ldots \quad (56)$$

These give a single differential equation of the first order to determine l^{λ}. It is

$$2\omega\left(a+\frac{1}{a}\right)\frac{d}{dt}(l^{\lambda}) = \left\{\frac{m(n+1)}{a^2} - \frac{8{\cdot}396M - m(6{\cdot}396n+2)}{a^3} - \omega^2\right\} l^{\lambda}$$
$$-\left(\frac{m}{a^2}+\frac{2m}{a^3}\right)\frac{l(l^n-1)}{l-1} - \frac{0{\cdot}157(M-mn)}{a^3}. \quad .. \quad (57)$$

The solution of (57) is given by

$$9{\cdot}396\omega\omega_0 e^{\mu T} = \left\{\frac{m(n+1)}{a^2} - \frac{8{\cdot}396M - m(6{\cdot}396n+2)}{a^3} - \omega^2\right\} l^{\lambda}$$
$$-\left(\frac{m}{a^2}+\frac{2m}{a^3}\right)\frac{l(l^n-1)}{l-1} - \frac{0{\cdot}157(M-mn)}{a^3} \quad .. \quad .. \quad (58)$$

where
$$\mu = \frac{\text{coeff. of } l^{\lambda}}{2\omega\left(a+\dfrac{1}{a}\right)}.$$

Let $\lambda = n$ when $t = T$, so that, (58) gives an algebraic equation of the nth degree to determine l. A positive root is approximately given by

$$l = 1+\left[\frac{\dfrac{9{\cdot}396M-3{\cdot}396mn-2m}{a^3}+\beta-\dfrac{m}{a^2}}{\dfrac{9{\cdot}396M-5{\cdot}396mn}{a^3}+\beta-\dfrac{mn}{a^2}}\right]^{\frac{1}{n-1}} \quad .. \quad .. \quad (59)$$

The expression for the radius of the spiral arm for this case is

$$R = a\left[2+n\left\{\frac{\dfrac{9{\cdot}396M-3{\cdot}396mn-2m}{a^3}+\beta-\dfrac{m}{a^2}}{\dfrac{9{\cdot}396M-5{\cdot}396mn}{a^3}+\beta-\dfrac{mn}{a^2}}\right\}^{\frac{1}{n-1}}\right] \quad .. \quad (60)$$

Comparing this with the expression given by (50) we see that the effect of the perturbing influence is to bring the spiral arms closer to the central ellipsoid. It can be very easily shown that the perturbing influence also puts a check on the number of particles moving along the spiral arms.

10. *Evolution of spiral nebulae.*

The whole of the theoretical research on the problem of the nebulae, in general, is based on the tacit assumption that these bodies are masses of gas in rotation with increasing angular velocity. This has also been borne out by observational evidence. The sequence of configurations through which these gaseous masses pass is governed partly by the gas laws and partly by dynamical principles. The elliptical, spiral and other types of nebulae come under distinct sets of configurations.* In the case of elliptical nebulae the fall of temperature is much more rapid than the spiral nebulae, so much so, that the shrinkage is not checked by the angular momentum. This accounts for the relatively smaller size of the elliptical nebulae. While, in the case of spiral nebulae, the balance between the shrinkage and the increasing angular momentum is kept up by the ejection of matter by the central rotating ellipsoid. The motion of these outgoing masses takes place under dynamical conditions only.† This fact is shown by the set of differential equations ((6) or (27) and (28) of the paper referred to above) for the motion of the ejected particles. It is, thus, clearly seen that the elliptical nebulae and the spiral nebulae pass through two distinct sequences of configurations, and, hence, the evolution leading to their present stage is also distinct.

11. *Summary.*

In the first part of this paper we have shown that the orbits of the fundamental particles—the extra-galactic nebulae—are spiral tracks given by the Lagrange equations of motion which is the same as geodesic motion of the general theory of relativity. The angle of the spiral is found to be $\alpha = -36^\circ\, 52'$ approximately. This relativistic treatment suggests that the law of force which governs the motion of particles along the spiral arms is of the form $f(r) = \frac{M}{r^2}+\beta r$ which is a combination of the law of inverse square and the law of direct distance. The characteristic exponents of the spiral orbits give the 'region of stability' for the orbits. In the second part, solutions of the equations of motion are found in terms of a single variable λ as a function of t. Incidentally, we get an algebraic equation of the nth degree to determine l, so that, the determination of the length of a spiral arm becomes an easy matter provided we know m and n. A table of the probable values of m and n

* (Added on January 30, 1941.) 'It follows from our discussion that the elliptical nebulae are not to be considered as younger than the spirals. They have followed another root of development only because of a different initial state of motion. The argument furnishes no reason for believing that the stars should not have developed in elliptical nebulae to the same extent as in spirals .... The question of how the systems acquire their original rotation remains unanswered.'—G. Randers: 'A note on the evolution of extragalactic nebulae.' *Ap. J.*, 92, 2, p, 246, (Sept. 1940).

† The nature of the dynamical problem is such that the mass of the central ellipsoid in rotation with the angular velocity increasing with time reaches a particular stage (partial instability) such that the ejection of particles from the antipodal points is the only criterion for a momentarily stable state of the system. Once this process continues as the time passes by it is shown in paper I that $(\omega^2-\beta)a^3$ is a decreasing function. An important conclusion is that after this stage ω^2 increases while a decreases; and, if we now regard β as a uniformly increasing function of time (which must be expected as the density in outer regions increases) the system acquires partial stability. It will be seen, therefore, that no consideration of hydrodynamical instability is required in this purely dynamical problem. It is only the dynamical (gravitational) instability which is responsible for the spiral motion of the ejected particles.

are given for the particular cases M31 and M33. The problem of perturbed motion is also solved. Lastly, it is shown that the elliptical and spiral nebulae have distinct evolutions.

12. *Appendix.**

Going back to the theory given in the previous paper by the present author it is interesting to point out that both Lindblad and Chandrasekhar [12] start with a gravitational potential function for the discussion of the spiral phenomena. Employing cylindrical co-ordinates Lindblad finds the gravitational potential to be of the form $V(r, z)$. The thin lenticular shape taken by the rotating gaseous mass gives rise to dynamical instability which is responsible for the catastrophe of the breaking up of matter into two spiral arms. Considering the high angular momentum the order of evolution seems to be from types like the Magellanic Cloud via spiral nebulae to elliptical nebulae. An advanced stage of the disc formation leads to partial instability resulting in the formation of closed arms, the ejected particles moving longer distances along the spirals. It is found that the central ellipsoid contains most of the mass of the system and rotates with fairly uniform angular velocity. Galactic and nebular structure is satisfactorily explained on classical dynamics by a suitable modification of Newtonian gravitation. The tidal effects of the outgone particles (that is, the spiral arms) and the formation of massive spiral arms by successive ejections are dealt with. Our theory also give the same results. Two opposite symmetrical spirals (logarithmic type) are favoured. In a recent paper Lindblad [13] finds that the central portion rotates more rapidly than the spiral structure which has been found to be the case with the Andromeda nebula. Theoretical considerations supported by observational evidence go to point out that the rotation is in the direction in which the spirals wind (that is, taking $\vec{OP}$ as the radius vector, in the direction of θ increasing, which is found to be the case also according to our theory).

In a recent paper referred to above Dr. S. Chandrasekhar has explained the spiral formation by means of spherically symmetrical gravitational potential $\mathfrak{M}$ expressed in terms of residual velocities. The axially symmetrical non-steady systems give a pair of spirals as the orbits of constant relative density in contrast to Lindblad's theory of the spiral arms as the orbits of individual stars. He deals with almost all points (e.g., permanence of the spiral structure, etc.) concerning the spiral nebulae in Part XIII of the paper cited above and gives the theory of open, barred and other types of spirals. It is remarked that for most of the well-resolved open spirals the value of α is almost 45° (equiangular spirals). It is found that Chandrasekhar's theory is more to be favoured as it is capable of explaining a variety of cases of the spirals which Lindblad's theory fails to do.

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