CONTRIBUTION TO THE THEORY OF STELLAR MODELS.

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ABSTRACT.

Stellar models for the energy generation law $\eta \propto \rho^{\alpha} T^{\delta}$ studied by Chandrasekhar have been examined as regards their dimensions and convective stability. It is found only the standard solutions represent finite configurations. Further, Chandrasekhar's γ-models are all radiatively unstable for $\gamma \geqslant \frac{3}{4}$. The stability condition for (α, δ)-model has also been studied.

1. INTRODUCTION.

Stellar models in which the average energy generation per unit mass of enclosed matter denoted by η is given by

$$\eta = \eta_0 \rho^{\alpha} T^{\delta} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

ρ and T being density and temperature respectively at the point concerned, have been considered by Chandrasekhar (1936, 1937). Taking the opacity to be given by a formula analogous to that of Kramers, Chandrasekhar has studied the relation between ρ and T within such stellar bodies, and also the conditions for the inversion of density gradient. The case $\alpha = 0$, called by him γ-model, has been considered in detail, and as regards the general character of density-temperature relationship proves to be typical of the more general case $\alpha > 0$. Denoting temperature and density measured in suitable units by τ and σ respectively, three types of $\tau - \sigma$ solution curves have been shown to exist. Of these there are two which may correspond to stellar conditions even up to the boundary. The first type of solution, called standard solution by Chandrasekhar (denoted here by S), can be represented by a curve starting from the origin ($\tau = 0$, $\sigma = 0$) at a definite slope rising gradually, in some cases going to infinity, and in others turning back at a certain stage and asymptotically approaching the τ-axis. A second type of solution curves starts from the τ-axis with a steep gradient, rises upwards to a maximum, and then bends back and quickly assumes an asymptotic form approaching the τ-axis. These may correspond to stellar bodies with finite surface temperatures (called in this paper I solutions). A third type of solution curves in which $\sigma \rightarrow$ const. or ∞, as $\tau \rightarrow 0$ will not be considered here.

Beyond isolating these solutions and studying their density gradients which bring to light the density temperature relationship within the models, further characteristics of these solutions have not been studied by Chandrasekhar. In the present paper we shall consider two other aspects of these solutions. Firstly, we shall study if S and I solutions give *finite* configurations and masses. Such solutions are of interest in the way that they *may* directly represent stellar body configurations. Secondly, there is the more important question of radiative stability. The above solutions have all been obtained on the assumption that the energy transfer within the models is mainly by radiation. But there are certain conditions which must be satisfied in order that this may be true. We shall test the S and I solutions of Chandrasekhar from this point of view.

Our examination of these solutions shows that whereas the S solutions give finite configurations,* I solutions, as one can easily understand, ultimately correspond to isothermal gas spheres with infinite radii and masses. Further, many of the S solutions are radiatively unstable, those for fairly high values of δ all along the solution curve, while others for lower values of δ for temperatures (τ values) higher than certain limiting ones. Values of α higher than 2 are inconsistent with stability except under some rather unimportant special conditions. The I solutions have portions starting somewhat below their maxima and extendnig up to the τ-axis (which they meet for finite values of τ) which are radiatively stable.

PART I.

2. THE EQUATIONS.

For the energy generation defined by (1) and for a law of opacity given by

$$\kappa = \kappa_0 \frac{\rho^b}{T^{3+s}}, \qquad \text{(2)}$$

we have from the equations

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho \qquad \text{(3)}$$

$$\frac{dp_r}{dr} = -\frac{\kappa\rho}{c}\frac{L(r)}{4\pi r^2} \qquad \text{(4)}$$

by following a well-known analysis of Chandrasekhar (1936)

$$\gamma x y^{\alpha+b}\frac{dy}{dx} = y^{\alpha+b}(1+y)-x \qquad \text{(5)}$$

where $y = p_g/p_r$ = (gas pressure)/(radiation pressure), $x = KT^{-4\gamma}$,

$$4\gamma = \delta+3\alpha+3b-s-3, \quad K = 4\pi cGM/\kappa_0\eta_0 L \cdot \left(\frac{k}{\mu H}\cdot\frac{3}{a}\right)^{\alpha+b} \qquad \text{(6)}$$

Using Chandrasekhar's variables

$$y = zq, \quad x = q^\gamma, \quad \nu = \alpha+b$$

we can write equation (5) as follows:

$$\frac{dz}{dq} = \frac{1}{q^2}\left(1-\frac{q^{\gamma-\nu}}{z^\nu}\right). \qquad \text{(7)}$$

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* Chandrasekhar has indeed numerically worked out the S solutions for certain values of γ. They give finite configurations.

We want an equation to determine r. For this purpose we rewrite the equation

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{d}{dr}(p_r+p_g)\right] = -4\pi G\,.\,\rho \qquad \ldots \qquad \ldots \qquad (8)$$

in terms of z and q, and replacing r by the variable ξ, where

$$r = \left[\frac{1}{4\pi G}\left(\frac{k}{\mu H}\right)^2 \cdot \frac{3}{a} \cdot K^{-\frac{1}{2}\gamma}\right]^{\frac{1}{2}} \xi$$

and using (7) to replace the derivative dz/dq, we have after some calculation the following equation between z and ξ

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\frac{\xi^2}{zq^{\frac{1}{4}}(1-z^\nu q^{\nu-\gamma})}\cdot\frac{dz}{d\xi}\right] = -zq^{\frac{1}{4}}. \qquad \ldots \qquad \ldots \qquad (9)$$

Further from (3) we obtain

$$M(r) \propto -\frac{r^2}{\rho}\frac{dP}{dr} \propto \frac{\xi^2}{zq^{\frac{1}{4}}(1-z^\nu q^{\nu-\gamma})}\cdot\frac{dz}{d\xi}. \qquad \ldots \qquad \ldots \qquad (10)$$

Equations (7) and (9) will enable us to find the dimensions of the stellar models.

Chandrasekhar transforms equation (5) in $\tau-\sigma$ plane by putting

$$\rho = \rho_0\sigma, \quad T = T_0\tau \qquad \ldots \qquad \ldots \qquad \ldots \qquad (11)$$

the constants ρ_0, T_0 being defined by

$$KT_0^{-4\gamma} = \frac{k}{\mu H}\cdot\frac{3}{a}\left(\rho_0 T_0/K^{\frac{1}{\gamma}}\right) = 1. \qquad \ldots \qquad \ldots \qquad (11a)$$

We have then the following connections between the variables

$$z = \sigma\tau, \quad q = \tau^{-4}, \quad x = \tau^{-4\gamma} = q^\gamma, \quad y = z/\tau^4 = \sigma/\tau^3. \qquad \ldots \qquad (11b)$$

We may note $\sigma \propto$ density, $\tau \propto$ temperature, $q^{-1} \propto$ (radiation pressure), $z \propto$ gas pressure, and $\xi \propto$ radial distance.

3. The γ-model. $\gamma = 1$.

The class of stellar models discussed in detail by Chandrasekhar corresponds to $\alpha = 0$, $b = 1$, $4\gamma = (8-s)$, and has been called by him γ-model. The nature of the integral curves of (5) on (τ, σ)-plane has been described. The case $\gamma = 1$ is specially interesting in consideration of the fact that the (τ, σ) differential equation can then be integrated in closed form, and as regards the behaviour of the solution curves is typical of the cases $\frac{3}{4} < \gamma < \frac{3}{2}$. We shall first discuss the case $\gamma = 1$, and later proceed to deal with the more general case rather briefly. Rigorous proofs will be given in the Appendix.

Case $\gamma = 1$, $b = 1$. We shall seek solutions of the equations

$$\frac{dz}{dq} = \frac{1}{q^2}\left(1-\frac{1}{z}\right) \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (7a)$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\frac{\xi^2}{zq^{\frac{1}{4}}(1-z)}\frac{dz}{d\xi}\right] = -zq^{\frac{1}{4}} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (9a)$$

such that as $\sigma \to 0$, $\tau \to 0$; these solutions are known to exist (S solutions). We shall be interested in the region where τ is small, and hence also z. The solution of (7a) in question is

$$\frac{1}{q} = -z - \log(1-z). \qquad \ldots \quad \ldots \quad \ldots \quad (12)$$

For *small* z we have

$$zq^{\frac{1}{2}} \sim 2^{\frac{1}{2}} z^{\frac{1}{2}} \left(1 - \frac{1}{6} z\right).$$

Substituting in (9a) and putting

$$z^{\frac{1}{2}} = u, \quad 2^{\frac{1}{2}} \xi = \xi_1$$

we obtain for *small* u

$$\frac{1}{\xi_1^2} \frac{d}{d\xi_1} \left[\xi_1^2 \left\{ 1 + \frac{7}{6} u^2 + O(u^4) \right\} \frac{du}{d\xi_1} \right] = -\left\{ 1 - \frac{1}{6} u^2 + O(u^4) \right\} u. \quad (13)$$

This has solution such that as $u \to 0$, $\xi_1 \to \xi_0 < \infty$. In fact it has solution of the type

$$u \sim (\xi_0 - \xi_1)^\alpha [A + B(\xi_0 - \xi_1) + C(\xi_0 - \xi_1)^2 + \cdots] \qquad (\xi_1 \to \xi_0)$$

with $\alpha > 0$. Substituting and keeping only relevant initial terms we have

$$\left[1 + \frac{7}{6} (\xi_0 - \xi_1)^{2\alpha} A^2 \right] [A\alpha(\alpha-1)(\xi_0-\xi_1)^{\alpha-2} + B\alpha(\alpha+1)(\xi_0-\xi_1)^{\alpha-1}$$

$$+ C(\alpha+1)(\alpha+2)(\xi_0-\xi_1)^\alpha] + \frac{7}{3} A(\xi_0-\xi_1)^\alpha \cdot A^2\alpha^2(\xi_0-\xi_1)^{2\alpha-2}$$

$$+ \frac{2}{\xi_0}\left(1 + \frac{\xi_0-\xi_1}{\xi_0}\right) [-A\alpha(\xi_0-\xi_1)^{\alpha-1} - B(\alpha+1)(\xi_0-\xi_1)^\alpha] = -A(\xi_0-\xi_1)^\alpha.$$

From this the initial coefficients can be calculated

$$\alpha = 1$$

$$2B - \frac{2}{\xi_0} A = 0, \quad 6C + \frac{7}{3} A^3 - \frac{4B}{\xi_0} - \frac{2A}{\xi_0^2} = -A$$

whence the asymptotic form (14) is found except for an arbitrary constant A. A determines the slope of the u curve at its zero ξ_0.

The form of (13) also shows that for every small u the solution of (13) will behave as the solution of

$$\frac{1}{\xi_1^2} \frac{d}{d\xi_1} \left(\xi_1^2 \frac{du}{d\xi_1} \right) = -u \qquad \ldots \quad \ldots \quad \ldots \quad (13a)$$

which represents a polytrope of index 1. The solution of (13a) is of the form $A(\sin \xi_1/\xi_1) + B(\cos \xi_1/\xi_1)$, which is oscillatory in character. The amplitude of the oscillation is of the order of $\frac{1}{\xi_1}$, while the difference in the solutions of (13) and (13a) in a sufficiently small interval round the zero of the solution (13a) can be shown to be at least of the order of $\frac{1}{\xi_1^2}$. So the solution in question of (13) will vanish just like that of (13a).

As regards I solutions we can show that (7a) and (9a) admit of solutions such that as $\sigma \to 0$, $\tau \to \tau_0 > 0$, while $\xi_1 \to \infty$.

Let, as $\tau \to \tau_0$, $q \to q_0$. Now since

$$\frac{dz}{dq} = \frac{d(\sigma\tau)}{d(\tau^{-4})} = -\frac{\tau^5}{4}\left(\tau\frac{d\sigma}{d\tau}+\sigma\right) \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

and as in I solutions $d\sigma/d\tau \to \infty$ as $\tau \to \tau_0$, we note that $dz/dq \to -\infty$ as $q \to q_0$, and $z \to 0$. By trial we get the following asymptotic solution

$$\left.\begin{aligned} z &\sim \frac{A}{\xi^\alpha}, \quad \alpha > 0. \quad (\xi \to \infty) \\ q &\sim q_0 - \frac{B}{\xi^{\alpha+c}}, \quad c > 0 \ (\xi \to \infty) \end{aligned}\right\}. \quad \ldots \quad \ldots \quad (15)$$

The condition $c > 0$ makes $dz/dq \to -\infty$ as $\xi \to \infty$. Substituting in (7a) we obtain

$$-\frac{A}{B}\frac{\alpha}{\alpha+c}\xi^c \sim -\frac{1}{Aq_0{}^2}\xi^\alpha$$

whence

$$c = \alpha, \ B = \frac{q_0{}^2A^2}{2}.$$

Further substitution in (9a) gives

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\frac{\alpha}{q_0{}^{\frac{1}{4}}}\xi\right) \sim A\xi^{-\alpha}q_0{}^{\frac{1}{4}} \ (\xi \to \infty)$$

whence

$$\alpha = 2, \text{ and } A = \frac{2}{q_0{}^{\frac{1}{2}}}.$$

Hence there is a solution of the type:

$$z \sim \frac{2}{q_0{}^{\frac{1}{2}}}\frac{1}{\xi^2}, \ q \sim q_0 - \frac{2q_0}{\xi^4}. \quad (\xi \to \infty) \quad \ldots \quad \ldots \quad (15a)$$

Substitution in (10) gives

$$M \sim \frac{\xi^2}{zq^{\frac{1}{4}}(z-1)}\frac{dz}{dq} \propto \xi \to \infty \text{ as } \tau \to \tau_0.$$

(15a) represents the asymptotic behaviour of I solutions for large ξ, and corresponds to configurations of infinite radii and masses.

On the other hand, it is not possible to construct a regular solution such that $z \to 0$, $q \to q_0$, while $\xi \to \xi_0 < \infty$. For instance, let us suppose

$$\left.\begin{aligned} z &\sim A(\xi_0-\xi)^\alpha, \ \alpha > 1 \\ \text{and} \quad q &\sim q_0 - B(\xi_0-\xi)^\beta, \ \beta > 0 \end{aligned}\right\}. \ (\xi \to \xi_0) \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

Then

$$\frac{dz}{dq} \sim -\frac{A\alpha}{B\beta}(\xi_0-\xi)^{\alpha-\beta}.$$

Hence $B > 0$ (A being positive), and $\beta > \alpha$. (17a)

Substituting in (7a) and comparing both sides of equality we obtain

$$\beta = 2\alpha,\ B = \frac{A^2\alpha\, q_0^2}{\beta} = \frac{A^2 q_0^2}{2}. \quad \ldots \quad \ldots \quad \ldots (17b)$$

If now we substitute in (9a), then among the leading terms we have the equivalence

$$\frac{\alpha}{q_0^{\frac{1}{2}}}(\xi_0-\xi)^{-2} \sim A(\xi_0-\xi)^{\alpha}\, q_0^{\frac{1}{2}} \quad \ldots \quad \ldots \quad \ldots (17c)$$

which is impossible as $\alpha > 1$.

An idea of the behaviour of the I solution as $z \to 0$ can be formed thus. We know that as $z \to 0$

$$\left.\begin{aligned} q &\sim q_0 - \frac{1}{2} q_0^2 z^2 \\ z q^{\frac{1}{2}} &\sim q_0^{\frac{1}{2}} z\left(1 - \frac{q_0}{8} z^2\right) \end{aligned}\right\} (z \to 0) \quad \ldots \quad \ldots (18)$$

and equation (9a) has the form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\frac{\xi^2}{z q_0^{\frac{1}{2}}}(1+z+O(z^2))\frac{dz}{d\xi}\right] = -z q_0^{\frac{1}{2}}\left[1 - \frac{q_0}{8} z^2 + O(z^2)\right] \ldots \quad (19)$$

Putting $\qquad \log z = -y, \quad q_0^{\frac{1}{2}}\xi = \xi_1$

we find as $y \to \infty$ the behaviour of the solution of (19) is the same as that of

$$\frac{1}{\xi_1^2}\frac{d}{d\xi_1}\left(\xi_1^2\frac{dy}{d\xi_1}\right) = e^{-y}\left[1 + \left(\frac{dy}{d\xi_1}\right)^2\right]. \quad \ldots \quad \ldots (19a)$$

We can convince ourselves that for $y \to \infty$ this again has solution behaving as that corresponding to an isothermal gas sphere given by the equation

$$\frac{1}{\xi_1^2}\frac{d}{d\xi_1}\left(\xi_1^2\frac{dy}{d\xi_1}\right) = e^{-y}. \quad \ldots \quad \ldots \quad \ldots (19b)$$

It is known that every solution of (19b) as $y \to \infty$ approaches its singular solution $e^{-y} = 2/\xi_1^2$, so that on every such solution as $y \to \infty$ and $\xi_1 \to \infty$, $(dy/d\xi_1)^2 \to 4/\xi_1^2$, which will ultimately be negligible compared to 1 on the right-hand side of (19a). Thus as $y \to \infty$ (19a) has solutions behaving as those of (19b). These are I solutions which ultimately correspond to isothermal gas spheres.

The asymptotic behaviour of the solution $e^{-y} = 2/\xi_1^2$ also throws light on our previous result (15a) where we found $z \propto 1/\xi^2$. We know in equation (19b) for isothermal spheres the term e^{-y} represents density, and in the present case where τ is ultimately constant (and $d\sigma/d\tau \to \infty$), $z = \sigma\tau$ ultimately will represent density and be expected to behave as e^{-y}.

4. The general case ($\gamma \neq 1$).

We have discussed the case $\gamma = 1$ in detail. The conclusions for this case apply to the more general cases also, but many of the simple arguments of the above section are not applicable in the more general cases. An analytical proof of the property of the S solution, constructed by a method followed by the author in a previous paper (Sen, 1942), has been given in the Appendix. As regards the I solutions we now consider the typical cases treated by Chandrasekhar, suppressing full arguments and details on every occasion.

Take the general γ-model. We put $\alpha = 0$, $b = 1$, $\nu = 1$ in (9) and (7), and have to satisfy the equations

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\frac{\xi^2}{zq^{\frac{1}{2}}}\cdot\frac{1}{(1-zq^{1-\gamma})}\cdot\frac{dz}{d\xi}\right] = -zq^{\frac{1}{2}} \qquad \ldots \qquad \ldots \qquad (9b)$$

and

$$\frac{dz}{dq} = \frac{1}{q^2}\left(1-\frac{q^{\gamma-1}}{z}\right). \qquad \ldots \qquad \ldots \qquad \ldots \qquad (7b)$$

For I solutions we seek the asymptotic forms of those solutions for which as $\tau \rightarrow \tau_0$, i.e. $q \rightarrow q_0$, $z \rightarrow 0$, and $dz/dq \rightarrow -\infty$.

We try the following solution

$$\left.\begin{aligned} z &\sim \frac{A}{\xi^\alpha}, \quad \alpha > 0 \\ q &\sim q_0 - \frac{B}{\xi^{\alpha+c}}, \quad c > 0 \end{aligned}\right\} (\xi \rightarrow \infty) \qquad \ldots \qquad \ldots \qquad (20)$$

A and B being positive. This gives

$$\frac{dz}{dq} \sim -\frac{A}{B}\cdot\frac{\alpha}{\alpha+c}\xi^c. \qquad (\xi \rightarrow \infty)$$

Substituting in (7b) and using expansions valid for small z, we obtain by comparing the terms of highest order

$$c = \alpha, \quad B = \frac{A^2}{2q_0{}^{\gamma-3}}. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (21a)$$

Again, the substitution of the asymptotic values (20) in (9b) shows that this latter equation can be satisfied by putting

$$\alpha = 2, \quad Aq_0{}^{\frac{1}{2}}/\alpha = 1. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (21b)$$

(21a) and (21b) determine the four constants in (20) which can be written as

$$\left.\begin{aligned} & z \sim \frac{2}{q_0{}^{\frac{1}{2}}}\cdot\frac{1}{\xi^2}, \quad q \sim q_0 - 2q_0{}^{2-\gamma}\frac{1}{\xi^4} \\ \text{so that} \qquad & \sigma \sim zq^{\frac{1}{2}} \sim \frac{2}{q_0{}^{\frac{1}{2}}}\frac{1}{\xi^2}. \end{aligned}\right\} \xi \rightarrow \infty \qquad \ldots \qquad \ldots \qquad (22)$$

The more general case ((α, δ)-model) may be treated similarly. In fact making the substitution (20) in the original equation (7) we obtain the relations

$$c = \nu\alpha, \quad B = \frac{A^{\nu+1}}{\nu+1}q_0{}^{2+\nu-\gamma}$$

and further substitution in (9) gives the same relation as (21*b*). The asymptotic form of the I solution in this general case is

$$z \sim \frac{2}{q_0^{\frac{1}{2}}} \cdot \frac{1}{\xi^2}, \quad q \sim q_0 - \frac{2^{\nu+1}}{\nu+1} \cdot q_0^{\frac{3+\nu-2\gamma}{2}} \cdot \frac{1}{\xi^4} \qquad (\xi \to \infty) \quad .. \quad (23)$$

The I solutions thus all correspond to infinite configurations which ultimately behave as isothermal gas spheres.

The case of the S solutions is treated rigorously in the Appendix. We can, however, get a physical insight into the nature of the solution near the boundary by an analysis similar to that followed for $\gamma = 1$. It has been shown by Chandrasekhar (1937) that for $\gamma > 2$, the behaviour of the S solution is given by

$$z \sim \tau^{4-2\gamma}\left(A - \frac{2}{4-\gamma}\tau^{2\gamma}\right) \qquad (\tau \to 0) \quad .. \quad .. \quad (24)$$

$$A = \sqrt{\frac{2}{2-\gamma}}$$

whence we calculate

$$\left.\begin{aligned} zq^{\frac{1}{2}} &= \sigma \sim A\tau^{3-2\gamma} - \frac{2}{4-\gamma}\tau^3 & \quad .. \quad .. \quad (25) \\ zq^{1-\gamma} &\sim A\tau^{2\gamma} - \frac{2}{4-\gamma}\tau^{4\gamma}\,. & \quad .. \quad .. \quad (26) \end{aligned}\right\} (\tau \to 0)$$

Substituting these in (9*b*), and using expansions valid near $\tau = 0$ it can be shown that as $\tau \to 0$, the standard solution (at least for $1 < \gamma < \frac{3}{2}$) behaves as the solution of the polytrope equation of index $3-2\gamma$.

PART II.

CONVECTIVE STABILITY OF THE MODELS.

5. THE STABILITY CONDITIONS.

We shall now examine how far the S and I solutions of Chandrasekhar conform to the stability conditions of the radiative gradient. Generally it will be useful to confine ourselves to those cases for which (τ, σ) curves have been obtained by Chandrasekhar. It will be necessary to introduce a new variable and connect it with the old ones.

(i) For *γ-models* defined by

$$\frac{L(r)}{M(r)} \cdot \frac{M}{L} = \eta = \eta_0 T^{\delta} \quad .. \quad .. \quad .. \quad (31)$$

let us put

$$4\pi cG\theta = \frac{\kappa L(r)}{M(r)}\,. \quad .. \quad .. \quad .. \quad (32)$$

Then for Kramers' law

$$\kappa = \kappa_0 \frac{\rho}{T^{3+s}} \qquad \ldots \qquad (2a)$$

we obtain from the three above equations

$$\theta = \left(\frac{a}{3}\frac{\mu H}{k} \cdot K\right)^{-1} \rho T^{4\gamma - 3}$$

where γ is defined by $4\gamma = (\delta - s)$, and K has the same value as in (6), except that we now put $\alpha = 0$, $b = 1$ in that formula. Further as

$$y = \beta/(1-\beta) = \frac{3}{a} \cdot \frac{k}{\mu H}\frac{\rho}{T^3}$$

we may write (32) as (introducing our previous variable x)

$$\theta = \frac{1}{K} y T^{4\gamma} = \frac{y}{x}. \qquad \ldots \qquad (33)$$

Transforming to (τ, σ) variables we may also write

$$\theta = \frac{y}{x} = \sigma \tau^{4\gamma - 3}, \; y = \frac{\sigma}{\tau^3}. \qquad \ldots \qquad (33')$$

(ii) *For the more general* (α, δ)*-model* of Chandrasekhar defined by (1) and (2), using variables

$$\rho = \rho_0 \sigma, \quad T = T_0 \tau \qquad \ldots \qquad (34)$$

where

$$\rho_0 = \left[\left(\frac{a}{3} \cdot \frac{\mu H}{k}\right)^{3+s+\delta} \left(\frac{\kappa_0 \eta_0 L}{4\pi c G M}\right)^3\right]^{\frac{1}{3(1-\alpha-b)+s-\delta}}$$

$$T_0 = \left[\left(\frac{a}{3} \cdot \frac{\mu H}{k}\right)^{\alpha+b} \left(\frac{\kappa_0 \eta_0 L}{4\pi c G M}\right)\right]^{\frac{1}{3(1-\alpha-b)+s-\delta}} \qquad \ldots \qquad (35)$$

we obtain after some calculation

$$\theta = \sigma^{\alpha+b} \tau^{\delta - s - 3}.$$

In our subsequent calculations we shall put $b = 1$ (*corresponding to cases integrated by Chandrasekhar*). We then obtain (for $b = 1$)

$$\theta = \sigma^{\alpha+1} \tau^{\delta - s - 3} = \frac{y^\nu}{x}, \; y = \frac{\sigma}{\tau^3}, \; x = \tau^{-4\gamma} \qquad \ldots \qquad (36)$$

where now

$$\alpha + 1 = \nu, \quad 4\gamma = \delta - s + 3\alpha.$$

It has been shown elsewhere (Sen, 1941) that *the radiative gradient is stable, or unstable and is to be replaced by an adiabatic gradient according as the expression*

$$f(y, \theta) \equiv \frac{\left(1 + \frac{1}{4}y\right)\theta}{1 - \left(1 + \frac{1}{4}y\right)\theta} - \frac{2}{3} \cdot \frac{(4+y)^2}{y(8+y)}$$

is negative or positive. Or simplifying, we conclude *the gradient is radiative or adiabatic, according as*

$$\theta \lesseqgtr \frac{8(4+y)}{5y^2+40y+3}. \qquad \ldots \quad \ldots \quad \ldots \quad (37)$$

We shall make use of the conditions (33), (36) and (37) to examine if the density temperature curves of Chandrasekhar's stellar models conform to the above condition of stable radiative gradient. Chandrasekhar's curves, it should be noted, have all been drawn on the assumption that the temperature gradient is radiative.

6 (γ-MODEL, $\alpha = 0$).

From (37) and (33′) it is evident that *for all members of the γ-model if we put*

$$F = \frac{y}{x} - \frac{8(4+y)}{5y^2+40y+32} \qquad \ldots \quad \ldots \quad \ldots \quad (38)$$

the gradient is radiative when $F<0$ and adiabatic if $F>0$. We shall henceforth always consider the curve $F = 0$ both in (x, y) and (τ, σ)-planes. We are interested only in the first quadrants of these planes. $F(x, y) = 0$ passes through the origin, and has $y = x$ as its tangent there. We further note that as $x \to \infty$, $F(x, y) = 0$ asymptotically approaches the parabola

$$y^2 = \frac{8}{5}x. \qquad (x \to \infty) \qquad \ldots \quad (39)$$

Again on $F(x, y) = 0$ we have from (38)

$$\frac{1}{x} = \left(\frac{1}{y} + \frac{5y^2+40y+128}{5y^2+40y+32}\right)\frac{dy}{dx}, \qquad \ldots \quad \ldots \quad \ldots \quad (40)$$

so that y is monotone increasing. For points in the area lying between $F(x, y) = 0$ and x-axis, $F<0$, and the stable gradient there must be radiative, whereas at every point above the curve $F = 0$ the stable gradient should be adiabatic. These results in (x, y)-plane are true for all members of the γ-model, but as soon as we go to the (τ, σ)-plane it becomes necessary to differentiate between cases for different values of γ. Let us first take the special case $\gamma = 1$. For this value of γ

$$\frac{dy}{dx} = \frac{3}{4}\frac{y}{x} - \frac{1}{4}\frac{1}{x^{\frac{1}{4}}}\frac{d\sigma}{d\tau}.$$

For $x \neq 0$ and at the point where $d\sigma/d\tau = 0$,

$$\frac{dy}{dx} = \frac{3}{4}\frac{y}{x}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (41)$$

Hence, if there be an extremum value of σ on $F = 0$ in (τ, σ)-plane (we shall presently show it is a maximum), the slope at the corresponding point of $F(x, y) = 0$ in the (x, y)-plane is given by (41).

Substituting (41) in (40), we note that $x(\neq 0)$ goes out of the equation altogether, and hence at the extremum on $F(\tau, \sigma) = 0$ the value of the ratio of the pressures is given by the equation

$$5y^3+30y+96y-64 = 0 . \quad .. \quad .. \quad .. \quad (42)$$

This equation has only one positive root, namely $y_0 = 0{\cdot}599$. There is thus *only one* extremum value of σ on $F(\tau, \sigma) = 0$. Now $x = 0$, $y = 0$ corresponds to $\sigma = 0$, $\tau = \infty$, and the tangent $y = x$ to $\sigma\tau = 1$. Thus in the (τ, σ)-plane $F = 0$ is asymptotic to $\sigma\tau = 1$ for $\tau \to \infty$ (which is also true of all solution curves in the (τ, σ)-plane).

Further, the asymptotic parabola (39) corresponds to

$$\sigma^2 = \frac{8}{5}\tau^2, \quad \text{i.e.} \quad \sigma = \sqrt{\frac{8}{5}}\tau \quad .. \quad .. \quad (39a)$$

on (τ, σ)-plane (in the first quadrant). Hence the slope of $F(\tau, \sigma) = 0$ at origin is given by (39*a*). It is thus evident that $F(\tau, \sigma) = 0$ has only one maximum in (τ, σ)-plane.

Substituting y_0 in $F = 0$ in (38) we find the corresponding value of x, namely $x_0 = 0{\cdot}858$. Then by substituting in $x = \tau^{-4}$, and $y = \sigma/\tau^3$, we find $\sigma_{\max} = 0{\cdot}628$, and the corresponding $\tau(\sigma_{\max}) = 1{\cdot}039$.

The curve $F(\tau, \sigma) = 0$ divides the (τ, σ)-plane (first quadrant) into two parts such that at all (τ, σ) points lying between the curve $F = 0$ and the τ-axis, the radiative gradient is stable, and at all points lying above this curve the radiative gradient is unstable and under suitable conditions can be replaced by a stable adiabatic gradient. Hence, our above analysis shows that for the γ-model with $\gamma = 1$, any configuration with a density value making $\sigma > 0{\cdot}628$ will be radiatively unstable. If we take the integral curves (for $\gamma = 1$) of Chandrasekhar in (τ, σ)-plane, and plot also the curve $F(\tau, \sigma) = 0$ in the same plane, then only those integral curves or parts of curves which lie below $F = 0$ will represent radiatively stable configurations. For instance we find the standard solution of Chandrasekhar (for $\gamma = 1$) as entirely radiatively unstable. Of the I solutions only the lower parts of the steep branch are radiatively stable. The case of the standard solution for $\gamma = 1$ can be roughly seen as follows. The standard solution $\sigma \sim \sqrt{2}\tau(\tau \to 0)$ leaves the origin at a slope $\sqrt{2}$, while $F(\tau, \sigma) = 0$ has the slope $\sqrt{\frac{8}{5}}$ at the origin. As $\sqrt{2} > \sqrt{\frac{8}{5}}$, near the origin the standard solution lies above $F = 0$. The maximum of the standard solution occurs at $\sigma = 0{\cdot}843$ which is greater than $\sigma_{\max} = 0{\cdot}628$ calculated above. We shall show presently that *for all* $\gamma > 0$ the standard solution curves lie above $F = 0$ as $\tau \to \infty$. This suggests that the standard solution for $\gamma = 1$, is probably wholly radiatively unstable. In fact the result of actual plotting is shown in fig. 2. *The standard solution for $\gamma = 1$ is thus entirely radiatively unstable.* We can now consider the general case of the γ-model. We first recapitulate the following results obtained from Chandrasekhar's integration of the (τ, σ) differential equation:—

(i) For $\gamma < 2$, *as* $\tau \to 0$, the solution curve has the form (Chandrasekhar, 1936)

$$\sigma \sim \sqrt{\frac{2}{2-\gamma}}\,\tau^{3-2\gamma}. \qquad \ldots \quad \ldots \quad \ldots \quad (43a)$$

Hence as $x = \tau^{-4\gamma}$, and $y = \sigma/\tau^3$ for $0 < \gamma < \frac{3}{4}$, $(x, y) \to \infty$ corresponds to $(\tau, \sigma) \to 0$, and further for $\frac{3}{4} < \gamma < 2$, $(x, y) \to \infty$ corresponds to $\tau \to 0$, $\sigma \to \infty$.

(ii) For every value of γ, *as* $\tau \to \infty$ the solution curve has the form

$$\sigma \sim \tau^{3-4\gamma}\left[1 + \frac{\gamma-1}{\tau^{4\gamma}} + \ldots\right] \qquad \ldots \quad \ldots \quad \ldots \quad (43b)$$

so that for $0 < \gamma < \frac{3}{4}$, $\quad \sigma \to \infty$, as $\tau \to \infty$

$\gamma > \frac{3}{4}$, $\quad \sigma \to 0$, as $\tau \to \infty$

$\gamma = \frac{3}{4}$, $\quad \sigma \to 1$, as $\tau \to \infty$.

We can now calculate the position of the curve $F(\tau, \sigma) = 0$ in (τ, σ)-plane. We first note that at the maximum σ we must have

$$\frac{dy}{dx} = \frac{3}{4\gamma}\cdot\frac{y}{x} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (41a)$$

whence, by substituting in (40) and after subsequent simplification, we obtain the following equation for the value of the pressure ratio at the maximum σ of $F(\tau, \sigma)$-curve in (τ, σ)-plane

$$5(2\gamma-3)y^3 + 30(4\gamma-5)y^2 + 96(4\gamma-5)y + 64(4\gamma-3) = 0. \qquad \ldots \quad (44)$$

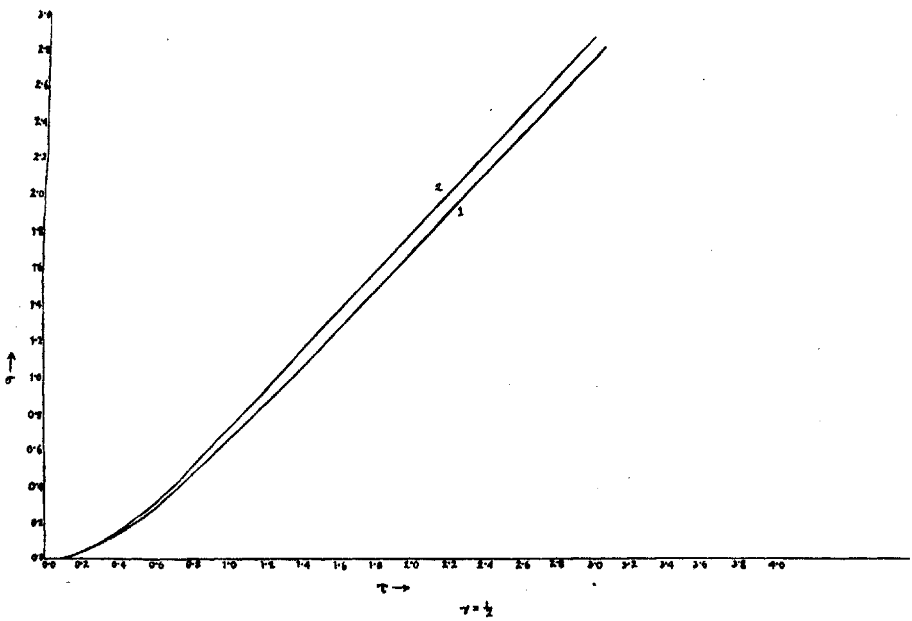

FIG. 1(*a*).—(2) is the standard solution curve for $\gamma = \frac{1}{2}$, and (1) is the curve $F = 0$ separating the regions of stable and unstable radiative gradients. The lower part has been magnified in fig. 1(*b*).

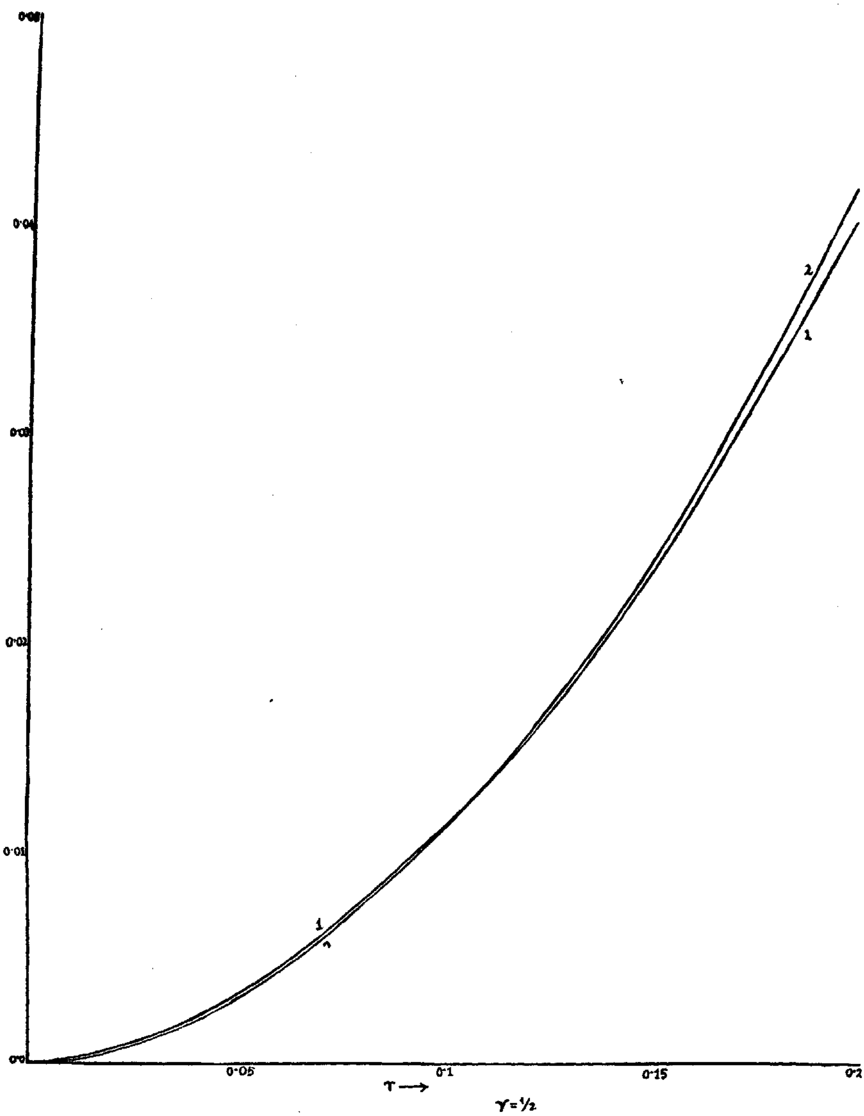

FIG. 1(*b*).—(2) is the standard solution curve for $\gamma = \frac{1}{2}$, and (1) is the curve $F = 0$ separating the region of unstable from that of stable radiative gradient which latter lies between this curve and τ-axis. The standard solution for $\gamma = \frac{1}{2}$ is radiatively stable only for τ values lying between 0 and 0·11 and σ values between 0 and 0·0137.

From a discussion of the roots of this cubic equation we obtain the following results:

For $\gamma > \frac{3}{2}$, *or* $\gamma < \frac{3}{4}$, there are *no* positive roots; for $\frac{3}{4} < \gamma < \frac{3}{2}$ there is *only one* positive root.

Hence for every value of γ satisfying $\frac{3}{4} < \gamma < \frac{3}{2}$ there exists a maximum value of σ, say $\sigma_{\max}(\gamma)$, such that a gas configuration in equilibrium with a density-value which makes the corresponding $\sigma > \sigma_{\max}(\gamma)$ cannot have a stable radiative gradient there.

We can now roughly locate the curve $F(\tau, \sigma) = 0$ among the integral curves thus.

For all γ-models, such that $0 < \gamma < \frac{3}{2}$, the behaviour of the S solution curves as $\tau \to 0$, $\sigma \to 0$ is given by (43a); whereas (39) shows as $\tau \to 0$, $\sigma \to 0$ the F curve in (τ, σ)-plane has the form

$$\sigma \sim \sqrt{\frac{8}{5}}\,\tau^{3-2\gamma}. \qquad (\tau \to 0)$$

Hence as $\sigma \to 0$, $\tau \to 0$ (close to the boundary), the S solution curve is steeper than the F curve if $2/(2-\gamma) > \frac{8}{5}$, i.e. if $\gamma > \frac{3}{4}$, so that $F = 0$ lies below the S solution. *Hence for $\gamma > \frac{3}{4}$ the radiative gradient is unstable near the boundary; for $\gamma < \frac{3}{4}$ it is stable.*

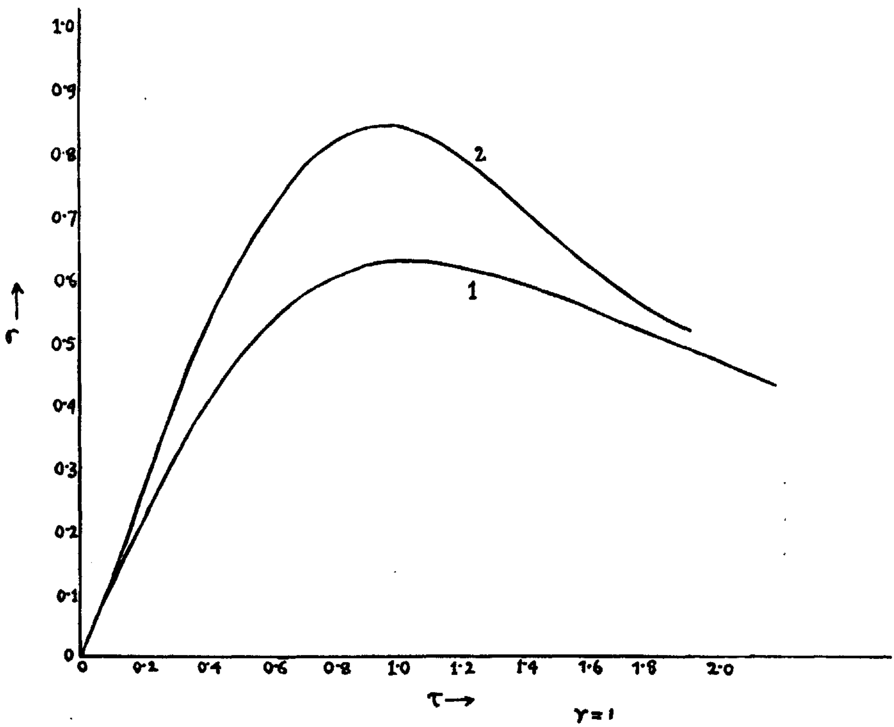

FIG. 2.—(2) is the standard solution curve for $\gamma = 1$, and (1) is the curve $F = 0$ dividing the region of unstable radiative gradient from that of stable radiative gradient which latter lies between this curve and τ-axis. The figure shows the standard solution for $\gamma = 1$ is radiatively unstable all throughout.

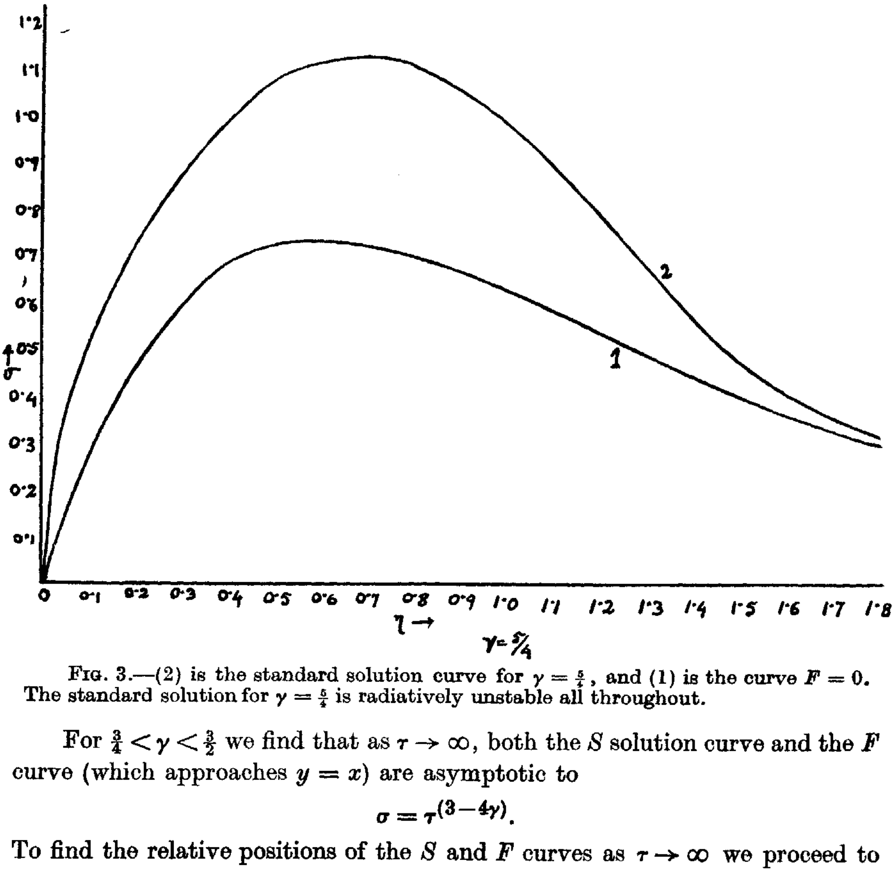

FIG. 3.—(2) is the standard solution curve for $\gamma = \frac{5}{4}$, and (1) is the curve $F = 0$. The standard solution for $\gamma = \frac{5}{4}$ is radiatively unstable all throughout.

For $\frac{3}{4} < \gamma < \frac{3}{2}$ we find that as $\tau \to \infty$, both the S solution curve and the F curve (which approaches $y = x$) are asymptotic to

$$\sigma = \tau^{(3-4\gamma)}.$$

To find the relative positions of the S and F curves as $\tau \to \infty$ we proceed to the next higher approximation. We find

$$\left.\begin{array}{ll} \text{for } F \text{ curve} & \sigma \sim \tau^{3-4\gamma}(1-\tau^{-4\gamma}) \\ \text{for } S \text{ curve} & \sigma \sim \tau^{3-4\gamma}[1-(\gamma-1)\tau^{-4\gamma}]. \end{array}\right\} (\tau \to \infty) \qquad \text{..} \quad (46)$$

Hence *so long as* $\gamma > 0$, $F(\tau, \sigma) = 0$ lies below the S solution curve as $\tau \to \infty$, so that *ultimately the S solutions all lie in the region of unstable radiative gradient.*

Summing up, for $\frac{3}{4} < \gamma < \frac{3}{2}$ the S solutions, as $\tau \to 0$, $\sigma \to 0$, all run in the region of unstable radiative gradient and also lie in the same region as $\tau \to \infty$. For $0 < \gamma < \frac{3}{4}$, the S solutions start at $\tau = 0$, $\sigma = 0$ with stable radiative gradients, but ultimately as $\tau \to \infty$ run into the region of unstable radiative gradient. As the figures 1(a) and 1(b) show then, the S and F curves in the (τ, σ)-plane cross only once, so that only a part of the solution starting from the boundary $\tau = 0$ to a maximum τ is radiatively stable. Hence, if $0 \leqslant \gamma < \frac{3}{4}$, there is for a given γ, a $\tau_{\max}(\gamma)$, such that at all temperatures for which the corresponding $\tau > \tau_{\max}(\gamma)$, the standard solution is radiatively unstable. Similarly, there is a minimum $y_{\min}(\gamma)$, such that whenever the ratio of the pressures $y < y_{\min}(\gamma)$, the standard solution is radiatively unstable. In the case of $\gamma = \frac{3}{4}$ equation (44) reduces to (leaving out the root $y = 0$)

$$15y^2+120y+384 = 0$$

which has no real positive root. Hence $F(\tau, \sigma) = 0$ in (τ, σ) plane has no maximum.

On the standard solution S for $\gamma = \frac{3}{4}$, we find σ behaves as

$$\sigma_S \sim \tau^{\frac{3}{2}}\left[\sqrt{\frac{8}{5}} - \frac{8}{13}\tau^{\frac{3}{2}}\right] \qquad (\tau \to 0) \qquad \ldots \quad \ldots (47a)$$

while on $F(\tau, \sigma) = 0$ we find after some calculation the behaviour of σ to be given by

$$\sigma_F \sim \tau^{\frac{3}{2}}\left[\sqrt{\frac{8}{5}} - 4\tau^{\frac{3}{2}}\right]. \qquad (\tau \to 0) \qquad \ldots \quad \ldots (47b)$$

Further, as $\tau \to \infty$

$$\sigma_F \sim 1-\tau^{-3}, \quad \sigma_S \sim 1-\frac{1}{4}\tau^{-3}. \qquad \ldots \quad \ldots (47c)$$

Thus at both ends $\sigma_S > \sigma_F$. The S solution curve for $\gamma = \frac{3}{4}$ (as is found by actual plotting of curves) lies indeed entirely above $F(\tau, \sigma) = 0$, and so *the radiative gradient must be unstable all throughout for* $\gamma = \frac{3}{4}$.

The F and S curves have been actually plotted and three cases are shown in figs. 1(a), 2 and 3 for $\gamma = \frac{1}{2}$, $\gamma = 1$ and $\gamma = \frac{5}{4}$ respectively. The crossing of the two curves for $\gamma = \frac{1}{2}$ is shown in fig. 1(b) drawn in a magnified scale. 1(a) and 2 are typical of the cases $0 < \gamma < \frac{3}{4}$ and $\frac{3}{4} < \gamma < \frac{3}{2}$ respectively.

Our result for the γ-model can be summarised thus. *For* $0 < \gamma < \frac{3}{4}$ *only a part of the standard solution from the boundary up to a definite minimum value of the ratio of gas to radiation pressure (to a maximum value of* τ*) depending on* γ *is radiatively stable; hence such a configuration is wholly radiatively stable only if this pressure ratio at the centre does not fall below a minimum critical value (and the central* τ *value does not exceed a critical maximum value); in the contrary case the configuration is radiatively unstable, the unstable region being an internal one involving the centre. For* $\frac{3}{4} \leqslant \gamma < \frac{3}{2}$*, the standard solution is entirely radiatively unstable.*

By examining the position of the F curve among I solutions it becomes apparent that *the lower portions of I solutions which fall steeply must be all radiatively stable. There exists a critical minimum pressure ratio in this case also.*

This result clearly points to the necessity of sometimes including a central convection zone (with adiabatic gradient) in the construction of stellar models.

7. (α, δ)-MODELS.

The analysis of the two previous sections discusses the stability of radiative equilibrium of a typical stellar model (γ-model) discussed by Chandrasekhar. We shall now very briefly describe the general characteristics of the more general (α, δ)-model in respect of the stability of the radiative transfer of energy.

We define in this connection a function F_1

$$F_1 = x - \frac{y^\nu (5y^2+40y+32)}{8(4+y)} \quad \ldots \ldots \ldots (48)$$

where x, y and ν are given by (36) (ν may be taken positive). We can show that for given (x, y) values the stable gradient is radiative or adiabatic according as

$$F_1 \gtrless 0. \quad \ldots \ldots \ldots \ldots (49)$$

The curve $F_1(x, y) = 0$ divides the (first quadrant of) (x, y)-plane into two regions of which the lower one between $F_1 = 0$ and x-axis represents the region of stable radiative gradient, while the upper one corresponds to the region of stable adiabatic gradient.

The gradient of $F_1(x, y) = 0$ is given by

$$\frac{dy}{dx}\left\{\frac{\nu}{y}+\frac{5y^2+40y+128}{(4+y)(5y^2+40y+32)}\right\}=\frac{1}{x} \quad \ldots \ldots (50)$$

so that (ν being positive) y is monotone increasing with x. From (36) we obtain when $d\sigma/d\tau = 0$

$$\frac{dy}{dx} = \frac{3}{4\gamma} \cdot \frac{y}{x}.$$

Substituting this value of dy/dx in (50), we obtain the value of y at that point of $F_1(\tau, \sigma) = 0$ where σ has an extreme value. Proceeding as before, we find this value of y to be given by

$$5(d-3)y^3+60(d-2)y^2+192(d-2)y+128d = 0 \quad \ldots \ldots (51)$$

where $$d = 4\gamma - 3\nu,$$

γ and ν being now given by (36).

A simple discussion shows that when $0 < d < 3$ equation (51) has only one positive root, and for $d \geqslant 3$ or $d \leqslant 0$, (51) has no positive root. Hence for $0 < 4\gamma - 3\nu < 3$, i.e. when $3+s < \delta < 6+s$, the equation $F_1(\tau, \sigma) = 0$ on (τ, σ) plane has only one maximum, say $\sigma_{\max}$, and for any $\sigma > \sigma_{\max}$, the corresponding radiative gradient cannot be stable. When $x \to 0$, $F_1 = 0$ has the asymptotic form

$$x = y^\nu, \quad \ldots \ldots \ldots \ldots (52a)$$

and for y large, $F_1 = 0$ has the form

$$x = \frac{5}{8} y^{\nu+1}$$

or

$$y = \left(\frac{8}{5}x\right)^{\frac{1}{\nu+1}}. \quad \ldots \ldots \ldots \ldots (52b)$$

Near the origin of (τ, σ)-plane, we have the following approximate forms of $F_1 = 0$, and of the standard solution S

$$F_1: \quad \sigma \sim \left(\frac{8}{5}\right)^{\frac{1}{\nu+1}} \tau^{3-\frac{4\gamma}{\nu+1}} \qquad \ldots \quad \ldots \quad \ldots \quad (52c)$$

$$S: \quad \sigma \sim \left(\frac{\nu+1}{\nu+1-\gamma}\right)^{\frac{1}{\nu+1}} \tau^{3-\frac{4\gamma}{\nu+1}}. \qquad \ldots \quad \ldots \quad (52d)$$

Hence the standard solution S lies below F_1 curve if

$$\frac{\nu+1}{\nu+1-\gamma} < \frac{8}{5}$$

i.e.

$$\gamma < \frac{3}{8}(\nu+1)$$

which by (36) is equivalent to

$$\delta < 3+s-\frac{3}{2}\alpha. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (53)$$

Thus, when (53) is satisfied the standard solution S runs in the region of stable radiative gradient near $\tau = 0$; for the opposite inequality it is in the region of unstable radiative gradient. In every case the $F_1(\tau, \sigma)$ curve has a maximum, the standard solution has unstable radiative gradient near the boundary.

We shall now find the position of $F_1 = 0$ as $\tau \to \infty$, i.e. $x \to 0$ while $(\gamma > 0)$. In this region we find the following asymptotic forms for F_1 and S:

$$F_1: \quad \sigma \sim \tau^{3-\frac{4\gamma}{3}} \left\{1 - \frac{1}{\nu}\tau^{-\frac{4\gamma}{\nu}}\right\} \qquad \ldots \quad \ldots \quad (54a)$$

$$S: \quad \sigma \sim \tau^{3-\frac{4\gamma}{3}} \left\{1 - \frac{1}{\nu}\frac{\left(1-\frac{\gamma}{\nu}\right)}{\tau^{4\gamma/\nu}}\right\}. \qquad \ldots \quad \ldots \quad (54b)$$

Hence for $4\gamma = \delta - s + 3\alpha > 0$ (ν being supposed positive) the standard solution is ultimately (i.e. as $\tau \to \infty$) in the region of unstable radiative gradient.

For $\delta - s + \frac{3\alpha}{2} > 3$, the configuration given by the standard solution is radiatively unstable; for $-\frac{3\alpha}{2} < \delta - s + \frac{3\alpha}{2} < 3$, the configuration given by the standard solution has stable radiative gradient up to a minimum value of the ratio of gas to radiation pressure. It is evident that for $\alpha \geqslant 2$ radiative stability is not possible if only $\delta > s$, which will be satisfied for energy generation law with even quite small power of temperature.

The I solutions, as before, will be all radiatively stable along lower parts of their vertical downward course, there being a critical minimum pressure ratio in this case also.

Appendix.

To bring the analysis in a line with our previous work (Sen, 1942) on the behaviour of the solutions of stellar equations near the boundary of the configurations, we introduce the notations used elsewhere, which are different from those used in the present paper. The Appendix forms a section by itself, and its equations, unless expressly stated, should not be compared with those of this paper.

We take a more general case, and suppose the opacity and energy generation to be given by the following laws:

$$\kappa = \kappa_0 \rho^a T^{-b} \qquad \ldots \quad (a)$$

$$\eta = \frac{L(r)}{M(r)} = \eta_0 \rho^\alpha T^\delta. \qquad \ldots \quad (b)$$

Consider equations (3), (4) of section 2, and

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho$$

$$p_g = \frac{k}{\mu H}\rho T, \quad p_r = \frac{1}{3} a T^4$$

and substitute in these equations

$$p_g = (p_g)_0 p, \quad p_r = (p_r)_0 q, \quad T = T_0 \tau, \quad \rho = \rho_0 \sigma, \quad r = r_0 x, \quad M = M_0 m,$$

where ρ_0, T_0 are respectively the central density and temperature. Then we can replace the above equations by the following set of equations among dimensionless quantities,

$$dp + \alpha' dq = -\beta \frac{\sigma m}{x^2} . dx \qquad \ldots \quad (c)$$

$$dq = -\sigma^{a+\alpha+1} \tau^{\delta-b} \frac{m}{x^2} dx \qquad \ldots \quad (d)$$

$$dm = \sigma x^2 \, dx \qquad \ldots \quad (e)$$

where the scale constants are connected by the equations

$$M_0 = 4\pi r_0^3 \rho_0, \quad \frac{\kappa_0 \eta_0}{4\pi c r_0} . \frac{\rho_0^{a+\alpha+1}}{(p_r)_0} T_0^{\delta-b} . M_0 = 1$$

and

$$(p_g)_0 = \frac{k}{\mu H} \rho_0 T_0, \quad (p_r)_0{}^* = \frac{1}{3} a T_0^4, \quad \alpha' = (p_r)_0/(p_g)_0, \quad \beta = -\frac{G M_0 \rho_0}{r_0 (p_g)_0}.$$

Now put

$$\xi = \frac{1}{x}, \quad \pi = p + \alpha' q, \quad a + \alpha = d, \quad \delta - b - \alpha - a = 4f$$

and

$$p q^{-\frac{1}{4}} d\xi = dt. \qquad \ldots \quad (g)$$

---

* a in this formula need not be confused with a in the expression for κ in (a).

Then we replace (*c*), (*d*), (*e*) by

$$\frac{d\pi}{dt} = \beta m \quad \dots \quad \text{(i)}$$

$$\frac{dq}{dt} = mp^d q^f \quad \dots \quad \text{(ii)}$$

$$\frac{dm}{dt} = -\xi^{-4} \quad \dots \quad \text{(iii)}$$

to which we add (*g*) as

$$\frac{d\xi}{dt} = \frac{q^{\frac{1}{4}}}{p}. \quad \dots \quad \text{(iv)}$$

Equations (i)–(iv) show that as t decreases ξ also decreases, and if as $t \to 0$, ξ also tends to a finite limit $\xi_0 > 0$, then m also tends to a finite limit m_0 (which in astrophysically significant solutions we take to be greater than zero), and then π also decreases to a finite limit π_0.

Since the right-hand sides of (i), (ii), (iii) and (iv) do not involve t, the solutions of the equations will not depend on the initial value of t. We shall take $t = 0$ to correspond to the boundary surface. Let us investigate the behaviour of a solution for which $m \to m_0 > 0$, as $t \to 0$. At the boundary of the configuration we take $\tau = 0$ and $\sigma = 0$, so that $\pi = 0$. Hence by (i)

$$\pi \sim \beta m_0 t. \quad (t \to 0) \quad \dots \quad (h)$$

Let us try the solution

$$q = At^n. \quad (n > 0) \quad (t \to 0) \quad \dots \quad (i)$$

Then (ii) gives

$$Ant^{n-1} \sim m_0(\beta m_0 t - \alpha At^n)^d . \, A^f t^{nf}.$$

This will be generally satisfied if q be a small quantity of higher order than p, and hence

$$n-1 = d+nf, \quad An = \beta^d m_0^{\ d+1} A^f$$

which give

$$n = \frac{d+1}{1-f}, \quad A = \left(\frac{d+1}{1-f} \cdot \frac{1}{\beta^d \, m_0^{\ d+1}}\right)^{\frac{1}{1-f}}. \quad \dots \quad (j)$$

Since $n > 0$, we must have

$$1 > f. \quad \dots \quad \text{(v)}$$

Equation (iv) gives

$$\frac{d\xi}{dt} \sim \frac{A^{\frac{1}{4}}}{\beta m_0} . \, t^{\frac{1}{4}\left(\frac{d+1}{1-f}\right)-1} \quad (t \to 0)$$

whence integrating

$$\xi \sim \xi_0 + \frac{A^{\frac{1}{4}}}{\beta m_0} \cdot \frac{4(1-f)}{d+1} t^{\frac{1}{4}\left(\frac{d+1}{1-f}\right)}. \quad \dots \quad (k)$$

Since the right-hand side of (iv) is σ^{-1}, $d\xi/dt \to \infty$ as $t \to 0$. Hence

$$\frac{1}{4}\left(\frac{d+1}{1-f}\right) < 1, \quad \text{i.e.} \quad d+4f < 3. \quad \dots \quad \text{(vi)}$$

Now the assumption that q is of higher order than p implies $n > 1$, i.e.

$$\frac{d+1}{1-f} > 1, \qquad \text{i.e.}\quad d+f > 0\,. \quad .. \qquad .. \qquad ... \text{(vii)}$$

Now suppose, if possible, $\xi_0 = 0$. Then we get from (iii)

$$\frac{dm}{dt} \sim \text{const.}\ t^{-\frac{d+1}{1-f}} \qquad (t \to 0)$$

so that

$$m \sim m_0 - \text{const.}\ t^{-\left(\frac{f+d}{1-f}\right)}. \quad (t \to 0) \quad .. \qquad .. \qquad .. \qquad (l)$$

But on account of (v) and (vii), (l) contradicts the assumption $m \to m_0$ as $t \to 0$. Hence $\xi_0 \neq 0$. We have then the solution

$$m \sim m_0 - \xi_0{}^{-4} t \qquad (t \to 0)$$

and

$$p = \beta m_0 t\,, \ q \sim A t^{\frac{d+1}{1-f}}, \ \xi \sim \xi_0 + \frac{A^{\frac{1}{4}}}{\beta m_0} \cdot \frac{4(1-f)}{d+1} t^{\frac{1}{4}\left(\frac{d+1}{1-f}\right)}$$

with the value of A in (j). Hence

$$p \sim \frac{\beta m_0}{A^{\frac{1-f}{d+1}}} \cdot \tau^{\frac{4(1-f)}{d+1}}$$

so that

$$\sigma \sim \frac{\beta m_0}{A^{(1-f)/(d+1)}} \cdot \tau^{\frac{4(1-f)}{d+1} - 1}. \quad (\tau \to 0) \qquad .. \qquad .. \quad (m)$$

These solutions are valid when

$$f < 1\,, \ d+f > 0\,, \text{ and } d+4f < 3$$

of which the second and the third relations together imply the first. The form (m) shows that it is Chandrasekhar's S solution. Replacing d, f by α, δ, etc., *we conclude that the S solutions correspond to finite configurations provided*

$$3(a+\alpha) > b - \delta\,, \quad \text{and } \delta - b < 3\,. \ .. \qquad .. \qquad \text{(vi, vii)}$$

In case of γ-models, we put $\alpha = 0$, $a = 1$, $4\gamma = \delta - b + 3$. The above conditions show that for γ-models the S configuration is finite when γ lies between 0 and $\frac{3}{2}$. In case of $\gamma = 0$, it can be easily shown that p and q are both of the first order in t; then an S solution corresponding to finite configuration can be constructed. Hence *for γ-models all S solutions give finite configurations so long as* $0 \leqslant \gamma < \frac{3}{2}$.

It can be shown that the behaviour of I solutions as $\sigma \to 0$, $\tau \to \tau_0$ is given by

$$p \sim A t^{\frac{2}{3}}, \quad q \sim q_0 + B t^{\frac{2}{3}(1+d)}, \quad \xi \sim D t^{\frac{1}{3}}, \quad m \sim C t^{-\frac{1}{3}} \qquad (t \to 0) \quad (n)$$

where A, B, C, D are the constants

$$A = \left(\frac{18 q_0}{\beta}\right)^{\frac{1}{3}}, \ B = \frac{3 q_0{}^f}{2(1+d)} \left(\frac{16 q_0{}^{d+1}}{3\beta^{4+d}} \,.\, 18^d\right)^{\frac{1}{3}}, \ C = \left(\frac{16 q_0}{3\beta^4}\right)^{\frac{1}{3}}, \ D = \left(\frac{3\beta}{2 q_0{}^{\frac{1}{2}}}\right)^{\frac{1}{2}}.$$

It should be noted that ξ now represents the reciprocal of the radial distance.

We can further show easily that there are no solutions which correspond to finite configurations with non-zero temperature at the boundary. Further calculations here are suppressed.

References.

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