ACCURATE CALCULATIONS ON THE CASCADE THEORY OF ELECTRONIC SHOWERS WITHOUT COLLISION LOSS.

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ABSTRACT.

A rigorous solution of the equations of the Cascade theory, taking the radiation and pair creation cross-sections to be those for complete screening and neglecting collision loss entirely is given. The solution is expressed in the form of an integral, from which the nature of the energy spectra of the shower particles as well as the total number of particles at any depth produced in a particle excited shower and also in a quantum excited shower for various energies of the primary have been obtained. The differences occurring between a particle excited shower and a quantum excited shower has been interpreted physically. The results of the present paper have been compared with similar results of the previous authors.

Following the physical ideas put forward by Bhabha and Heitler (1937) and Carlson and Oppenheimer (1937), the theory of the cascade process in cosmic rays has been worked out by several authors with a view to making a more rigorous treatment of the problem both from the standpoint of physical assumptions and mathematical treatments. Landau and Rumer (1938) tried to work out the problem, neglecting ionisation or collision loss. Starting in a rigorous way, they ultimately made such approximations that instead of giving more accurate figures than those of the preceding authors, their results are often in error by as much as a factor thirty. Later on Serber (1938) tried to work out the same problem and extended the calculations of Snyder (1938). But their analysis is defective for various reasons *.

In a paper by Bhabha and the present author † a complete theory of the cascade production of cosmic ray showers together with the effect of the collision loss on it has been given. In the present paper we give the results obtained

* The function $k(y, s)$, used by Serber, is easily seen to be given by

$$k_{\mu}(y, s) = \prod_{r=1}^{s} \frac{r\mu_y}{(\mu_y - \mu_{y+r})(\mu_y - \nu_{y+r})}; \quad k_{\nu}(y, s) = \prod_{r=1}^{s} \frac{r\nu_y}{(\nu_y - \mu_{y+r})(\nu_y - \nu_{y+r})}.$$

With these values of $k(y, s)$, the expression for Z, as given by eq. (6) is an infinite series which obviously *diverges for all values of E*. Consequently, Z cannot be a proper solution of the problem. Their series does not satisfy the correct boundary conditions at $t = 0$, and due to the divergence of the series it is impossible to say that the boundary conditions are satisfied even approximately. The expression given for N without the term $k_{\mu}(y, -y)$ will, however, give the total number of particles having energies greater than β, if collision loss is neglected, as will be evident from the present paper. .

† In course of publication, referred to in this paper as (A).

when one neglects the collision loss completely. A comparison with similar results obtained by the previous authors has also been made at the end.

In making such calculations the exact cross-sections for radiation loss and pair creation, for the case of complete screening as given by Bethe and Heitler (1934) have been taken. The variation of these cross-sections due to incompleteness of screening and also the effect of collision loss have been neglected. Denoting by t all lengths measured in terms of a characteristic length such that a length of 1 c.m. in a material corresponds to

$$t = \frac{4NZ^2}{137} \cdot \left(\frac{e^2}{mc^2}\right)^2 \log(183Z^{-\frac{1}{3}}), \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

let $P(E, t)dE$ and $Q(E, t)dE$ be the number of particles and quanta respectively at any depth t and having energies lying between E and $E+dE$. Also let $N(E, t)$ be the total number of particles at the depth t and having energies greater than E. Following the work of Landau and Rumer, Bhabha and the present author have shown that neglecting collision loss, the total number of particles of energy greater than E, produced by a primary particle of energy E_0, is given exactly by,

$$N(E, t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E}\right)^{S-1} \cdot \frac{1}{S-1} \cdot \left\{\frac{D-\lambda}{\mu-\lambda} e^{-\lambda t} + \frac{\mu-D}{\mu-\lambda} e^{-\mu t}\right\} dS \qquad (2)$$

and also the energy spectrum for particles and quanta are given by

$$P(E, t) = \frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E}\right)^{S} \cdot \left\{\frac{D-\lambda}{\mu-\lambda} e^{-\lambda t} + \frac{\mu-D}{\mu-\lambda} e^{-\mu t}\right\} dS \quad \ldots \quad (3a)$$

$$Q(E, t) = \frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E}\right)^{S} \cdot \frac{C}{\mu-\lambda} \cdot \left\{e^{-\lambda t} - e^{-\mu t}\right\} dS \quad \ldots \quad \ldots \quad (3b)$$

where $$A_S = \left(\frac{4}{3}+\alpha\right)\left\{\frac{d}{dS}\log\Gamma(S)+\gamma-1+\frac{1}{S}\right\}+\frac{1}{2}-\frac{1}{S(S+1)}, \quad \ldots \quad (4a)$$

$$B_S = 2\left[\frac{1}{S}-\left(\frac{4}{3}+\alpha\right)\frac{1}{(S+1)(S+2)}\right], \quad \ldots \quad \ldots \quad \ldots \quad (4b)$$

$$C_S = \left[\frac{1}{S+1}+\left(\frac{4}{3}+\alpha\right)\frac{1}{S(S-1)}\right], \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4c)$$

$$D = \frac{7}{9}-\frac{1}{6}\alpha, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4d)$$

and $\alpha = 1/9 \log(183Z^{-\frac{1}{3}})$, γ the Euler-Mascheroni constant.

λ, μ are the roots of the equation

$$x^2-(A_S+D)x+(A_SD-B_SC_S) = 0. \quad \ldots \quad \ldots \quad (5)$$

σ is some real number greater than 1.

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We, therefore, have

$$N(E,t) = N_\lambda(E,t)+N_\mu(E,t) \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where

$$N_\lambda(E,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \exp\{y(S-1)-\lambda t\} \cdot \frac{1}{S-1}\cdot\frac{D-\lambda}{\mu-\lambda}\, dS. \qquad (7a)$$

$$N_\mu(E,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \exp\{y(S-1)-\mu t\} \cdot \frac{1}{S-1}\cdot\frac{\mu-D}{\mu-\lambda}\, dS. \ldots \quad (7b)$$

where $\quad y = \log_e\left(\dfrac{E_0}{E}\right).$

The integral (7*a*) can be evaluated by the saddle point method, with an error which is less than 2 per cent, and is given by

$$N_\lambda(E,t) = \frac{\exp \psi(S_0)}{\sqrt{2\pi\psi''(S_0)}}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

where

$$\psi(S) = y(S-1)-\lambda t + \log\left(\frac{1}{S-1}\cdot\frac{D-\lambda}{\mu-\lambda}\right) \quad \ldots \quad \ldots \quad (9)$$

and S_0 is the saddle point, such that $\psi'(S_0) = 0$. A dash represents differentiation with respect to S.

The values of λ, μ, $\dfrac{d\lambda}{dS}$, $\dfrac{d\mu}{dS}$, $\dfrac{d^2\lambda}{dS^2}$ and $\dfrac{d^2\mu}{dS^2}$ can be calculated from equations (4) and (5) for all real values of S. A complete table of such values has been given in (A). From this table the values of $\psi(S)$, $\psi'(S)$ and $\psi''(S)$ can be obtained for all real values of S at intervals of 0·1. For any other value of S lying between any two tabulated values, the values of $\psi(S)$ and $\psi''(S)$ are to be obtained by using Newton's formula for forward interpolation, viz.

$$\phi(S) = \phi(S_1)+\frac{S-S_1}{h}\cdot\Delta\,\phi(S_1)+\frac{(S-S_1)\,(S-S_2)}{2!\,h^2}\,\Delta^2\,\phi(S_1)+\ldots \quad \ldots \quad (10)$$

where S lies between S_1 and S_2 which are the tabulated points and $S_2-S_1 = h$ and $\Delta\,\phi(S_1)$, $\Delta^2\,\phi(S_1)$, etc. are the successive differences. Putting $\dfrac{S-S_1}{h} = x$, we have

$$\phi(S_1+xh) = \phi(S_1)+x\Delta\,\phi(S_1)+\frac{x(x-1)}{2!}\cdot\Delta^2\,\phi(S_1)+\ldots \quad \ldots \quad \ldots \quad (11)$$

Equation (11) has been used for the determinations of $\psi(S_0)$ and $\psi''(S_0)$.

To get the value of S_0 we use (11) taking $\psi'(S)$ instead of $\phi(S)$ and putting $S_0 = S_1+xh$, we make $\psi'(S_1+xh) = 0$, so that we have

$$0 = \psi'(S_1)+x\Delta\,\psi'(S_1)+\frac{x(x-1)}{2!}\,\Delta^2\,\psi'(S_1)+\,\ldots$$

So that,

$$x = -\,\frac{\psi'(S_1)}{\Delta\,\psi(S_1)-\frac{1}{2}\Delta^2\,\psi'(S_1)} - \frac{x^2}{2}\cdot\frac{\Delta^2\,\psi'(S_1)}{\{\Delta\,\psi'(S_1)-\frac{1}{2}\Delta^2\,\psi'(S_1)\}},$$

and consequently to a second approximation,

$$x = -\,\frac{\psi'(S_1)}{\Delta\,\psi'(S_1)-\frac{1}{2}\Delta^2\psi'(S_1)} - \frac{1}{2}\cdot\frac{\Delta^2\,\psi'(S_1)\cdot\{\psi'(S_1)\}^2}{\{\Delta\,\psi'(S_1)-\frac{1}{2}\Delta^2\,\psi'(S_1)\}^3}. \quad \ldots \quad (12)$$

The value of S_1 for given values of y and t can be easily obtained by noting the change in sign of the tabulated values of $\psi'(S)$. Since the values of S_1 and S_2 are known to the first place of decimal and to the order of accuracy we require, one more place in the value of S_0 is sufficient, it is enough to consider up to a second approximation. The value of x given by (12) can thus be considered as sufficiently accurate. Thus knowing x we get S_0 immediately.

The values of N_λ thus obtained from (8) are given in Table I. It can also be shown as in (A) that

$$N_\mu(E,t) = \exp\left[\left\{\tfrac{1}{2}-\alpha'(\gamma-1)\right\}t\right].\left(\frac{2y}{3\alpha't}\right)^{\alpha't/2}.J_{\alpha't}\left(\sqrt{6\alpha'ty}\right) \quad \cdots \quad (13)$$

where $$\alpha' = (\tfrac{4}{3}+\alpha).$$

The values of N_μ are also given in Table I, from which it will be clear that for $t > 3$, N_μ is negligible.

TABLE I.

Values of N_λ and N_μ as functions of y and t for a particle excited shower.

$y \backslash t$		1·0	2·0	3·0	4·0	6·0	8·0	10·0	15·0	25·0
2	N_λ	1·14	0·723	0·445	0·247					
	N_μ	0·12	0·02							
3	N_λ	1·59	2·05	1·73	1·19	0·453	0·168	0·0548		
	N_μ	−0·30	−0·22							
4	N_λ	2·32	4·07	4·58	4·06	2·13	0·889	0·328	0·0223	
	N_μ	−0·54	−0·34							
5	N_λ	3·19	7·20	9·91	10·6	7·81	4·07	1·77	0·148	$5{\cdot}48\times10^{-3}$
	N_μ	−0·53	−0·22							
6	N_λ	4·20	11·64	19·26	24·56	23·52	15·45	7·99	0·909	$4{\cdot}84\times10^{-3}$
	N_μ	−0·38	0·18	0·20						
7	N_λ	5·70	17·96	34·05	50·70	63·21	50·45	31·34	4·870	$3{\cdot}39\times10^{-2}$
	N_μ	−0·10	0·56	0·30						
8	N_λ	6·81	25·82	57·51	98·14	157·0	149·6	108·4	22·82	0·230
	N_μ	0·28	0·72	0·22						
9	N_λ	7·65	36·83	93·64	174·4	349·8	401·9	339·2	98·09	1·43
	N_μ	0·56	0·30	−0·17						
10	N_λ	9·40	48·37	142·9	305·4	720·2	1000	981·5	371·1	8·24

If the primary be a quantum of energy E_0 instead of a particle, we have for the boundary conditions,

$$P(E, 0) = 0 \text{ and } Q(E, 0) = \delta(E_0 - E). \quad \ldots \quad \ldots \quad (14)$$

In this case we have

$$P(E, t) = \frac{1}{2\pi i E_0} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_0}{E}\right)^S \cdot \frac{B}{\mu - \lambda} \left\{ e^{-\lambda t} - e^{-\mu t} \right\} dS \quad \ldots \quad \ldots \quad (15a)$$

$$Q(E, t) = \frac{1}{2\pi i E_0} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_0}{E}\right)^S \cdot \left\{ \frac{\mu - D}{\mu - \lambda} e^{-\lambda t} + \frac{D - \lambda}{\mu - \lambda} e^{-\mu t} \right\} dS. \quad \ldots \quad (15b)$$

Consequently, the total number of particles $N_Q(E, t)$ produced in a quantum excited shower is given by

$$N_Q(E, t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_0}{E}\right)^{S-1} \cdot \frac{B}{\mu - \lambda} \cdot \frac{1}{S-1} \cdot \left(e^{-\lambda t} - e^{-\mu t} \right) dS,$$

$$\approx \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_0}{E}\right)^{S-1} \cdot \frac{B}{\mu - \lambda} \cdot \frac{1}{S-1} \cdot e^{-\lambda t} dS, \quad \ldots \quad \ldots \quad (16)$$

except for very small t.

The values of $P(E, t)$, $Q(E, t)$ and $N_Q(E, t)$ can similarly be calculated by the saddle-point method. The values of $N_Q(E, t)$ for different values of y and t are given in Table II.

TABLE II.

Values of $N_Q(E, t)$ as functions of t for a quantum excited shower.

$y \backslash t$	1·0	2·0	3·0	4·0	6·0	8·0	10·0	15·0	25·0
2	0·784	0·854	0·662	0·430					
3	1·07	1·70	1·75	1·47	0·735	0·293			
4	1·39	2·94	3·90	4·02	2·76	1·38	0·575		
5	1·81	4·78	7·73	9·36	8·67	5·41	2·68	0·287	
6	2·20	7·12	13·7	19·6	21·9	18·0	10·7	1·55	
7	2·66	10·0	23·2	37·9	58·2	54·1	38·3	7·58	0·0687
8	2·95	14·3	37·1	68·6	131	149	123	32·6	0·441
9	3·54	18·4	57·9	119	279	374	362	130	22·4
10	4·38	25·9	86·5	201	563	901	993	473	13·8

It will be evident from Tables I and II that the curves giving N as a function of t are almost the same for a shower produced by a particle or a quantum of the same initial energy, with the difference that the latter curve is shifted to a greater thickness by nearly one unit of length over the whole of its range. It has been shown in (A) that for a given y the maximum of N

and P will occur at such a value of t that the corresponding value of S_0 is 2. From this it can be deduced that for a given y the maximum of N for a particle excited shower will occur at a thickness, 0·82 in characteristic units less than that for a quantum excited shower. The value of $N_{max.}$ for the former, however, will be greater than that of the latter, although the relative difference decreases as y increases. This will be clear from Fig. 1 where the N–t curve for two different values of y, viz. 4 and 10, have been given both for the particle excited shower (P) and quantum excited shower (Q). This is exactly what is to be expected from the physical nature of the problem. A high energy quanta, on entering the material, creates a pair within a very short distance in which one particle either has an energy *nearly* equal to that of the quanta, which then produces the cascade, or two particles, each with about half the energy, both of which produce smaller cascades.

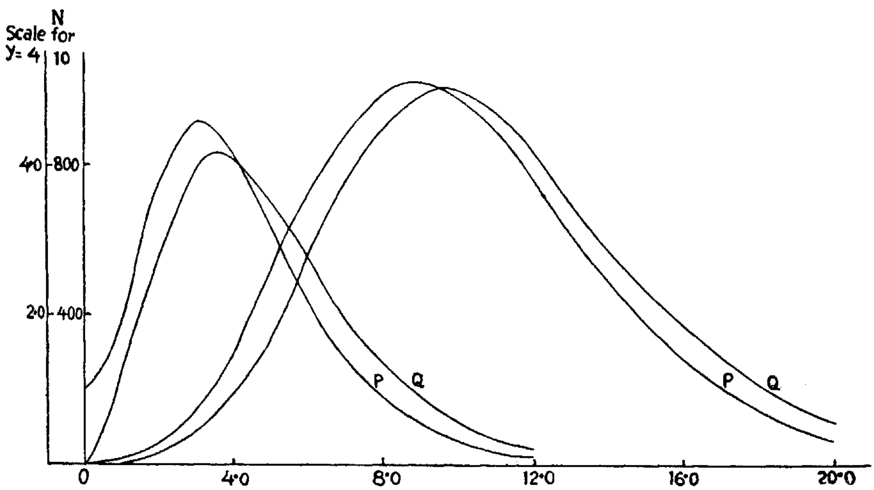

FIG. 1.

A comparison of the values of N obtained from (7) with similar figures obtained from the work of the previous authors has been made in Table III. It is clear from Table III that the results obtained by Bhabha and Heitler are correct to within thirty per cent at the position of the maximum and also to the left of it in the transition curve. For a particular value of y, the values of N as obtained by Bhabha and Heitler are less than that of the present paper before the maximum of the transition curve is attained, after which the values of Bhabha and Heitler are higher than those of the present paper. The values of Carlson and Oppenheimer are very much higher than those of the present paper. This is due to the approximate forms of the expressions assumed for the cross-sections of the different processes.

TABLE III.

Comparison of the values of N for a particle excited shower.
B. & H. = Bhabha and Heitler.
C. & O. = Carlson and Oppenheimer.
C. = Present author.

y \ t		2	4	6	8	10	12
3	B. & H.	1·50	1·64	0·60	0·22	0·06	
	C. & O.	3·30	3·00	1·30	0·60	0·20	
	C.	1·83	1·13	0·45	0·17	0·055	
5	B. & H.	6·40	12·00	10·60	6·80	3·20	1·40
	C.	6·98	10·61	7·81	4·07	1·77	0·689
10	B. & H.	40·0	250	640	1130	1360	1040
	C.	48·37	305·4	720·2	1000	981·5	752·0

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